MATH 222 (Lectures 1,2,4) **Worksheet 7** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Find a solution to the initial value problem

$$\frac{dy}{dx} = e^y x^3$$
$$y(0) = 0$$

Solution: In what follows, the value of the constant of integration may change from line to line.

$$\begin{aligned}
\frac{dy}{dx} &= e^y x^3 \\
e^{-y} dy &= x^3 dx \\
\int e^{-y} dy &= \int x^3 dx \\
-e^{-y} &= \frac{1}{4}x^4 + C \\
e^{-y} &= -\frac{1}{4}x^4 + C \\
y &= -\ln(C - \frac{1}{4}x^4)
\end{aligned}$$

Substituting the initial condition $0 = y(0) = -\ln(C)$, we find that $C = 1$ and $y(x) = -\ln(1 - \frac{1}{4}x^4)$

2. Find a solution to the initial value problem

$$\frac{dy}{dx} = (1 + y^2)e^x$$
$$y(0) = 0$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= (1 + y^2)e^x \\
\frac{dy}{1 + y^2} &= e^x dx \\
\int \frac{dy}{1 + y^2} &= \int e^x dx \\
\arctan(y) &= e^x + C \\
y &= \tan(e^x + C)
\end{aligned}$$

Substituting in the initial condition, we find that $0 = Y(0) = \tan(1 + C)$. A possible choice of C is $C = -1$. Our final answer is then $y(x) = \tan(e^x - 1)$.

3. Find a solution to the initial value problem

$$\begin{aligned}\frac{dy}{dx} &= y\sqrt{y^2-1}\cos(x)\\ y(0) &= 1\end{aligned}$$

Solution: First, we can observe that one solution to this problem is given by $y(x) = 1$. We can find another solution by separating variables.

$$\begin{aligned}\frac{dy}{dx} &= y\sqrt{y^2-1}\cos(x)\\ \frac{dy}{y\sqrt{y^2-1}} &= \cos(x)dx\\ \int \frac{dy}{y\sqrt{y^2-1}} &= \int \cos(x)dx\\ \operatorname{arcsec}(y) &= \sin(x) + C\\ y &= \sec(\sin(x) + C)\end{aligned}$$

Substituting in the initial condition $y(0) = 1$ we find that

$$1 = y(0) = \sec(C)$$

So we may take, for example, $C = 0$. Our final solution is then either of $y(x) = 1$ or $y(x) = \sec(\sin(x))$.

4. Find the general solution to the differential equation

$$\frac{dy}{dx} = x^2 + y^2x^2$$

Solution:

$$\begin{aligned}\frac{dy}{dx} &= x^2 + y^2x^2\\ \frac{dy}{dx} &= x^2(1+y^2)\\ \frac{dy}{1+y^2} &= x^2dx\\ \int \frac{dy}{1+y^2} &= \int x^2dx\\ \arctan(y) &= \frac{x^3}{3} + C\\ y(x) &= \tan\left(\frac{x^3}{3} + C\right)\end{aligned}$$

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5. Find the general solution to the differential equation

$$\frac{1}{2x}\frac{dy}{dx} = y + e^{x^2}$$

Solution: We begin by writing the problem in standard form as

$$\frac{dy}{dx} - 2xy = 2xe^{x^2}$$

The integrating factor for this problem is $m(x) = e^{\int -2x dx} = e^{-x^2}$. If we multiply through by e^{-x^2}, then the equation becomes separable and we can find the general solution directly directly.

$$\begin{aligned}
e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y &= 2x \\
\frac{d(e^{-x^2}y)}{dx} &= 2x \\
\int d(e^{-x^2}y) &= \int 2x dx \\
e^{-x^2}y &= x^2 + C \\
y(x) &= x^2e^{x^2} + Ce^{x^2}
\end{aligned}$$

6. Find a solution to the initial value problem

$$\begin{aligned}
\frac{dy}{dx} &= (y-1)\frac{1}{x} \\
y(-1) &= 0
\end{aligned}$$

Solution:

$$\frac{dy}{dx} = (y-1)\frac{1}{x}$$

$$\begin{aligned}
\frac{1}{y-1}\frac{dy}{dx} &= \frac{1}{x} \\
\frac{1}{y-1}dy &= \frac{1}{x}dx \\
\int \frac{1}{y-1}dy &= \int \frac{1}{x}dx \\
\ln|y-1| &= ln|x| + c \\
y - 1 &= \pm|x|e^c \\
y &= 1 \pm |x|e^c
\end{aligned}$$

We are working near -1, so $|x| = -x$. Plugging in $y(-1) = 0$,

$$0 = 1 \pm e^c \underbrace{(-(-1))}_{|-1|}$$

e^c is always positive, so we must have

$$0 = y(-1) = 1 - e^c$$

Thus $1 = e^c$ and we get as our final answer

$$\begin{aligned} y(x) &= 1 - e^c(-x) \\ y(x) &= 1 + x \end{aligned}$$

7. Find a solution to the initial value problem

$$\begin{aligned} x\frac{dy}{dx} + 2y &= -\frac{\sin(x)}{x} \\ y(\frac{\pi}{2}) &= 1 \end{aligned}$$

Solution: We being by writing the differential equation in standard form as

$$\frac{dy}{dx} + \frac{2}{x}y = -\frac{\sin(x)}{x^2}$$

The integrating factor for this problem is $m(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = x^2$. Multiplying through by x^2 converts this problem to

$$\begin{aligned} x^2\frac{dy}{dx} + 2xy &= -\sin(x) \\ \frac{d(x^2y)}{dx} &= -\sin(x) \\ \int d(x^2y) &= -\int \sin(x)dx \\ x^2y &= \cos(x) + C \\ y(x) &= \frac{\cos(x)}{x^2} + \frac{C}{x^2} \end{aligned}$$

Susbtituting in the initial condition, we find that

$$1 = y(\frac{\pi}{2}) = \underbrace{\frac{\cos(\frac{\pi}{2})}{(\frac{\pi}{2})^2}}_{0} + \frac{C}{(\frac{\pi}{2})^2}$$

so that $y(x) = \frac{\cos(x)}{x^2} + \frac{\pi^2}{4}\frac{1}{x^2}$.