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1 Foundations of geometric group theory

1.1 Word metrics and Cayley graphs

Let S be a finite generating set for a group G . We assume S is *symmetric*, i.e., if $s \in S$, then $s^{-1} \in S$.

Definition 1.1. A *word* in S is a concatenation of elements of S : $W = s_1 s_2 \dots s_n$ where $s_i \in S$. We let $\|W\|$ denote the length of the word, so here $\|W\| = n$. If two words W, U are letter-for-letter equivalent, we write $W \equiv U$. If W, U represent the same element of the group G , we write $W =_G U$. If $g \in G$, write $|g|_S$ for the length of the shortest word in S that represents g in G . Given $g, h \in G$, let $d_S(g, h) = |g^{-1}h|_S$. The metric d_S is the *word metric* on G with respect to S .

Definition 1.2. The *Cayley graph* of G with respect to S , which we denote by $\Gamma(G, S)$ is the graph whose vertex set is $\{g \in G\}$ with a (directed) edge from a vertex g to a vertex h whenever $h = gs$ for some $s \in S$.

Each edge has an orientation and a label. But if $gs = h$, then $g = hs^{-1}$ as well, so there is also a directed edge from h to g labeled by s^{-1} . This is rather cumbersome, so we typically don't draw the edge labeled by s^{-1} . We just keep in mind that if we traverse an edge labeled by s going in the opposite direction as its orientation, we think of that edge as labeled by s^{-1} .

Given a path p in $\Gamma(G, S)$, let $\mathbf{Lab}(p)$ be its label in $\Gamma(G, S)$. We denote the initial point of p by p_- and the terminal point of p by p_+ .

Remark 1.3. d_S is a metric on the vertex set of $\Gamma(G, S)$. Identifying each edge with the unit interval $[0, 1]$ extends this metric to the whole graph.

Exercises 1.4. Draw the following Cayley graphs.

1. $\Gamma(\mathbb{Z}/4\mathbb{Z}, \{1\})$
2. $\Gamma(\mathbb{Z}, \{1\})$
3. $\Gamma(\mathbb{Z}, \{2, 3\})$
4. $\Gamma(\mathbb{Z}^2, \{a = (1, 0), b = (0, 1)\})$
5. $\Gamma(\mathbb{F}_2, \{a, b\})$, where $\mathbb{F}_2 = \langle a, b \rangle$ (i.e., the group with two generators and no relations)
6. $\Gamma(BS(1, 2), \{a, t\})$, where $BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$.

1.2 Metric spaces

Let (X, d) denote a metric space X with its metric d . If the metric is clear, we may just write X .

If $x \in X$ and $r \geq 0$, then $B_R(x) = \{y \in X \mid d(x, y) \leq r\}$ is the (closed) ball of radius r about x . If $A \subseteq X$, then we write $\mathcal{N}_r(A) = \{y \in X \mid d(y, a) \leq r\} = A^{+r}$ for the (closed) r -neighborhood of A in X .

Definition 1.5. A *path* in X is a continuous map $p: [a, b] \rightarrow X$ for some $[a, b] \subseteq \mathbb{R}$. By an abuse of notation, we also call the image of this map p . Then $p_- = p(a)$ and $p_+ = p(b)$. A *ray* is a continuous map $p: [a, \infty) \rightarrow X$, and a *bi-infinite path* is a continuous map $p: (-\infty, \infty) \rightarrow X$.

Given a path p , the *length* of p is

$$\ell(p) := \sup_{a \leq t_1 \leq t_2 \leq \dots \leq t_n \leq b} \sum_{i=1}^{n-1} d(p(t_i), p(t_{i+1})), \quad (1)$$

where the supremum is taken over all n and all choices of t_1, \dots, t_n . This supremum is often infinite. We typically consider *rectifiable* paths, which are those with $\ell(p) < \infty$.

