MATH 222 (Lectures 1,2,4) **Worksheet 11.5 (supplement)** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Suppose that $y(x)$ solves

$$\begin{aligned} 0 &= xy''(x) + y(x) - x \\ y'(0) &= -1. \end{aligned}$$

Find the degree three Taylor polynomial around zero for $y(x)$ and use this to compute an estimate to $y(.5)$.

Solution: Write

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + o(x^3) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3) \\ &= a_0 + a_1x + a_2x^2 + o(x^2) \\ y''(x) &= 2a_2 + 6a_3x + o(x). \end{aligned}$$

Plugging into the differential equation, we have

$$\begin{aligned} 0 &= x\left(2a_2 + 6a_3x + o(x)\right) + \left(a_0 + a_1x + a_2x^2 + o(x^2)\right) - x \\ &= 2a_2x + 6a_3x^2 + o(x^2) + a_0 + a_1x + a_2x^2 + o(x^2) - x \\ &= a_0 + \left(2a_2 + a_1 - 1\right)x + (6a_3 + a_2)x^2 + o(x^2) \end{aligned}$$

Equating coefficients, we see that

$$0 = a_0 \tag{1}$$
$$0 = 2a_2 + a_1 - 1 \tag{2}$$
$$0 = 6a_3 + a_2 \tag{3}$$

It follows from (1) that $a_0 = 0$. We are given that $a_1 = -1$, so plugging this into (2) gives $a_2 = 1$. Plugging this into (3) gives $a_3 = \frac{-1}{6}$. We conclude that the degree three Taylor polynomial for this $y(x)$ around zero is $0 - 1 \cdot x + 1 \cdot x^2 - \frac{1}{6}x^3 = -x + x^2 - \frac{1}{6}x^3$. Our approximation for $y(.5)$ is then

$$y\left(\frac{1}{2}\right) \approx -\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{6}\left(\frac{1}{2}\right)^3 = -0.270833$$

2. Suppose that $y(x)$ solves

$$\begin{aligned} 0 &= y''(x) + e^{2x} + xy(x) \\ y(0) &= 1 \qquad y'(0) = 1. \end{aligned}$$

Compute the degree three Taylor polynomial of $y(x)$ around zero.

Solution: Write

$$
\begin{aligned}
y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + o(x^3) \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + o(x^3) \\
&= a_0 + a_1 x + o(x) \\
y''(x) &= 2a_2 + 6a_3 x + o(x)
\end{aligned}
$$

and recall that

$$
e^{2x} = 1 + 2x + o(x).
$$

We have

$$
\begin{aligned}
0 &= y''(x) + e^{2x} + xy(x) \\
&= (2a_2 + 6a_3 x + o(x)) + (1 + 2x + o(x)) + x\left(a_0 + a_1 x + o(x)\right) \\
&= (2a_2 + 6a_3 x + o(x)) + (1 + 2x + o(x)) + (a_0 x + o(x)) \\
&= (2a_2 + 1) + (6a_3 + 2 + a_0)\, x + o(x).
\end{aligned}
$$

Equating coefficients, we see that

$$
\begin{aligned}
0 &= 2a_2 + 1 \qquad (1) \\
0 &= 6a_3 + 2 + a_0. \qquad (2)
\end{aligned}
$$

We are given that $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$. Solving in (1), we see that $a_2 = \frac{f''(0)}{2} = \frac{-1}{2}$. Plugging $a_0 = 1$ into (2) gives $a_3 = \frac{f'''(0)}{3!} = \frac{-1}{2}$. We conclude that the degree three Taylor polynomial of $y(x)$ around zero is $1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3$.

3. Suppose that $y(x)$ solves

$$
\begin{aligned}
0 &= y''(x) + y'(x) + e^{x^2} \\
y(0) &= 1 \qquad y'(0) = 1.
\end{aligned}
$$

Compute the degree three Taylor polynomial of $y(x)$ around zero.

Solution: Write

$$
\begin{aligned}
y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + o(x^3) \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + o(x^3) \\
y'(x) &= a_1 + 2a_2 x + o(x) \\
y''(x) &= 2a_2 + 6a_3 x + o(x)
\end{aligned}
$$

and recall that

$$
\begin{aligned}
e^{x^2} &= 1 + x^2 + o(x^2) \\
&= 1 + o(x).
\end{aligned}
$$

Plugging in, we have

$$\begin{aligned}0 &= y''(x) + y'(x) + e^{x^2} \\ &= (2a_2 + 6a_3x + o(x)) + (a_1 + 2a_2x + o(x)) + (1 + o(x))\,.\end{aligned}$$

Equating coefficients, we see that

$$0 = 2a_2 + a_1 + 1 \tag{1}$$
$$0 = 6a_3 + 2a_2. \tag{2}$$

We are given that $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$. Plugging into (1), we see that $a_2 = -1$. Plugging this into (2), we see that $a_3 = \frac{1}{3}$. We conclude that the degree three Taylor polynomial of $y(x)$ around zero is $y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 = 1 + x - x^2 + \frac{1}{3}x^3$.

4. Suppose that $y(x)$ solves the differential equation

$$\begin{gathered}0 = y''(x) + y(x) \\ y(0) = 0 \qquad y'(0) = 1.\end{gathered}$$

Compute the degree five Taylor polynomial of $y(x)$ around zero.

Solution: Write

$$\begin{aligned}y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + o(x^5) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3) \\ y''(x) &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + o(x^3)\end{aligned}$$

and recall that we are given $a_0 = 0$ and $a_1 = 1$. We substitute into the differential equation

$$\begin{aligned}0 &= y''(x) + y(x) \\ &= \left(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + o(x^3)\right) + \left(a_0 + a_1x + a_2x^2 + a_3x^3 + o(x^3)\right) \\ &= (a_0 + 2a_2) + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + (20a_5 + a_3)x^3 + o(x^3).\end{aligned}$$

Equating coefficients, we have

$$0 = a_0 + 2a_2 \tag{1}$$
$$0 = 6a_3 + a_1 \tag{2}$$
$$0 = 12a_4 + a_2 \tag{3}$$
$$0 = 20a_5 + a_3. \tag{4}$$

We are given $a_0 = 0$, so plugging this into (1) implies that $a_2 = 0$. Plugging this into (3) implies that $a_4 = 0$. Similarly, plugging $a_1 = 1$ into (2) implies that $a_3 = -\frac{1}{6} = \frac{-1}{3!}$. Plugging this into (4), we see that $a_5 = \frac{1}{20 \cdot 6} = \frac{1}{5!}$. We conclude that the degree five Taylor polynomial of $y(x)$ around zero is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$.