

Please inform your TA if you find any errors in the solutions.

1. Suppose that  $y(x)$  solves

$$\begin{aligned} 0 &= xy''(x) + y(x) - x \\ y'(0) &= -1. \end{aligned}$$

Find the degree three Taylor polynomial around zero for  $y(x)$  and use this to compute an estimate to  $y(.5)$ .

**Solution:** Write

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + o(x^3) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3) \\ &= a_0 + a_1x + a_2x^2 + o(x^2) \\ y''(x) &= 2a_2 + 6a_3x + o(x). \end{aligned}$$

Plugging into the differential equation, we have

$$\begin{aligned} 0 &= x(2a_2 + 6a_3x + o(x)) + (a_0 + a_1x + a_2x^2 + o(x^2)) - x \\ &= 2a_2x + 6a_3x^2 + o(x^2) + a_0 + a_1x + a_2x^2 + o(x^2) - x \\ &= a_0 + (2a_2 + a_1 - 1)x + (6a_3 + a_2)x^2 + o(x^2) \end{aligned}$$

Equating coefficients, we see that

$$0 = a_0 \tag{1}$$

$$0 = 2a_2 + a_1 - 1 \tag{2}$$

$$0 = 6a_3 + a_2 \tag{3}$$

It follows from (1) that  $a_0 = 0$ . We are given that  $a_1 = -1$ , so plugging this into (2) gives  $a_2 = 1$ . Plugging this into (3) gives  $a_3 = \frac{-1}{6}$ . We conclude that the degree three Taylor polynomial for this  $y(x)$  around zero is  $0 - 1 \cdot x + 1 \cdot x^2 - \frac{1}{6}x^3 = -x + x^2 - \frac{1}{6}x^3$ . Our approximation for  $y(.5)$  is then

$$y\left(\frac{1}{2}\right) \approx -\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{6}\left(\frac{1}{2}\right)^3 = -0.270833$$

2. Suppose that  $y(x)$  solves

$$\begin{aligned} 0 &= y''(x) + e^{2x} + xy(x) \\ y(0) &= 1 \quad y'(0) = 1. \end{aligned}$$

Compute the degree three Taylor polynomial of  $y(x)$  around zero.

**Solution:** Write

$$\begin{aligned}y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + o(x^3) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3) \\ &= a_0 + a_1x + o(x) \\ y''(x) &= 2a_2 + 6a_3x + o(x)\end{aligned}$$

and recall that

$$e^{2x} = 1 + 2x + o(x).$$

We have

$$\begin{aligned}0 &= y''(x) + e^{2x} + xy(x) \\ &= (2a_2 + 6a_3x + o(x)) + (1 + 2x + o(x)) + x(a_0 + a_1x + o(x)) \\ &= (2a_2 + 6a_3x + o(x)) + (1 + 2x + o(x)) + (a_0x + o(x)) \\ &= (2a_2 + 1) + (6a_3 + 2 + a_0)x + o(x).\end{aligned}$$

Equating coefficients, we see that

$$0 = 2a_2 + 1 \tag{1}$$

$$0 = 6a_3 + 2 + a_0. \tag{2}$$

We are given that  $a_0 = y(0) = 1$  and  $a_1 = y'(0) = 1$ . Solving in (1), we see that  $a_2 = \frac{f''(0)}{2} = \frac{-1}{2}$ . Plugging  $a_0 = 1$  into (2) gives  $a_3 = \frac{f'''(0)}{3!} = \frac{-1}{3}$ . We conclude that the degree three Taylor polynomial of  $y(x)$  around zero is  $1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3$ .

3. Suppose that  $y(x)$  solves

$$\begin{aligned}0 &= y''(x) + y'(x) + e^{x^2} \\ y(0) &= 1 \quad y'(0) = 1.\end{aligned}$$

Compute the degree three Taylor polynomial of  $y(x)$  around zero.

**Solution:** Write

$$\begin{aligned}y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + o(x^3) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3) \\ y'(x) &= a_1 + 2a_2x + o(x) \\ y''(x) &= 2a_2 + 6a_3x + o(x)\end{aligned}$$

and recall that

$$\begin{aligned}e^{x^2} &= 1 + x^2 + o(x^2) \\ &= 1 + o(x).\end{aligned}$$

Plugging in, we have

$$\begin{aligned} 0 &= y''(x) + y'(x) + e^{x^2} \\ &= (2a_2 + 6a_3x + o(x)) + (a_1 + 2a_2x + o(x)) + (1 + o(x)). \end{aligned}$$

Equating coefficients, we see that

$$0 = 2a_2 + a_1 + 1 \tag{1}$$

$$0 = 6a_3 + 2a_2. \tag{2}$$

We are given that  $a_0 = y(0) = 1$  and  $a_1 = y'(0) = 1$ . Plugging into (1), we see that  $a_2 = -1$ . Plugging this into (2), we see that  $a_3 = \frac{1}{3}$ . We conclude that the degree three Taylor polynomial of  $y(x)$  around zero is  $y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 = 1 + x - x^2 + \frac{1}{3}x^3$ .

4. Suppose that  $y(x)$  solves the differential equation

$$\begin{aligned} 0 &= y''(x) + y(x) \\ y(0) &= 0 \quad y'(0) = 1. \end{aligned}$$

Compute the degree five Taylor polynomial of  $y(x)$  around zero.

**Solution:** Write

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + o(x^5) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3) \\ y''(x) &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + o(x^3) \end{aligned}$$

and recall that we are given  $a_0 = 0$  and  $a_1 = 1$ . We substitute into the differential equation

$$\begin{aligned} 0 &= y''(x) + y(x) \\ &= (2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + o(x^3)) + (a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3)) \\ &= (a_0 + 2a_2) + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + (20a_5 + a_3)x^3 + o(x^3). \end{aligned}$$

Equating coefficients, we have

$$0 = a_0 + 2a_2 \tag{1}$$

$$0 = 6a_3 + a_1 \tag{2}$$

$$0 = 12a_4 + a_2 \tag{3}$$

$$0 = 20a_5 + a_3. \tag{4}$$

We are given  $a_0 = 0$ , so plugging this into (1) implies that  $a_2 = 0$ . Plugging this into (3) implies that  $a_4 = 0$ . Similarly, plugging  $a_1 = 1$  into (2) implies that  $a_3 = -\frac{1}{6} = \frac{-1}{3!}$ . Plugging this into (4), we see that  $a_5 = \frac{1}{20 \cdot 6} = \frac{1}{5!}$ . We conclude that the degree five Taylor polynomial of  $y(x)$  around zero is  $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$ .