1. (5 pts.) Find all the fourth roots of the complex number i.

Answer: We have $i = e^{(\pi/2)i}$. So using the formula from class, the fourth roots are $e^{(\pi/8+k\pi/2)i}$ for $k = 0, 1, 2, 3$.

2. Circle the word True or False to indicate your answer. (No explanation needed.)

(a) (2 pts.) Suppose $f(x, y)$ is a function of two variables that is differentiable at the point $(1, 1)$. If the gradient $\nabla f(1, 1)$ is not zero, there are exactly two direction unit vectors $\vec{u}$ such that $D_{\vec{u}} f(1, 1) = 0$.

Answer: True. The directional derivative is $\vec{u} \cdot \nabla f(1, 1)$. This is zero exactly when $\vec{u}$ is orthogonal to $\nabla f(1, 1)$. There are two unit vectors orthogonal to a given nonzero vector.

(b) (2 pts.) Suppose $\vec{r}(t)$ is a smooth parametrization of a curve and that $|\vec{r}(t)| = 1$ for all t. Then $\vec{r}(t)$ and $\vec{r}\,'(t)$ are always orthogonal to one another for each value of t.

Answer: True. We have $0 = \frac{d}{dt}|\vec{r}(t)|^2 = \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = 2\vec{r}(t) \cdot \vec{r}\,'(t)$, which proves the orthogonality.

(c) (2 pts.) Every continuous function on the domain $D = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 < 1\}$ attains an absolute maximum on D.

Answer: False. D is not closed. For instance, the function $x^2 + y^2$ does not have an absolute maximum on D since it gets close to 1 but not equal to 1.

(d) (2 pts.) Let $f(x, y)$ be a differentiable function on the entire xy-plane. Then $f(x, y)$ must have at least one critical point.

Answer: False. If $f(x, y) = x$, then $f_x = 1$, which has no zeroes.

(e) (2 pts.) The binormal component of acceleration is always 0.

Answer: True. This was shown in class.

3.

(a) (5 pts.) Find an equation for the plane with x-intercept 2, y-intercept 3, and z-intercept 4.

Answer: $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

(b) (5 pts.) Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS, where

$$P = (2, 0, 3), Q = (2, 3, 4), R = (3, 1, 5), \text{ and } S = (4, 4, 3).$$

Answer: This is the absolute value of $\det \begin{pmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 4 & 0 \end{pmatrix} = 3 \cdot 2 \cdot 2 + 1 \cdot 1 \cdot 4 - 1 \cdot 1 \cdot 2 = 14$, so the volume is 14.

4. (5 pts.) Calculate the following limits or show that they do not exist.

(a) (5 pts.)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 y^2 z^2}{x^2 + y^2 + z^2}$$

Answer: We have $x^2 \leq x^2 + y^2 + z^2$, so $|\frac{x^2y^2z^2}{x^2+y^2+z^2}| \leq |y^2 z^2|$. By the squeeze theorem, the limit is 0 since $|y^2 z^2| \to 0$ as $(x, y, z) \to (0, 0, 0)$.

(b) (5 pts.)

$$\lim_{(x,y)\to(0,0)} \frac{y^4 \cos^2 x}{2x^4 + y^4}$$

Answer: When $y = 0$ we get 0 for the limit. When $x = 0$ we get 1 for the limit. So the limit does not exist.

5.

(a) (10 pts.) Suppose that the acceleration of an object is given by $\vec{a}(t) = \langle -\cos t, -\sin t \rangle$ and that $\vec{r}(\frac{\pi}{2}) = \langle 0, 1 \rangle$ and $\vec{r}(\pi) = \langle -1, 0 \rangle$, where $\vec{r}(t)$ denotes the position vector of the object. Find $\vec{r}(t)$.

Answer: We have $\vec{v}(t) = \langle -\sin t + C_1, \cos t + D_1 \rangle$ and $\vec{r}(t) = \langle \cos t + C_1 t + C_2, \sin t + D_1 t + D_2 \rangle$. Plugging in the two values of t, we deduce that $C_1 \frac{\pi}{2} + C_2 = 0$ and $C_1 \pi + C_2 = 0$, $D_1 \frac{\pi}{2} + D_2 = 0$ and $D_1 \pi + D_2 = 0$. The only solution is that all these constants are 0, so $\vec{r}(t) = \langle \cos t, \sin t \rangle$.

(b) (5 pts.) What is the osculating place to the curve $\vec{r}(t) = \langle 1, t, t^2 \rangle$ when $t = 2$?
Hint: It is possible to avoid having to do much calculation.
Answer: The curve is in the plane $x = 1$, so this is the osculating plane for all t. Alternatively, we calculate $\vec{r}'(t) = \langle 0, 1, 2t \rangle$ and $\vec{T}(t) = \frac{\langle 0,1,2t \rangle}{\sqrt{1+4t^2}}$. The vector $\vec{N}(t)$ is proportional to $\vec{T}'(t) \cdot (1+4t^2)$ (to clear the denominator). One could write out the calculation of this vector (and get $\langle 0, -\frac{8}{\sqrt{17}}, 2\sqrt{17} - \frac{32}{\sqrt{17}} \rangle$), but since the $\vec{i}$ component is clearly 0, we see that $\vec{N}(t)$ and $\vec{T}(t)$ are both in the yz-plane. It follows that $\vec{B}$ is along the x-axis and the osculating plane is $x = 1$.

6. (10 pts.) Find the distance between the planes $-2x - 4y - 4z = -18$ and $x + 2y + 2z = 0$.
Answer: Write the first plane as $x + 2y + 2z = 9$. We can use the distance formula $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}} = \frac{9}{\sqrt{1+4^2+4^2}} = 3$. Alternatively, compute the distance of the plane $x + 2y + 2z = 9$ to the origin, which gives the same answer.

