Contents

1 Foundations of geometric group theory

1.1 Word metrics and Cayley graphs

Let S be a finite generating set for a group G. We assume S is *symmetric*, i.e., if $s \in S$, then $s^{-1} \in S$.

Definition 1.1. A *word* in S is a concatenation of elements of S: $W = s_1 s_2 \dots s_n$ where $s_i \in S$. We let $\|W\|$ denote the length of the word, so here $\|W\| = n$. If two words W, U are letter-for-letter equivalent, we write $W \equiv U$. If W, U represent the same element of the group G, we write $W =_G U$. If $g \in G$, write $|g|_S$ for the length of the shortest word in S that represents g in G. Given $g, h \in G$, let $d_S(g,h) = |g^{-1}h|_S$. The metric d_S is the *word metric* on G with respect to S.

Definition 1.2. The *Cayley graph* of G with respect to S, which we denote by $\Gamma(G,S)$ is the graph whose vertex set is $\{g \in G\}$ with a (directed) edge from a vertex g to a vertex h whenever $h = gs$ for some $s \in S$.

Each edge has an orientation and a label. But if $gs = h$, then $g = hs^{-1}$ as well, so there is also a directed edge from h to g labeled by s^{-1}. This is rather cumbersome, so we typically don't draw the edge labeled by s^{-1}. We just keep in mind that if we traverse an edge labeled by s going in the opposite direction as its orientation, we think of that edge as labeled by s^{-1}.

Given a path p in $\Gamma(G,S)$, let $\mathbf{Lab}(p)$ be its label in $\Gamma(G,S)$. We denote the initial point of p by p_- and the terminal point of p by p_+.

Remark 1.3. d_S is a metric on the vertex set of $\Gamma(G,S)$. Identifying each edge with the unit interval $[0,1]$ extends this metric to the whole graph.

Exercises 1.4. *Draw the following Cayley graphs.*

1. $\Gamma(\mathbb{Z}/4\mathbb{Z}, \{1\})$
2. $\Gamma(\mathbb{Z}, \{1\})$
3. $\Gamma(\mathbb{Z}, \{2,3\})$
4. $\Gamma(\mathbb{Z}^2, \{a = (1,0), b = (0,1)\})$
5. $\Gamma(\mathbb{F}_2, \{a,b\})$*, where* $\mathbb{F}_2 = \langle a, b\rangle$ *(i.e., the group with two generators and no relations)*
6. $\Gamma(BS(1,2), \{a,t\})$*, where* $BS(1,2) = \langle a, t \mid tat^{-1} = a^2\rangle$.

1.2 Metric spaces

Let (X,d) denote a metric space X with its metric d. If the metric is clear, we may just write X.

If $x \in X$ and $r \geq 0$, then $B_R(x) = \{y \in X \mid d(x,y) \leq r\}$ is the (closed) ball of radius r about x. If $A \subseteq X$, then we write $\mathcal{N}_r(A) = \{y \in X \mid d(y,a) \leq r\} = A^{+r}$ for the (closed) r–neighborhood of A in X.

Definition 1.5. A *path* is X is a continuous map $p\colon [a,b] \to X$ for some $[a,b] \subseteq \mathbb{R}$. By an abuse of notation, we also call the image of this map p. Then $p_- = p(a)$ and $p_+ = p(b)$. A *ray* is a continuous map $p\colon [a,\infty) \to X$, and a *bi-infinite path* is a continuous map $p\colon (-\infty,\infty) \to X$.

Given a path p, the *length* of p is

$$\ell(p) := \sup_{a \le t_1 \le t_2 \le \cdots \le t_n \le b} \sum_{i=1}^{n-1} d(p(t_i), p(t_{i+1})), \tag{1}$$

where the supremum is taken over all n and all choices of $t_1, \ldots, t_n$. This supremum is often infinite. We typically consider *rectifiable* paths, which are those with $\ell(p) < \infty$.

