

## Quiz 6 RM Solutions

Please inform your TA if you find any errors in the quiz solutions.

**1.** (4 points)

1. (2 points) For  $f(x) = \int_0^{\sin(x)} e^{-t^5} dt$ , find the first order Taylor polynomial of  $f(x)$  around zero and use this to find an approximation for  $f(.15)$ .

**Solution:**

$$f(x) = \int_0^{\sin(x)} e^{-t^5} dt$$

$$f'(x) = e^{-\sin^5(x)} \cos(x)$$

so that

$$f(0) = \int_0^{\sin(0)} e^{-t^5} dt = \int_0^0 e^{-t^5} dt = 0$$

$$f'(0) = e^{-\sin^5(0)} \cos(0) = 1$$

so the first order Taylor polynomial for  $f(x)$  is given by  $x$  and our approximation for  $f(.15)$  is .15.

2. (2 points) For  $f(x) = x^6 + x + 1$ , find the degree two Taylor polynomial of  $f(x)$  around one. (In other words, find  $T_4^1 f(x)$ .)

**Solution:**

$$\begin{array}{ll} f(x) = x^6 + x + 1 & f(1) = 3 \\ f'(x) = 6x^5 + 1 & f'(1) = 7 \\ f''(x) = 30x^4 & f''(1) = 30 \end{array}$$

$$\text{so } T_4^1 f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 3 + 7(x-1) + 15(x-1)^2.$$

**2.** (6 points)

A tank begins with 100 litres of salt water in it. Fresh water is pumped in at a rate of ten litres

per minute and the mixed water is pumped out at a rate of five litres per minute. If the tank initially has ten kilograms of salt in it, how many kilograms of salt will remain in the tank in thirty minutes? Assume the water is always perfectly mixed.

**Solution:** If  $S(t)$  is the amount of salt in the tank in kilograms, the amount of salt left in the tank changes according to

$$\text{change in salt amount} = -\text{proportion of salt in the water} \times \text{water removed}$$

Recall that the proportion of salt in the water is given by  $\frac{S(t)}{V(t)}$ , so we will need to know what  $V(t)$  is. Since we have a net increase of 5 litres per minute and we start at 100 litres, we know that  $V(t) = 100 + 5t$ . Rewriting the previous line, we find that

$$\begin{aligned}\frac{dS}{dt} &= -\frac{S(t)}{100 + 5t} \quad (5) \\ &= -\frac{S(t)}{20 + t}\end{aligned}$$

With the initial condition that  $S(0) = 10$ . This equation is separable:

$$\frac{dS}{S} = -\frac{dt}{20 + t}$$

(If we wanted the general condition we would be concerned as well with the possibility  $S = 0$ , but that would not satisfy  $S(0) = 10$ ). Integrating both sides gives

$$\ln|S| = -\ln|20 + t| + C$$

Negative salt makes no sense, and we are only interested in positive values of  $t$ , so we can remove the absolute values. Setting  $D = e^C$  we get  $S = D(20 + t)^{-1}$ , and by the initial condition  $D = 200$ .

Thus  $S(t) = \frac{200}{20+t}$  and therefore  $S(30) = 4$ . We conclude that in thirty minutes, there will be four kilograms of salt left in the tank.