Example 1.6 (From BH I.1 Ex. 1.19). Let $X = [0, 1]$, and let $0 = t_0 < t_1 < \dots < t_n < \dots$ be an infinite sequence in $[0, 1]$ converging to 1. Let $c: [0, 1] \rightarrow X$ be any path such that $c(0) = 0$, $c(t_n) = \sum_{k=1}^n \frac{(-1)^{k+1}}{k}$. Then c is not rectifiable because its length is bounded below by the harmonic series $\sum_{i=1}^{\infty} \frac{1}{k}$, which diverges to infinity.

Definition 1.7. A path is *geodesic* if $\ell(p) = d(p_-, p_+)$. A ray is *geodesic* if for all $a \leq s \leq t < \infty$, $\ell(p|_{[s,t]}) = d(p(s), p(t))$, and similarly for a bi-infinite path. A metric space is *geodesic* if every pair of points can be connected by a geodesic.

A weaker condition one can put on a metric space is that it is a *length space*, which means that it is a path-connected space such that for all $x, y \in X$, $d(x, y) = \inf\{\ell(p) \mid p_- = x, p_+ = y\}$. For example, consider a space with two points, x and y , with countably many edges between them indexed by the positive integers so that the n -th edge has length $1 + \frac{1}{n}$. Here, $d(x, y) = \inf(1 + \frac{1}{n})$, but there is no path of length 1 between them. Most statements in this class will hold for length spaces, but the proofs tend to be a bit more technical, so we will mostly work with geodesic spaces.

Definition 1.8. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f: X \rightarrow Y$ is:

- an *isometry* if for all $x, y \in X$,

$$d_Y(f(x), f(y)) = d_X(x, y)$$

and f is surjective;

- a *biLipschitz equivalence* if there exists $\lambda \geq 1$ such that for all $x, y \in X$,

$$\frac{1}{\lambda} d_X(x, y) \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y)$$

and f is surjective;

- a (λ, c) -*quasi-isometry* if there exists $\lambda \geq 1$ and $c \geq 0$ such that for all $x, y \in X$

$$\frac{1}{\lambda} d_X(x, y) - c \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + c$$

and $f(X)^{+c} = Y$, i.e., f is *coarsely surjective*.

We write $X \sim_{Lip} Y$ or $X \sim_{qi} Y$ if there exists a biLipschitz equivalence or (λ, c) -quasi-isometry from X to Y , respectively.

If the assumption of (coarse) surjectivity is dropped, then f is a(n) isometric, biLipschitz, or quasi-isometric *embedding*, respectively. Note that quasi-isometries are not required to be continuous, though it will follow from the definitions that isometries and biLipschitz equivalences are continuous.

Exercises 1.9. 1. Show that \sim_{Lip} and \sim_{qi} are equivalence relations.

2. Let X, Y be bounded metric spaces. Show that $X \sim_{qi} Y$.

3. Show that $\Gamma(\mathbb{Z}, \{1\})$ is quasi-isometric to the real line \mathbb{R} .

4. Show that $\Gamma(\mathbb{Z}, \{1\}) \sim_{qi} \Gamma(\mathbb{Z}, \{2, 3\})$.

5. In general, show that if $S, T \subseteq G$ are two finite generating sets, then $(G, d_S) \sim_{Lip} (G, d_T)$, and so $\Gamma(G, S) \sim_{qi} \Gamma(G, T)$.

1.3 Group actions

Let G act on a metric space X , that is, there exists a map $\varphi: G \times X \rightarrow X$ such that $\varphi(gh, x) = \varphi(g, \varphi(h, x))$. We write $G \curvearrowright X$, and, to save space, we often drop the φ notation simply write gx (or $g \cdot x$ or $g \cdot x$) for $\varphi(g, x)$. In this class, we will ALWAYS assume the action is by isometries, that is, for all $g \in G$, $d_X(x, y) = d_X(gx, gy)$.

There is a bijection between actions of G on X and homomorphisms $\rho: G \rightarrow \text{Isom}(X)$, where $\text{Isom}(X)$ is the group of isometries of the space X .