Example 1.6 (From BH I.1 Ex. 1.19). Let $X = [0,1]$, and let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be an infinite sequence in $[0,1]$ converging to 1. let $c\colon [0,1] \to X$ be any path such that $c(0) = 0$, $c(t_n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Then c is not rectifiable because its length is bounded below by the harmonic series $\sum_{i=1}^{\infty} \frac{1}{k}$, which diverges to infinity.

Definition 1.7. A path is *geodesic* if $\ell(p) = d(p_-, p_+)$. A ray is *geodesic* if for all $a \le s \le t < \infty$, $\ell(p|_{[s,t]}) = d(p(s), p(t))$, and similarly for a bi-infinite path. A metric space is *geodesic* if every pair of points can be connected by a geodesic.

A weaker condition one can put on a metric space is that it is a *length space*, which means that it is a path-connected space such that for all $x, y \in X$, $d(x,y) = \inf\{\ell(p) \mid p_- = x, p_+ = y\}$. For example, consider a space with two points, x and y, with countably many edges between them indexed by the positive integers so that the n-th edge has length $1 + \frac{1}{n}$. Here, $d(x,y) = \inf(1 + frac1n)$, but there is no path of length 1 between them. Most statements in this class will hold for length spaces, but the proofs tend to be a bit more technical, so we will mostly work with geodesic spaces.

Definition 1.8. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is:

- an *isometry* if for all $x, y \in X$,
$$d_Y(f(x), f(y)) = d_X(x,y)$$
and f is surjective;
- a *biLipschitz equivalence* if there exists $\lambda \ge 1$ such that for all $x, y \in X$,
$$\frac{1}{\lambda} d_X(x,y) \le d_Y(f(x), f(y)) \le \lambda d_X(x,y)$$
and f is surjective;
- a (λ, c)*–quasi-isometry* if there exists $\lambda \ge 1$ and $c \ge 0$ such that for all $x, y \in X$
$$\frac{1}{\lambda} d_X(x,y) - c \le d_Y(f(x), f(y)) \le \lambda d_X(x,y) + c$$
and $f(X)^{+c} = Y$, i.e., f is *coarsely surjective.*

We write $X \sim_{Lip} Y$ or $X \sim_{qi} Y$ if there exists a biLipschitz equivalence or (λ, c)–quasi-isometry from X to Y, respectively.

If the assumption of (coarse) surjectivity is dropped, then f is a(n) isometric, biLipschitz, or quasi-isometric *embedding*, respectively. Note that quasi-isometries are not required to be continuous, though it will follow from the definitions that isometries and biLipschitz equivalences are continuous.

Exercises 1.9. 1. *Show that $\sim_{Lip}$ and $\sim_{qi}$ are equivalence relations.*

2. *Let X, Y be bounded metric spaces. Show that $X \sim_{qi} Y$.*

3. *Show that $\Gamma(\mathbb{Z}, \{1\})$ is quasi-isometric to the real line $\mathbb{R}$.*

4. *Show that $\Gamma(\mathbb{Z}, \{1\}) \sim_{qi} \Gamma(\mathbb{Z}, \{2,3\})$.*

5. *In general, show that if $S, T \subseteq G$ are two finite generating sets, then $(G, d_S) \sim_{Lip} (G, d_T)$, and so $\Gamma(G, S) \sim_{qi} \Gamma(G, T)$.*

1.3 Group actions

Let G act on a metric space X, that is, there exists a map $\varphi\colon G \times X \to X$ such that $\varphi(gh, x) = \varphi(g, \varphi(h, x))$. We write $G \curvearrowright X$, and, to save space, we often drop the φ notation simply write gx (or $g.x$ or $g \cdot x$) for $\varphi(g, x)$. In this class, we will ALWAYS assume the action is by isometries, that is, for all $g \in G$, $d_X(x, y) = d_X(gx, gy)$.

There is a bijection between actions of G on X and homomorphims $\rho\colon G \to \mathrm{Isom}(X)$, where $\mathrm{Isom}(X)$ is the group of isometries of the space X.

