MATH 222 (Lectures 1,2,4) **Worksheet 6** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Compute $\int_{-\infty}^{0} \frac{x}{1+x^2} dx$ and $\int_{0}^{\infty} \frac{x}{1+x^2} dx$. What does this say about $\int_{-\infty}^{\infty} \frac{x dx}{1+x^2}$?

   Solution:

$$\begin{aligned}
\int_{-\infty}^{0} \frac{x}{1+x^2} dx &= \lim_{A \to -\infty} \int_{A}^{0} \frac{x dx}{1+x^2} \\
&= \lim_{A \to -\infty} \left[\frac{1}{2} \ln(1+x^2) \right]_{A}^{0} \\
&= \lim_{A \to -\infty} \ln(1) - \ln(1+A^2) \\
&= -\infty \\
\int_{0}^{\infty} \frac{x}{1+x^2} dx &= \lim_{B \to \infty} \int_{0}^{B} \frac{x dx}{1+x^2} \\
&= \lim_{B \to \infty} \left[\frac{1}{2} \ln(1+x^2) \right]_{0}^{B} \\
&= \lim_{B \to \infty} \ln(1+B^2) - \ln(1) \\
&= \infty
\end{aligned}$$

   Notice that this problem shows that $\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$ does not exist.

2. Show that $\int_{1}^{\infty} \frac{dx}{x^2-4}$ is not a finite number. What answer do you get if you forget that the integrand has an asymptote at 2 and fail to split the integral up there?

   Solution: You can use partial fractions to compute that

$$\int \frac{dx}{x^2-4} = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$$

   If we write

$$\int_{1}^{\infty} \frac{dx}{x^2-4} = \int_{1}^{2} \frac{dx}{x^2-4} + \int_{2}^{3} \frac{dx}{x^2-4} + \int_{3}^{\infty} \frac{dx}{x^2-4}$$

   it suffices to show that one of these integrals is infinite. For example

$$\begin{aligned}
\int_{1}^{2} \frac{dx}{x^2-4} &= \lim_{A \uparrow 2} \int_{1}^{2} \frac{dx}{x^2-4} \\
&= \lim_{A \uparrow 2} \left[\frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| \right]_{1}^{A} \\
&= \lim_{A \uparrow 2} \frac{1}{4} \ln \left| \frac{A-2}{A+2} \right| - \frac{1}{4} \ln \left| \frac{1}{2} \right| \\
&= -\infty
\end{aligned}$$

Now we will see what happens if we forget that the integrand has an asymptote at 2.

$$\lim_{A\to\infty} \frac{1}{4}\ln|\frac{A-2}{A+2}| - \frac{1}{4}\ln|\frac{-1}{3}| = \frac{1}{4}\ln(3)$$

3. Compute $\int_1^2 \frac{dt}{t\sqrt{t^2-1}}$.

   Solution:

$$\begin{aligned}\int_1^2 \frac{dt}{t\sqrt{t^2-1}} &= \lim_{B\to 1}\int_B^2 \frac{dt}{t\sqrt{t^2-1}}\\ &= \lim_{B\to 1}\ \text{arcsec}(2) - \ \text{arcsec}(B)\\ &= \text{arcsec}\ (2)\end{aligned}$$

4. Compute $\int_{10}^{\infty} \frac{dx}{x^2-9}$.

   Solution:

$$\begin{aligned}\int_{10}^{\infty}\frac{dx}{x^2-9} &= \lim_{L\to\infty}\int_{10}^{L}\frac{dx}{x^2-9}\\ &= \lim_{L\to\infty}\int_{10}^{L}\frac{1}{9(x-3)} - \frac{1}{9(x+3)}dx\\ &= \lim_{L\to\infty}\left[\frac{1}{9}\ln(x-3) - \frac{1}{9}\ln(x+3)\right]_{10}^{L}\\ &= \lim_{L\to\infty}\left[\frac{1}{9}\ln\left(\frac{x-3}{x+3}\right)\right]_{10}^{L}\\ &= \underbrace{\lim_{L\to\infty}\frac{1}{9}\ln\left(\frac{L-3}{L+3}\right)}_{0} - \frac{1}{9}\ln\left(\frac{7}{13}\right)\\ &= \frac{1}{9}\ln\left(\frac{13}{7}\right)\end{aligned}$$

5. Find $\int_0^{\infty}\frac{1}{e^t-1}dt$.

   Solution: First notice that the endpoints 0 and ∞ are both improper, so we need to break the integral up.

$$\int_0^{\infty}\frac{1}{e^t-1}dt = \lim_{a\downarrow 0}\int_a^1 \frac{1}{e^t-1}dt + \lim_{b\to\infty}\int_1^b \frac{1}{e^t-1}dt$$

We compute the antiderivative

$$\int \frac{1}{e^t - 1} dt = \int \frac{1}{u(u-1)} du \qquad u = e^t \quad \frac{1}{u} du = dt$$
$$= \int \frac{1}{u-1} du - \int \frac{1}{u} du$$
$$= \ln|u-1| - \ln|u| + C$$
$$= \ln|\frac{u-1}{u}| + C$$
$$= \ln|1 - \frac{1}{e^t}| + C$$

and therefore

$$\lim_{a \downarrow 0} \int_a^1 \frac{1}{e^t - 1} dt + \lim_{b \to \infty} \int_1^b \frac{1}{e^t - 1} dt = \lim_{a \downarrow 0} \left[\ln|1 - \frac{1}{e^t}| \right]_a^1 + \lim_{b \to \infty} \left[\ln|1 - \frac{1}{e^t}| \right]_1^b$$
$$= \lim_{a \downarrow 0} \ln|1 - \frac{1}{e}| - \underbrace{\ln|1 - \frac{1}{e^a}|}_{\text{Limit DNE}} + \underbrace{\lim_{b \to \infty} \ln|1 - \frac{1}{e^b}|}_{\text{Limit is 0}} - \ln|1 - \frac{1}{e}|$$

so the integral does not exist.

6. Compute $\int_1^3 \frac{dt}{\sqrt{9-t^2}}$.

Solution:

$$\int_1^3 \frac{dt}{\sqrt{9-t^2}} = \lim_{L \to 3} \int_1^3 \frac{dt}{\sqrt{9-t^2}}$$
$$= \lim_{L \to 3} \left[\arcsin(\frac{t}{3}) \right]_1^L$$
$$= \lim_{L \to 3} \arcsin(\frac{L}{3}) - \arcsin(\frac{1}{3})$$
$$= \frac{\pi}{2} - \arcsin(\frac{1}{3})$$