Midterm 2 Solutions

1.

(a) (6 pts.) Find a vector-valued function $\vec{r}(t)$ that represents the curve of intersection of the surfaces

$$z = x^2 + 4y^2 \text{ and } y = x^2.$$

Solution: We have y determined by x and z determined by x and y, so we can just use $\vec{r}(t) = \langle t, t^2, t^2 + 4t^4 \rangle$.

(b) (10 pts.) Let C be the smooth curve parameterized by the vector-valued function

$$\vec{p}(t) = \langle t, \sin(2t), \cos(2t) \rangle.$$

Find the $\vec{T}$-$\vec{N}$-$\vec{B}$ frame at the point $(\pi, 0, 1)$ on the curve C.

Solution: We have $\vec{p}\,'(t) = \langle 1, 2\cos(2t), -2\sin(2t) \rangle$ so $|\vec{p}\,'(t)| = \sqrt{5}$ and thus

$$\vec{T}(t) = 5^{-\frac{1}{2}} \langle 1, 2\cos(2t), -2\sin(2t) \rangle.$$

We have $\vec{T}'(t) = 5^{-\frac{1}{2}} \langle 0, -4\sin(2t), -4\cos(2t) \rangle$, so $|\vec{T}'(t)| = \frac{4}{\sqrt{5}}$ and thus

$$\vec{N}(t) = \frac{1}{4} \langle 0, -4\sin(2t), -4\cos(2t) \rangle = \langle 0, -\sin(2t), -\cos(2t) \rangle.$$

For $\vec{B}$ we can evaluate at $t = \pi$ to get $\vec{T}(\pi) = \frac{1}{\sqrt{5}} \langle 1, 2, 0 \rangle$, $\vec{N}(\pi) = \langle 0, 0, -1 \rangle$, so $\vec{B}(\pi) = \vec{T}(\pi) \times \vec{N}(\pi) = \frac{4}{5} \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{5}}(-2\vec{i} + \vec{j})$.

2. Circle the word True or False to indicate your answer. (No explanation needed.)

(a) Suppose that $\vec{r}_1(s)$ and $\vec{r}_2(t)$ are two smooth parametrizations of the same *oriented* smooth curve C. (Recall that by using the word oriented, we mean that the two parametrizations move along C in the same direction as s and t increase.)

(i) (3 pts.) The curvature at a point P on C is the same whether it is calculated using $\vec{r}_1$ or $\vec{r}_2$.

True. The defining formula $\kappa = |\frac{d\vec{T}}{ds}|$ proves this.

(ii) (3 pts.) The $\vec{T}$-$\vec{N}$-$\vec{B}$-frame at a point P on C the same whether it is calculated using $\vec{r}_1$ or $\vec{r}_2$.

True. Since $\vec{T}$ is intrinsic to the curve, so is $\vec{T}'$, which determines $\vec{N}$, and $\vec{B}$ is determined by these.

(b) (3 pts.) For any real numbers a, b, and c, the linearization $L(x, y)$ of the function $f(x, y) = ax + by + c$ at the point $(x_0, y_0) = (3, 2)$ is equal to $f(x, y)$.

True. Linearizing does nothing to a linear function.

(c) (3 pts.) Let $\vec{a}(t), \vec{b}(t)$, and $\vec{c}(t)$ be three vector-valued functions. Then

$$\frac{d}{dt}(\vec{a}(t) \cdot (\vec{b}(t) \times \vec{c}(t))) = \vec{a}\,'(t) \cdot (\vec{b}(t) \times \vec{c}(t)) + \vec{a}(t) \cdot (\vec{b}\,'(t) \times \vec{c}(t)) + \vec{a}(t) \cdot (\vec{b}(t) \times \vec{c}\,'(t)).$$

True. This is just the application of the two multiplication rules for $\cdot$ and $\times$ one after the other.