Definition 1.10. The action is *faithful* if ρ is injective, i.e., for all $g \in G$, there exists $x \in X$ such that $gx \neq x$. The action is *free* if for all $x \in X$, $\mathrm{Stab}_G(x) = \{1\}$, where $\mathrm{Stab}_G(x) = \{g \in G \mid gx = x\}$, i.e., for all $x \in X$ for all $g \in G$, $gx \neq x$.

The action is *proper* if for all bounded subsets $B \subseteq X$,

$$\#\{g \in G \mid gB \cap B \neq \emptyset\} < \infty.$$

This is also called *metric properness.* There is a topological version where bounded is replaced by compact.

The action is *cobounded* (resp. *cocompact*) if there exists a cobounded (resp. cocompact) subset $K \subseteq X$ such that

$$\bigcup_{g \in G} gK = X.$$

Group actions can be used to prove algebraic results. Here is just one example:

Theorem 1.11. *Suppose G acts freely on $\mathbb{R}^n$ (where $\mathbb{R}^n$ has the Euclidean metric). Then G is torsion free.*

Proof. Exercise. □

[Hint: Use the fact that any finite set of points $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^n$ has a unique *centroid*, that is, a point $w \in \mathbb{R}^n$ that minimizes the distance $\sum d_{\mathbb{R}^n}(w, x_i)$.]

The following is one of the the foundational results of geometric group theory. A *proper* metric space is one in which closed balls are compact.

Theorem 1.12 (Milnor–Swarz Lemma)**.** *Let G be a group acting properly and coboundedly on a propoer geodesic metric space X. Then G has a finite generating set S and $\Gamma(G, S) \sim_{qi} X$.*

Proof. Fix $x \in X$. Choose a bounded subset K of X so that $\bigcup_{g \in G} gK = X$. Since K is bounded, there exists a closed ball B of radius D such that $K \subseteq B$. Thus we have $\bigcup_{g \in G} gB = X$. Since X is proper, B is compact. Let

$$S = \{g \in G \setminus \{1\} \mid B \cap gB \neq \emptyset\}.$$

Since $G \curvearrowright X$ is proper, the set S is finite, and $S = S^{-1}$ by the definition of S.

Consider $\inf\{d(B,gB) \mid g \in G \setminus (S \cup \{1\})\}$. For some $g \in G \setminus (S \cup \{1\})$, the distance $d(B,gB)$ is a positive constant R, by the definition of S. The set H of elements of G such that $d(B,gB) \leq R$ is contained in

$$\{g \in G \mid gB(x, D+R) \cap B(x, D+R)\}$$

and hence is finite. Since

$$\inf\{d(B,gB) \mid g \in G \setminus (S \cup \{1\})\} = \inf\{d(B,gB) \mid g \in H \setminus (S \cup \{1\})\},$$

and the latter is taken over a finite set, we have that the infimum is realized by some $h_0 \in H \setminus (S \cup \{1\})$. For this h_0, we have that $d(B,hB) = 2L$ for some positive constant L. By definition, if $d(B,gB) < 2L$, then $g \in S$.

We first show that S generates G. Let $g \in G$ and consider a geodesic $[x, gx]$ in X. let $k = \left\lfloor \frac{d(x,gx)}{L} \right\rfloor$. Then there exists a finite sequence a points along the geodesic $x = x_0, x_1, \ldots x_{k+1} = gx$ such that $d(x_i, x_{i+1}) \leq L$ for every $i \in \{0, \ldots, k\}$. For each such i, let $h_i \in G$ be such that $x_i \in h_i B$. We take $h_0 = 1$ and $h_{k+1} = g$. As $d(B, h_i^{-1}h_{i+1}B) = d(h_iB, h_{i+1}B) \leq d(x_i, x_{i+1}) = L$, it follows that $h_i^{-1}h_{i+1} \in S$. Also,

$$h_0(h_0^{-1}h_1)(h_1^{-1}h_2)\cdots(h_k^{-1}h_{k+1}) = h_{k+1} = g,$$

and so $g \in \langle S \rangle$. Therefore the finite set S generates G.

