1 Foundations of geometric group theory

1.1 Word metrics and Cayley graphs

Let S be a finite generating set for a group G. We assume S is *symmetric*, i.e., if $s \in S$, then $s^{-1} \in S$.

Definition 1.1. A *word* in S is a concatenation of elements of S: $W = s_1 s_2 \dots s_n$ where $s_i \in S$. We let $\|W\|$ denote the length of the word, so here $\|W\| = n$. If two words W, U are letter-for-letter equivalent, we write $W \equiv U$. If W, U represent the same element of the group G, we write $W =_G U$. If $g \in G$, write $|g|_S$ for the length of the shortest word in S that represents g in G. Given $g, h \in G$, let $d_S(g,h) = |g^{-1}h|_S$. The metric d_S is the *word metric* on G with respect to S.

Definition 1.2. The *Cayley graph* of G with respect to S, which we denote by $\Gamma(G,S)$ is the graph whose vertex set is $\{g \in G\}$ with a (directed) edge from a vertex g to a vertex h whenever $h = gs$ for some $s \in S$.

Each edge has an orientation and a label. But if $gs = h$, then $g = hs^{-1}$ as well, so there is also a directed edge from h to g labeled by s^{-1}. This is rather cumbersome, so we typically don't draw the edge labeled by s^{-1}. We just keep in mind that if we traverse an edge labeled by s going in the opposite direction as its orientation, we think of that edge as labeled by s^{-1}.

Given a path p in $\Gamma(G,S)$, let $\mathbf{Lab}(p)$ be its label in $\Gamma(G,S)$. We denote the initial point of p by p_- and the terminal point of p by p_+.

Remark 1.3. d_S is a metric on the vertex set of $\Gamma(G,S)$. Identifying each edge with the unit interval $[0,1]$ extends this metric to the whole graph.

Exercises 1.4. *Draw the following Cayley graphs.*

1. $\Gamma(\mathbb{Z}/4\mathbb{Z}, \{1\})$
2. $\Gamma(\mathbb{Z}, \{1\})$
3. $\Gamma(\mathbb{Z}, \{2,3\})$
4. $\Gamma(\mathbb{Z}^2, \{a = (1,0), b = (0,1)\})$
5. $\Gamma(\mathbb{F}_2, \{a,b\})$*, where* $\mathbb{F}_2 = \langle a, b\rangle$ *(i.e., the group with two generators and no relations)*
6. $\Gamma(BS(1,2), \{a,t\})$*, where* $BS(1,2) = \langle a, t \mid tat^{-1} = a^2\rangle$.

1.2 Metric spaces

Let (X,d) denote a metric space X with its metric d. If the metric is clear, we may just write X.

If $x \in X$ and $r \geq 0$, then $B_R(x) = \{y \in X \mid d(x,y) \leq r\}$ is the (closed) ball of radius r about x. If $A \subseteq X$, then we write $\mathcal{N}_r(A) = \{y \in X \mid d(y,A) \leq r\} = A^{+r}$ for the (closed) r–neighborhood of A in X.

Definition 1.5. A *path* is X is a continuous map $p\colon [a,b] \to X$ for some $[a,b] \subseteq \mathbb{R}$. By an abuse of notation, we also call the image of this map p. Then $p_- = p(a)$ and $p_+ = p(b)$. A *ray* is a continuous map $p\colon [a,\infty) \to X$, and a *bi-infinite path* is a continuous map $p\colon (-\infty,\infty) \to X$.

Given a path p, the *length* of p is

$$\ell(p) := \sup_{a \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq b} \sum_{i=1}^{n-1} d(p(t_i), p(t_{i+1})), \tag{1}$$

where the supremum is taken over all n and all choices of $t_1, \ldots, t_n$. This supremum is often infinite. We typically consider *rectifiable* paths, which are those with $\ell(p) < \infty$.

Example 1.6 (From BH I.1 Ex. 1.19). Let $X = [0,1]$, and let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be an infinite sequence in $[0,1]$ converging to 1. let $c\colon [0,1] \to X$ be any path such that $c(0) = 0$, $c(t_n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Then c is not rectifiable because its length is bounded below by the harmonic series $\sum_{i=1}^{\infty} \frac{1}{k}$, which diverges to infinity.

Definition 1.7. A path is *geodesic* if $\ell(p) = d(p_-, p_+)$. A ray is *geodesic* if for all $a \leq s \leq t < \infty$, $\ell(p|_{[s,t]}) = d(p(s), p(t))$, and similarly for a bi-infinite path. A metric space is *geodesic* if every pair of points can be connected by a geodesic.