Definition 1.10. The action is *faithful* if ρ is injective, i.e., for all $g \in G$, there exists $x \in X$ such that $gx \neq x$. The action is *free* if for all $x \in X$, $\text{Stab}_G(x) = \{1\}$, where $\text{Stab}_G(x) = \{g \in G \mid gx = x\}$, i.e., for all $x \in X$ for all $g \in G$, $gx \neq x$.

The action is *proper* if for all bounded subsets $B \subseteq X$,

$$\#\{g \in G \mid gB \cap B \neq \emptyset\} < \infty.$$

This is also called *metric properness*. There is a topological version where bounded is replaced by compact.

The action is *cobounded* (resp. *cocompact*) if there exists a cobounded (resp. cocompact) subset $K \subseteq X$ such that

$$\bigcup_{g \in G} gK = X.$$

Group actions can be used to prove algebraic results. Here is just one example:

Theorem 1.11. *Suppose G acts freely on \mathbb{R}^n (where \mathbb{R}^n has the Euclidean metric). Then G is torsion free.*

Proof. Exercise. □

[Hint: Use the fact that any finite set of points $\{x_1, \dots, x_n\}$ in \mathbb{R}^n has a unique *centroid*, that is, a point $w \in \mathbb{R}^n$ that minimizes the distance $\sum d_{\mathbb{R}^n}(w, x_i)$.]

The following is one of the the foundational results of geometric group theory. A *proper* metric space is one in which closed balls are compact.

Theorem 1.12 (Milnor–Swarz Lemma). *Let G be a group acting properly and coboundedly on a proper geodesic metric space X . Then G has a finite generating set S and $\Gamma(G, S) \sim_{qi} X$.*

Proof. Fix $x \in X$. Choose a bounded subset K of X so that $\bigcup_{g \in G} gK = X$. Since K is bounded, there exists a closed ball B of radius D such that $K \subseteq B$. Thus we have $\bigcup_{g \in G} gB = X$. Since X is proper, B is compact. Let

$$S = \{g \in G \setminus \{1\} \mid B \cap gB \neq \emptyset\}.$$

Since $G \curvearrowright X$ is proper, the set S is finite, and $S = S^{-1}$ by the definition of S .

Consider $\inf \{d(B, gB) \mid g \in G \setminus (S \cup \{1\})\}$. For some $g \in G \setminus (S \cup \{1\})$, the distance $d(B, gB)$ is a positive constant R , by the definition of S . The set H of elements of G such that $d(B, gB) \leq R$ is contained in

$$\{g \in G \mid gB(x, D + R) \cap B(x, D + R)\}$$

and hence is finite. Since

$$\inf \{d(B, gB) \mid g \in G \setminus (S \cup \{1\})\} = \inf \{d(B, gB) \mid g \in H \setminus (S \cup \{1\})\},$$

and the latter is taken over a finite set, we have that the infimum is realized by some $h_0 \in H \setminus (S \cup \{1\})$. For this h_0 , we have that $d(B, h_0B) = 2L$ for some positive constant L . By definition, if $d(B, gB) < 2L$, then $g \in S$.

We first show that S generates G . Let $g \in G$ and consider a geodesic $[x, gx]$ in X . let $k = \left\lfloor \frac{d(x, gx)}{L} \right\rfloor$. Then there exists a finite sequence a points along the geodesic $x = x_0, x_1, \dots, x_{k+1} = gx$ such that $d(x_i, x_{i+1}) \leq L$ for every $i \in \{0, \dots, k\}$. For each such i , let $h_i \in G$ be such that $x_i \in h_i B$. We take $h_0 = 1$ and $h_{k+1} = g$. As $d(B, h_i^{-1} h_{i+1} B) = d(h_i B, h_{i+1} B) \leq d(x_i, x_{i+1}) = L$, it follows that $h_i^{-1} h_{i+1} \in S$. Also,

$$h_0(h_0^{-1} h_1)(h_1^{-1} h_2) \cdots (h_k^{-1} h_{k+1}) = h_{k+1} = g,$$

and so $g \in \langle S \rangle$. Therefore the finite set S generates G .