It remains to show that $\Gamma(G,S)$ is quasi-isometric to X. let $f\colon G \to X$ be defined by $f(g) = gx$ (so f is the orbit map). Since $G \curvearrowright X$ is cobounded, this map is $2D$–coarsely surjective.

The argument above shows that

$$|g|_S \leq k+1 \leq \frac{1}{L}d(x,gx) + 1.$$

For the other half of the inequality, let $|g|_S = m$, and let $w = s_1 s_2 \ldots s_m$ be a word in S such that $w =_G g$. By the triangle inequality,

$$\begin{aligned} d(x,gx) &= d(x, s_1 s_2 \ldots s_m) \\ &\leq d(x, s_1 x) + d(s_1 x, s_1 s_2 x) + \cdots + d(s_1 \ldots s_{m-1} x, s_1 \ldots s_m x) \\ &= \sum_{i=1}^{m} d(x, s_i x) \\ &\leq 2Dm \\ &= 2D|g|_S. \end{aligned}$$

Thus, we have for any $g \in G$:

$$L|g|_s - L \leq d(x,gx) \leq 2D|g|_S.$$

Note that since both the word metric and the metric on X are left-invariant (under the action of G), this suffices to prove the statement for any $h, g \in G$, rather than the pair $1, g \in G$ for which we have proven the result. □

Corollary 1.13. *The following follow immediately from the Milnor–Schwarz Lemma:*

1. *If G is finitely generated and H is a finite-index subgroup of G, then $H \sim_{qi} G$.*
2. *If $N \trianglelefteq G$ is a finite normal subgroup and G/N is finitely generated, then G is finitely generated and $G \sim_{qi} G/N$.*
3. *If M is a closed Riemannian manifold and $\tilde{M}$ is its universal cover, then $\pi_1(M) \sim_{qi} \tilde{M}$.*
4. *If G is a connected Lie group with left-invariant Riemannian metric and $\Gamma \leq G$ is a uniform lattice, then Γ is finitely generated and $\Gamma \sim_{qi} G$.*

2 Geometry of hyperbolic metric spaces

Let X be a metric space. Given two points $x, y \in X$, let $[x, y]$ be a geodesic from X to Y.

2.1 Four equivalent definitions of hyperbolicity

Assume X is geodesic. For any geodesic triangle T in X with vertices x, y, z the following hold.

1. There exist points $i_z \in [x, y], i_y \in [x, z], i_x \in [y, z]$ such that $a := d(x, i_z) = d(x, i_y)$, $b := d(y, i_z) = d(y, i_x)$ and $c := d(z, i_x) = d(z, i_y)$. See Figure **??**. The points i_x, i_y, i_z are called *internal points* of T.

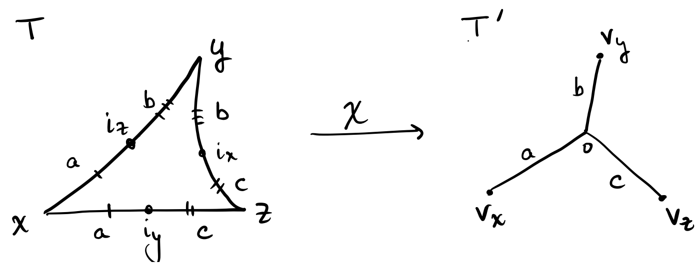

Figure 1: A geodesic triangle T in X and its comparison tripod T'.

2. Let T' be a *comparison tripod*, that is, a graph with 3 vertices of valence one labeled v_x, v_y, and v_z, and a central vertex of valence 3 labeled o, so that the lengths of the edges from v_x, v_y, and v_z to o are a, b, and c, respectively. The map $T \to T'$ which sends x, y, z to v_x, v_y, v_z respectively, extends uniquely to a map $\chi\colon T \to T'$ which restricts to an isometry on each side of T. This map sends i_x, i_y, i_z to o, and every other point in T' except v_x, v_y, and v_z will have a preimage in T consisting of exactly two points.