A weaker condition one can put on a metric space is that it is a *length space*, which means that it is a path-connected space such that for all $x, y \in X$, $d(x,y) = \inf\{\ell(p) \mid p_- = x, p_+ = y\}$. For example, consider a space with two points, x and y, with countably many edges between them indexed by the positive integers so that the n-th edge has length $1 + \frac{1}{n}$. Here, $d(x,y) = \inf(1 + frac1n)$, but there is no path of length 1 between them. Most statements in this class will hold for length spaces, but the proofs tend to be a bit more technical, so we will mostly work with geodesic spaces.

Definition 1.8. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is:

- an *isometry* if for all $x, y \in X$,
  $$d_Y(f(x), f(y)) = d_X(x,y)$$
  and f is surjective;
- a *biLipschitz equivalence* if there exists $\lambda \geq 1$ such that for all $x, y \in X$,
  $$\frac{1}{\lambda} d_X(x,y) \leq d_Y(f(x), f(y)) \leq \lambda d_X(x,y)$$
  and f is surjective;
- a (λ, c)–*quasi-isometry* if there exists $\lambda \geq 1$ and $c \geq 0$ such that for all $x, y \in X$
  $$\frac{1}{\lambda} d_X(x,y) - c \leq d_Y(f(x), f(y)) \leq \lambda d_X(x,y) + c$$
  and $f(X)^{+c} = Y$, i.e., f is *coarsely surjective.*

We write $X \sim_{Lip} Y$ or $X \sim_{qi} Y$ if there exists a biLipschitz equivalence or (λ, c)–quasi-isometry from X to Y, respectively.

If the assumption of (coarse) surjectivity is dropped, then f is a(n) isometric, biLipschitz, or quasi-isometric *embedding*, respectively. Note that quasi-isometries are not required to be continuous, though it will follow from the definitions that isometries and biLipschitz equivalences are continuous.

Exercises 1.9.

1. *Show that $\sim_{Lip}$ and $\sim_{qi}$ are equivalence relations.*
2. *Let X, Y be bounded metric spaces. Show that $X \sim_{qi} Y$.*
3. *Show that $\Gamma(\mathbb{Z}, \{1\})$ is quasi-isometric to the real line $\mathbb{R}$.*
4. *Show that $\Gamma(\mathbb{Z}, \{1\}) \sim_{qi} \Gamma(\mathbb{Z}, \{2,3\})$.*
5. *In general, show that if $S, T \subseteq G$ are two finite generating sets, then $(G, d_S) \sim_{Lip} (G, d_T)$, and so $\Gamma(G,S) \sim_{qi} \Gamma(G,T)$.*

1.3 Group actions

Let G act on a metric space X, that is, there exists a map $\varphi\colon G \times X \to X$ such that $\varphi(gh, x) = \varphi(g, \varphi(h, x))$. We write $G \curvearrowright X$, and, to save space, we often drop the φ notation simply write gx (or $g.x$ or $g \cdot x$) for $\varphi(g, x)$. In this class, we will ALWAYS assume the action is by isometries, that is, for all $g \in G$, $d_X(x, y) = d_X(gx, gy)$.

There is a bijection between actions of G on X and homomorphims $\rho\colon G \to \mathrm{Isom}(X)$, where $\mathrm{Isom}(X)$ is the group of isometries of the space X.

Definition 1.10. The action is *faithful* if ρ is injective, i.e., for all $g \in G \setminus \{1\}$, there exists $x \in X$ such that $gx \neq x$. The action is *free* if for all $x \in X$, $\mathrm{Stab}_G(x) = \{1\}$, where $\mathrm{Stab}_G(x) = \{g \in G \mid gx = x\}$, i.e., for all $x \in X$ for all $g \in G$, $gx \neq x$.

The action is *proper* if for all bounded subsets $B \subseteq X$,

$$\#\{g \in G \mid gB \cap B \neq \emptyset\} < \infty.$$

This is also called *metric properness.* There is a topological version where bounded is replaced by compact.

The action is *cobounded* (resp. *cocompact*) if there exists a cobounded (resp. cocompact) subset $K \subseteq X$ such that

$$\bigcup_{g \in G} gK = X.$$

Group actions can be used to prove algebraic results. Here is just one example:

Theorem 1.11. *Suppose G acts freely on $\mathbb{R}^n$ (where $\mathbb{R}^n$ has the Euclidean metric). Then G is torsion free.*

Proof. Exercise. □

[Hint: Use the fact that any finite set of points $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^n$ has a unique *centroid*, that is, a point $w \in \mathbb{R}^n$ that minimizes the distance $\sum d_{\mathbb{R}^n}(w, x_i)$.]