Topics

1. Gradients
2. Chain Rule
3. Implicit Function Theorem
4. Higher Order Partial Derivatives
5. Clairaut's Theorem
6. Optimization
   (a) First and second derivative test
   (b) Constrained opitimization
      i. Lagrange Multipliers
      ii. Solve boundary for one variable and substitute into the original function to get a function of one variable
      iii. Parametrize boundary and substitute into the original function to get a function of one variable
7. Double integrals in Cartesian coordinates (i.e., no polar coordinates)
   (a) Over rectangles
   (b) Over regions that are not rectangles

Problems:

1. Find the direction of maximal increase of $f(x,y) = 2x^3y + 5\ln(y)x^2$ at the point $(1,1)$. In what directions could you move so that the value of f does not change? Give your answer as a vector.

2. Let $f(x,y) = x^2 + y^2$. Where on $x^2 + y^2 = 1$ is $\vec{\nabla} f$ parallel to $\langle 2, -3 \rangle$?

3. Graph the zero set of $f(x,y) = x^3 - x^2y$. Explain, using the Implicit Function Theorem, precisely how you know from the zero set that the point $(0,0)$ is a critical point. (i.e., it is not sufficient to simply say: "because the zero set intersects itself")

4. Consider the surface $f(x,y) = x^2 + y^2$. For which values of x do the level sets of f define y as an implicit function of x? For these values, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. For which values of y do the level sets of f define x as an implicit function of y? For these values, find $\frac{\mathrm{d}x}{\mathrm{d}y}$.

5. (a) If $g(x,y) = f(u,v,z)$, where $u = 3x+5$, $v = 2x + y^2$, and $z = xy$, find all first- and second-order partial derivatives of g.

   (b) If $g(x,y) = f(x^2 + 2xy + y^2)$, where $f(t)$ is a function of one variable, find all first- and second-order partial derivatives of g.

6. Find and classify all critical points of $f(x,y) = x^3 + 2x^2y^2$. Are any of these local extrema also global extrema? Why/why not?

7. Consider the curve $g(x,y) = x^2 - y^3 + y^4 = 0$. Its graph is the loop in the following figure.

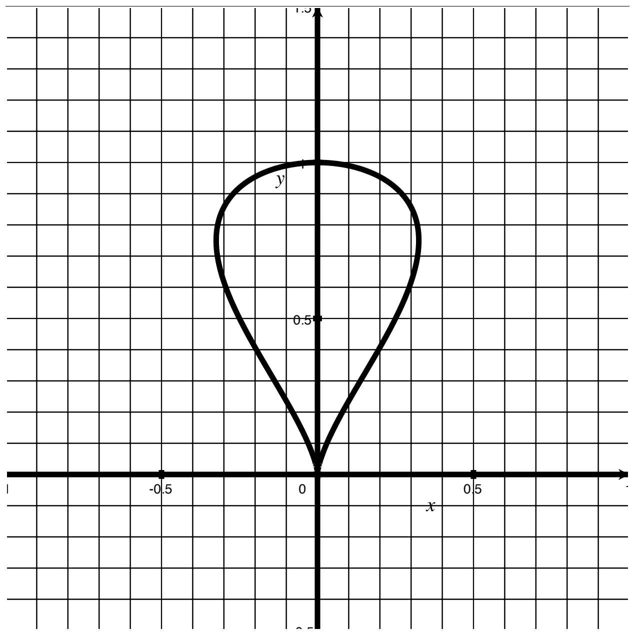
   Let $f(x,y) = x - y$.

   (a) Without doing any computations, is there any guarantee that $f(x,y)$ actually attains a global minimum value and a global maximum value somewhere on the constraint $g(x,y) = 0$? Explain!

   (b) In the above figure, draw an assortment of level sets for $f(x,y)$. (Label the levels.)

   (c) Mark the **approximate** location where $f(x,y)$ attains its minimum value on $g(x,y) = 0$.

   (d) Mark the **exact** location where $f(x,y)$ attains its maximum value on $g(x,y) = 0$.



(e) Set up (but don't solve!) the Lagrange system $\begin{cases} \vec{\nabla} f(x,y) = \lambda\, \vec{\nabla} g(x,y), \\ g(x,y) = 0. \end{cases}$

(f) Is $(0,0)$ a solution of the Lagrange system from **(e)**?

(g) What is $\vec{\nabla} g(0,0)$?

(h) What is the moral of this problem?

8. Find the global extrema of $f(x,y) = x^2 + 3xy + y^2$ on the rectangle $\{(x,y) | -2 \leq x \leq 2, 0 \leq y \leq 3\}$.

9. Find $\iint\limits_R (4 - y^2)\, \mathrm{d}A$ where $R = \{(x,y) | 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

10. Find the volume under $f(x,y) = 1$ over the region D bounded by $y = x$ and $x = y^2$.

11. Find the volume under $f(x,y) = 6x$ over the region D bounded by $x^2 + y^2 = 4$, $x \geq 0$.

12. Reverse the limits of integration (you do not need to evaluate): $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, \mathrm{d}x\mathrm{d}y$.