Definition 2.1 (Thin triangles)**.** Let $\delta \geq 0$. A geodesic metric spaces satisfies $\mathrm{Hyp}_1(\delta)$ if for any geodesic triangle T in X and any $a, b \in T$ which map to the same point in T', we have $d(a, b) \leq \delta$. The triangle T is called *δ–thin*.

Definition 2.2 (Insize)**.** Let $\delta \geq 0$. A geodesic metric spaces satisfies $\mathrm{Hyp}_2(\delta)$ if for any geodesic triangle T in X, $\mathrm{diam}(\{i_x, i_y, i_z\}) \leq \delta$. The quantity $\mathrm{diam}(\{i_x, i_y, i_z\})$ is called the *insize* of the triangle T.

The following is the most common definition of a (geodesic) hyperbolic metric space. It is attributed to Rips, and is sometimes called the Rips condition.

Definition 2.3 (Slim triangles)**.** Let $\delta \geq 0$. A geodesic metric spaces satisfies $\mathrm{Hyp}_3(\delta)$ if for any geodesic triangle T in X, we have $[x, y] \subseteq \mathcal{N}_\delta([x, z] \cup [y, z])$. Equivalently, given any point $a \in [x, y]$, there exists a point $b \in [x, z] \cup [y, z]$ such that $d(a, b) \leq \delta$.

Now let X be a (not necessarily geodesic) metric space and $x_0 \in X$.

Definition 2.4. The *Gromov product* of $x, y \in X$ with respect to x_0 is

$$(x \mid y)_{x_0} = \frac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y)).$$

If X is a tree, then $(x \mid y)_{x_0}$ measures the distance from x_0 to the geodesic $[x, y]$ (draw a picture of this!). Intuitively, $(x \mid y)_{x_0}$ will in general measure how long geodesics from x_0 to x and from x_0 to y "fellow travel." But note that we did not assume X was geodesic. Another way to think about the Gromov product is that it measures the failure of the triangle inequality to be an equality: in general we have that $d(x_0, x) + d(x_0, y) \geq d(x, y)$, and we call $d(x_0, x) + d(x_0, y) - d(x, y) \geq 0$ the *defect*. The quantity $(x \mid y)_{x_0}$ is half the defect.

The following is Gromov's original definition of a hyperbolic metric space.

Definition 2.5. Let $\delta \geq 0$. A metric spaces satisfies $\mathrm{Hyp}_4(\delta)$ if for any four points $x, y, z, w \in X$,

$$(x \mid y)_w \geq \min\{(x \mid z)_w, (y \mid z)_w\} - \delta.$$

Exercises 2.6. *If X is a tree, then X satisfies $Hyp_4(0)$. (We discussed this in class, but see if you can reconstruct the reasoning!)*

We can reformulate $\mathrm{Hyp}_4(\delta)$ as a more symmetric 4–*point condition*:

$$d(x, w) + d(y, z) \leq \max\{d(x, y) + d(z, w), d(x, z) + d(y, w)\} + 2\delta. \tag{2}$$

If you consider $x, y, z,$ and w to be the vertices of a tetrahedron, then the three sums in (**??**) are the sums of the lengths of the pairs of opposite sides. In a tree, a tetrahedron is a subtree with 6 vertices: the vertices x, y, z, w have valence 1, and there are two vertices with valence three. In such a tetrahedron, you can check that there is a tie for the two largest sums of opposite sides, and so $d(x, w) + d(y, z) \leq \max\{d(x, y) + d(z, w), d(x, z) + d(y, w)$. The 4–point condition makes this coarse by requiring only that there is a *coarse* tie for the two largest sums.

Up to modifying δ by a constant multiple, all four conditions $\mathrm{Hyp}_i(\delta)$ are equivalent:

Theorem 2.7. *For any geodesic metric space X, any $\delta \geq 0$, and any geodesic triangle T, the following hold:*

(1) T is δ–thin $\implies$ T is δ–slim $\implies$ the insize of T is at most 2δ $\implies$ T is 2δ–thin; and

(2) the insize of T is at most δ $\implies$ $Hyp_4(\delta)$ $\implies$ the insize of T is at most 6δ.