(d) (3 pts.) Suppose that $f(x, y)$ is a continuous function on the xy-plane with the origin $(0, 0)$ removed. Also suppose that $\lim_{(x,y)\to(0,0)} f(x, y)$ exists and is equal to 0. Then it is possible that $\lim_{(x,y)\to(0,0)} f(x, y)^2$ does not exist.

False. If $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, f has a continuous extension to all of the xy-plane and so its square is continuous at 0. *Alternative solution*: use the squeeze theorem on a region around the origin small enough that $|f^2(x, y)| \leq |f(x, y)|$.

3. Evaluate the limit or show that it does not exist.

(a) (7 pts.) Find

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+2y^2}.$$

Solution: We use the squeeze theorem to see that the limit is 0. We have $|\frac{x^2}{x^2+2y^2}| \leq 1$ so $|\frac{3x^2y}{x^2+2y^2}| \leq |3y|$, and $|3y| \to 0$.

(b) (7 pts.) Find

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}.$$

Solution: The paths of the form $x = my^3$ have limits depending on m, so the limit does not exist.

(c) (7 pts.) Find

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2).$$

Solution: In polar coordinates, this is $r^2\ln(r^2)$. Use l'Hopital's rule to see that $\lim_{r\to 0^+} \frac{\ln r^2}{r^{-2}} = \lim_{r\to 0^+} \frac{r^{-2} \cdot 2r}{-2r^{-3}} = \lim_{r\to 0^+} -r^2 = 0$.

4. (14 pts.) Find the osculating circle of the parabola $x = y^2$ at the point $(0,0)$.

Solution: This is just Example 8 from Section 13.3 in the textbook with x and y switched. We can just use the same argument, switching x and y everywhere, to get the circle $y^2 + (x-\frac{1}{2})^2 = \frac{1}{4}$ for the circle. For convenience we repeat the argument here.

We can use the formula for curvature of a graph by thinking of x as the dependent and y as the independent variable. So we have $x = y^2 = f(y)$, $f'(y) = 2y$, $f''(y) = 2$, so $\kappa(y) = \frac{|f''(y)|}{[1+(f'(y))^2]^{3/2}}$, which is 2 if $y = 0$. The osculating circle is in the xy-plane and its center is $\frac{1}{\kappa(0)}$ away from $(0,0)$ in the direction of $\vec{N}(0)$. The parametrization $\vec{r}(t) = \langle t^2, t\rangle$ gives $\vec{r}'(t) = \langle 2t, 1\rangle$ or $\vec{r}'(0) = \vec{T}'(0) = \langle 0, 1\rangle$ since this is of unit length. Since $\vec{N}(0)$ is orthogonal to $\vec{T}(0)$ and in the xy-plane (since the whole curve is), the only possibilities for $\vec{N}(0)$ are $\pm\langle 1, 0\rangle$. We can see that the positive sign is correct by looking at a graph of the curve (or by calculating the direction of $\vec{T}'(t)$ at $t = 0$). So the circle is centered at $(\frac{1}{2}, 0)$ and has radius $\frac{1}{2}$.

5.

(a) (8 pts.) The volume of a cylinder of height h and radius r is given by $V = \pi r^2 h$. Find the rate at which the volume V is changing per second if the height is 3, the radius is 4, the height is increasing by $\frac{1}{2}$ a unit per second, and the radius is decreasing by $\frac{1}{2}$ a unit per second.

Solution: The chain rule gives

$$\frac{dV}{dt} = \frac{\partial V}{\partial h}\frac{dh}{dt} + \frac{\partial V}{\partial r}\frac{dr}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}.$$

Substituting $r = 4$, $\frac{dh}{dt} = \frac{dr}{dt} = \frac{1}{2}$, and $h = 3$, we get $8\pi - 12\pi = 4\pi$ cubed units/second.

(b)

(i) (8 pts.) Suppose that the function $F(x,y)$ is defined by

$$F(x,y) = g(x^2+y^2, y-x)$$

where $g(r,s) = rs$. Calculate the first partial derivatives of F with respect to x and y.