It remains to show that $\Gamma(G, S)$ is quasi-isometric to X . let $f: G \rightarrow X$ be defined by $f(g) = gx$ (so f is the orbit map). Since $G \curvearrowright X$ is cobounded, this map is $2D$ -coarsely surjective.

The argument above shows that

$$|g|_S \leq k + 1 \leq \frac{1}{L} d(x, gx) + 1.$$

For the other half of the inequality, let $|g|_S = m$, and let $w = s_1 s_2 \dots s_m$ be a word in S such that $w =_G g$. By the triangle inequality,

$$\begin{aligned} d(x, gx) &= d(x, s_1 s_2 \dots s_m x) \\ &\leq d(x, s_1 x) + d(s_1 x, s_1 s_2 x) + \cdots + d(s_1 \dots s_{m-1} x, s_1 \dots s_m x) \\ &= \sum_{i=1}^m d(x, s_i x) \\ &\leq 2Dm \\ &= 2D|g|_S. \end{aligned}$$

Thus, we have for any $g \in G$:

$$L|g|_S - L \leq d(x, gx) \leq 2D|g|_S.$$

Note that since both the word metric and the metric on X are left-invariant (under the action of G), this suffices to prove the statement for any $h, g \in G$, rather than the pair $1, g \in G$ for which we have proven the result. \square

Corollary 1.13. *The following follow immediately from the Milnor–Schwarz Lemma:*

1. *If G is finitely generated and H is a finite-index subgroup of G , then $H \sim_{qi} G$.*
2. *If $N \trianglelefteq G$ is a finite normal subgroup and G/N is finitely generated, then G is finitely generated and $G \sim_{qi} G/N$.*
3. *If M is a closed Riemannian manifold and \tilde{M} is its universal cover, then $\pi_1(M) \sim_{qi} \tilde{M}$.*
4. *If G is a connected Lie group with left-invariant Riemannian metric and $\Gamma \leq G$ is a uniform lattice, then Γ is finitely generated and $\Gamma \sim_{qi} G$.*

2 Geometry of hyperbolic metric spaces

Let X be a metric space. Given two points $x, y \in X$, let $[x, y]$ be a geodesic from X to Y .

2.1 Four equivalent definitions of hyperbolicity

Assume X is geodesic. For any geodesic triangle T in X with vertices x, y, z the following hold.

1. There exist points $i_z \in [x, y], i_y \in [x, z], i_x \in [y, z]$ such that $a := d(x, i_z) = d(x, i_y), b := d(y, i_z) = d(y, i_x)$ and $c := d(z, i_x) = d(z, i_y)$. See Figure 1. The points i_x, i_y, i_z are called *internal points* of T .

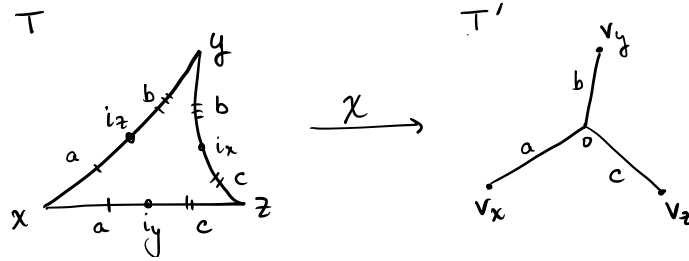


Figure 1: A geodesic triangle T in X and its comparison tripod T' .

2. Let T' be a *comparison tripod*, that is, a graph with 3 vertices of valence one labeled $v_x, v_y,$ and v_z , and a central vertex of valence 3 labeled o , so that the lengths of the edges from $v_x, v_y,$ and v_z to o are $a, b,$ and c , respectively. The map $T \rightarrow T'$ which sends x, y, z to v_x, v_y, v_z respectively, extends uniquely to a map $\chi: T \rightarrow T'$ which restricts to an isometry on each side of T . This map sends i_x, i_y, i_z to o , and every other point in T' except $v_x, v_y,$ and v_z will have a preimage in T consisting of exactly two points.