Proof. We first prove (1). It is clear that the first implication holds.

For the second implication, let $x, y, z \in X$, and consider a geodesic triangle T. Suppose T is δ–slim. Then there exists a point p on (without loss of generality) $[x, y]$ at distance at most δ from i_x. By the triangle inequality, $|d(y, p) - d(y, i_x)| \leq \delta$. As $d(y, i_x) = d(y, i_z)$, we have $d(p, i_z) \leq \delta$. Thus $d(i_x, i_z) \leq 2\delta$. Similarly for the other pairs.

For the third implication, suppose p lies on $[y, z]$ is such that $d(y, p) < d(y, i_x)$. Then the fiber of $\chi\colon T \to T'$ containing p is $\{p, q\}$, where $q \in [y, x]$ and $d(y, p) = d(y, q)$. We will show that $d(p, q) \leq 2\delta$ by building a geodesic triangle with p, q as internal points.

let $c\colon [0, 1] \to X$ be a monotone parametrization of $[y, z]$, and for each $t \in [0, 1]$, consider a geodesic triangle $T_t = T(x, y, c(t))$, two of whose sides are $[y, x]$ and $c([0, t])$. The interal point of T_t on $c([0, t])$ varies continuously (though maybe not monotonically) as a function of t. At $t = 0$, it is y, while at $t = 1$ it is i_x, so for some $t \in [0, 1]$ it is p. Since $d(y, p) = d(y, q)$, q must also be an internal point for T_t. As we assume the insize of all triangles is at most 2δ, we have $d(p, q) \leq 2\delta$.

We now prove (2). For the first implication, assume that the insize of any geodesic triangle in X is at most δ. Given $x, y, z, w \in X$, we may assume without loss of generality that $S := d(x, z) + d(y, w) \leq M := d(x, y) + d(z, w) \leq L := d(x, w) + d(y, z)$. To show the 4–point condition holds, we must show that $L \leq M + 2\delta$. Let $T = T(x, w, y)$ and $T' = T'(x, w, z)$ be geodesic triangles, and denote their internal points by i_x, i_w, i_y and i'_x, i'_w, i'_z, respectively. Consider the path from y to z which passes through i_x, i_y, i'_z, and i'_w, in that order. By the triangle inequality, we have

$$\begin{aligned} d(y,z) &\le d(y,i_x) + d(i_x,i_y) + d(i_y,i'_z) + d(i'_z,i'_w) + d(i'_w,z) \\ &\le d(y,i_x) + \delta + d(i_y,i'_z) + \delta + d(i'_w,z) \\ &= d(y,i_w) + d(i_y,i'_z) + d(z,i'_x) + 2\delta. \end{aligned}$$

We also have that

$$d(x,w) = d(x,i_y) + d(i'_z,w) - d(i'_z,i_y) = d(x,i_w) + d(i'_x,w) + d(i'_z,i_y).$$

Thus

$$\begin{aligned} L &\le d(y,i_w) + d(i_y,i'_z) + d(z,i'_x) + 2\delta + d(x,i_w) + d(i'_x,w) + d(i'_z,i_y) \\ &= (d(x,i_w) + d(i_w,y)) + (d(w,i'_x) + d(z,i'_x)) + 2\delta \\ &= M + 2\delta, \end{aligned}$$

as desired.

For the second implication, assume that X satisfies $\mathrm{Hyp}_4(\delta)$. Consider a triangle $T = T(x,y,z)$. We will apply the four-point condition to the points x, y, z, and $i_x \in [y,z]$. The largest sum of pairs of distances in the four-point condition will be $d(x,i_x) + d(y,z)$. To see this, notice that $2(d(x,i_x) + d(y,z)) = (d(x,i_x) + d(i_x,y)) + (d(x,i_x) + d(i_x,z)) + d(y,z)$ is greater than the perimeter P of T, while the other sums are $d(z,i_x)+d(x,y) = d(y,i_x)+d(x,z) = P/2$. Therefore the four-point condition implies that $d(x,i_x)+d(y,z) \le P/2 + 2\delta$. Since $d(y,z) + d(x,i_z)$ is also equal to $P/2$, we conclude that $|d(x,i_x) - d(x,i_z)| \le 2\delta$. Similarly, we have $|d(z,i_x) - d(z,i_z)| \le 2\delta$.