Solution: If we compose, we get $F(x,y) = (x^2+y^2)(y-x) = -x^3 + x^2y - xy^2 + y^3$, which has $F_x = -3x^2 + 2xy - y^2$ and $F_y = x^2 - 2xy + 3y^2$. *Alternative solution*: Use the chain rule to get

$$F_x = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial x} \text{ and } F_y = \frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial s}\frac{\partial s}{\partial y}$$

and calculate $\frac{\partial g}{\partial r} = s = y - x$, $\frac{\partial g}{\partial s} = r = x^2 + y^2$, $\frac{\partial r}{\partial x} = 2x$, $\frac{\partial r}{\partial y} = 2y$, $\frac{\partial s}{\partial x} = -1$, and $\frac{\partial s}{\partial y} = 1$, which gives the same answer.
(ii) (5 pts.) What is the linearization of $F(x, y)$ at $(1, 1)$?
Solution: We have $L(x, y) = F(0, 0) + F_x(1, 1)(x - 1) + F_y(1, 1)(y - 1) = 0 + (-3 + 2 - 1)(x - 1) + (1 - 2 + 3)(y - 1) = -2x + 2y$.
(iii) (3 pts.) Suppose that y is implicitly defined as a function of x by the equation

$$F(x, y) = 0.$$

In terms of F_x and F_y, what is $\frac{dy}{dx}$? (You do not need to substitute your answer from part (i); just leave your answer in terms of F_x and F_y.)
Solution: This is $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

6.

(a) (10 pts.) Write down and evaluate the arc length integral from $\vec{r}(0)$ to $\vec{r}(t)$ of the curve

$$\vec{r}(t) = \left(\frac{1}{t^2 + 1} - \frac{1}{2}\right)\vec{i} + \frac{t}{t^2 + 1}\vec{j}.$$

Hint: The formula sheet at the end of the booklet may be helpful.
Solution: We have $\vec{r}'(t) = \langle \frac{-2t}{(t^2+1)^2}, \frac{-2t^2+t^2+1}{(t^2+1)^2} \rangle = \langle \frac{-2t}{(t^2+1)^2}, \frac{-t^2+1}{(t^2+1)^2} \rangle$. The arc length is the integral of

$$|\vec{r}'(t)| = \sqrt{\frac{(-2t)^2 + (-t^2 + 1)^2}{(t^2 + 1)^4}} = \sqrt{\frac{t^4 + 2t^2 + 1}{(t^2 + 1)^4}} = \sqrt{\frac{(t^2 + 1)^2}{(t^2 + 1)^4}} = \frac{1}{1 + t^2}.$$

This integral is $\tan^{-1} t$.
(b) (10 pts., extra credit) Reparametrize $\vec{r}(t)$ with respect to arc length and fully simplify. What is the curve?
Solution: We have $s(t) = \tan^{-1} t$, so $t(s) = \tan s$. We plug in $\tan s$:

$$\vec{r}(t(s)) = \left(\frac{1}{\tan^2 s + 1} - \frac{1}{2}\right)\vec{i} + \frac{\tan s}{\tan^2 s + 1}\vec{j} = \left(\frac{1}{\sec^2 s} - \frac{1}{2}\right)\vec{i} + \frac{\tan s}{\sec^2 s}\vec{j}$$
$$= (\cos^2 s - \frac{1}{2})\vec{i} + \sin s \cos s\vec{j} = \frac{1}{2}\cos(2s)\vec{i} + \frac{1}{2}\sin(2s)\vec{j}$$

by the double-angle formulas. So this is just the circle of radius $\frac{1}{2}$ centered at the origin, which can be seen directly. However, $s(t)$ never makes it to $\pi/2$, so one point is in fact inaccessible by the original parametrization, namely $(-\frac{1}{2}, 0)$.