We now consider the four points $\{x, z, i_x, i_z\}$. In this case the three sums of pairs of distances are $d(x,z) = d(x,i_z) + d(z,i_x)$, $d(i_x,i_z) + d(x,z)$, and $d(x,i_x) + d(z,i_z)$. By the previous paragraph, the third sum is at most $d(x,i_z) + d(z,i_x) + 4\delta = d(x,z) + 4\delta$. Thus we see that the largest two pairs of sums are the last two listed above. Applying the four-point condition, we have $d(i_x,i_z) \le 6\delta$. Analogous arguments yield that $d(i_x,i_y) \le 6\delta$ and $d(i_y,i_z) \le 6\delta$, as well, completing the proof. □

Definition 2.8. We say that a metric space X is δ–hyperbolic (or Gromov hyperbolic) if it satisfies $\mathrm{Hyp}_4(\delta)$. If X is a geodesic metric space, we will also assume that δ has been chosen so that $\mathrm{Hyp}_1(\delta)$–$\mathrm{Hyp}_3(\delta)$ also hold.

Note: In a geodesic δ–hyperbolic metric space, the geometric interpretation of the Gromov product is coarsely the same as in the case of a tree: $(x \mid y)_{x_0}$ and $d(x_0,[x,y])$ will differ by at most δ.

Example 2.9. Examples of hyperbolic metric spaces:

1. If X is a bounded diameter metric space, then X is δ–hyperbolic with $\delta = \mathrm{diam}(X)$.

2. If X is a simplicial tree, then X is 0–hyperbolic.

3. The classical hyperbolic space $\mathbb{H}^2$ is δ–hyperbolic. (Exercise: Find δ! Hint: first show that it suffices to consider ideal triangles – those with all vertices at infinity. Then use the (isometric) action of $PSL_2(\mathbb{R})$ on $\mathbb{H}^2$ to show that it suffices to consider the ideal triangle with vertices at $-1, 1$, and ∞, in the upper half plane model.)

4. $\mathbb{H}^n$ for $n \ge 3$.

5. A space X is an *ultrametric space* if the following strong version of the triangle inequality holds for all $x, y, z \in X$: $d(x,y) \le \max\{d(x,z), d(y,z)\}$. (In other words, a (geodesic) space is an ultrametric

space is there is always a tie for the two longest sides of a triangle.) Ultrametric spaces often come up in relation to number theory; for example, the p–adic numbers $\mathbb{Q}_p$ with the p–adic metric space is an ultrametric space. All ultrametric spaces are 0–hyperbolic (this can be shown using the fourth definition of hyperbolicity).

6. An $\mathbb{R}$–tree is a generalization of a simplicial tree which can be defined as a space in which every geodesic triangle is a tripod. $\mathbb{R}$–trees are 0–hyperbolic.

7. Non-example: $\mathbb{R}^2$ (and more generally $\mathbb{R}^n$) is not hyperbolic for any δ. In fact, more is true: no δ–hyperbolic space can contain an isometrically copy of $\mathbb{R}^2$ (called a *flat*), or even a quasi-isometrically copy (called a *quasi-flat*).

Exercises 2.10. *Let X be a δ–hyperbolic metric space and $P = p_1 \dots p_n$ be a geodesic n–gon in X with $n \geq 3$. Let a be a point on p_i for some $1 \leq i \leq n$. Prove that there exists $j \neq i$ and $b \in p_j$ such that $d(a,b) \leq (n-2)\delta$. (In fact, $n-2$ can be replaced by $\log_2(n)$.)*