7. (10 pts.) Bob is at the origin, $(0, 0)$, in the xy-coordinate plane, and the temperature $T(x, y)$ is a differentiable function of x and y. The directional derivative of T at the origin in the direction of the point $(1, \sqrt{3})$ is 1 degree Celsius per second. The directional derivative at the origin in the direction of $(-1, 0)$ is -1 degree Celsius per second. What direction should Bob walk so that the temperature will increase as quickly as possible? (Give your answer in the form of a unit direction vector.) What is the rate of increase?
Answer: Let $\nabla T = a\vec{i} + b\vec{j}$. Then the second given derivative shows $\nabla T \cdot (-\vec{i}) = -a = -1$, so $a = 1$. The first needs $\langle 1, \sqrt{3} \rangle$ to be scaled by $\frac{1}{2}$ so that it is a unit vector, and gives $\nabla T \cdot (\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}) = \frac{a}{2} + \frac{\sqrt{3}}{2} b = 1$, so plugging in $a = 1$, we get $b = \frac{1}{\sqrt{3}}$. The magnitude of ∇T is $\frac{2}{\sqrt{3}}$ and the direction is $\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$.

8. (15 pts.) Let D be the region bounded by $x \geq 0$, $y \geq 0$, and $y + x^2 \leq 1$. Find the absolute minimum and absolute maximum of $f(x, y) = x^2 - x + y^2 - y$ on D. Specify the points where this minimum/maximum is obtained. Also, classify all the critical points of $f(x, y)$.
Hint: Two of the values you find will be close to one another. If you aren't sure which is smaller, just provide both (no need to simplify) and you will receive full credit.
Answer: We remark that writing the function as $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 - \frac{1}{2}$ gives a very short solution to the question of the absolute maximum/minimum; it is clear that the minimum on all of $\mathbf{R}^2$ is $-\frac{1}{2}$, attained at $(\frac{1}{2}, \frac{1}{2})$, and that even on the larger region bounded by $0 \leq x \leq 1$ and $0 \leq y \leq 1$ the function is nonpositive, so the maxima on D are at $(0, 0)$, $(1, 0)$, and $(0, 1)$, where it is 0. To solve the problem by using the course material instead, note that we have $f_x = 2x - 1$ and $f_y = 2y - 1$, so the only critical point is $(\frac{1}{2}, \frac{1}{2})$, and the second derivative test shows that it is a local minimum. It is inside D, so $f(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{2}$ is a possible absolute extremum. Along the x-axis, we have x ranging from 0 to 1 and $f(x, 0) = x^2 - x$. The first derivative of this is $2x - 1$, so $x = \frac{1}{2}$ is the only critical point. We get the boundaries $(0, 0)$, $(1, 0)$, and $(\frac{1}{2}, 0)$ as the possible extrema, with values of $0, 0$, and $-\frac{1}{4}$, respectively. The analysis of the y-axis is completely the same, giving $(0, 1)$ and $(0, \frac{1}{2})$ as additional possible extrema. Finally we get the curve $y = 1 - x^2$, along which $f(x, 1 - x^2) = x^2 - x + (1 - x^2)^2 - (1 - x^2) = x^4 - x$. Taking the first derivative gives $4x^3 - 1$ and $(4^{-\frac{1}{3}}, 1 - 8^{-\frac{1}{3}})$ as a possible extrema. The value there is $256^{-\frac{1}{3}} - 4^{-\frac{1}{3}} \approx -0.47$, which is larger than $-\frac{1}{2}$. The absolute maximum is 0 (at $(0, 0), (1, 0), (0, 1)$) and the absolute minimum is $-\frac{1}{2}$ (at $(\frac{1}{2}, \frac{1}{2})$).

9. (15 pts.) Suppose that a rectangular cardboard box has a long diagonal (i.e. corner to furthest opposite corner) of length 1 unit. Find the maximum possible surface area of the box, making sure to show all your work.
Answer: If the side lengths are x, y, z, the long diagonal is $\sqrt{x^2 + y^2 + z^2}$. We can work with the square to simplify calculations, giving $g(x, y, z) = x^2 + y^2 + z^2 = 1$. Then $f(x, y, z) = 2xy + 2yz + 2zx$, $\nabla f = \langle 2y + 2z, 2x + 2z, 2x + 2y \rangle$ and $\nabla g = \langle 2x, 2y, 2z \rangle$. If we add the three components of $\nabla f = \lambda \nabla g$, we get $4(x + y + z) = 2\lambda(x + y + z)$, and we note that since x, y, z are nonnegative and $x^2 + y^2 + z^2 = 1$,

we must have $x + y + z \neq 0$, so $\lambda = 2$. We deduce from the three components of $\nabla f = \lambda \nabla g$ that $x = \frac{y+z}{2}$ and similarly, y and z are the averages of the other two. The largest of x, y, z can only be the average of the other two if $x = y = z$. Plugging this into the constraint, we get $x = \frac{1}{\sqrt{3}}$ and $2xy + 2yz + 2zx = 2$.

Extra credit (10 pts.)

The smooth curve defined by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ intersects the sphere $x^2 + y^2 + z^2 = 3$ when $t = \pm 1$. What are the cosines of the angles of intersection at these points?

Answer: We have $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$, which is $\langle 1, 2, 3$ when $t = 1$ and $\langle 1, -2, 3 \rangle$ when $t = -1$. The gradient of $f(x, y, z) = x^2 + y^2 + z^2$ is $\langle 2x, 2y, 2z \rangle$, which is $\langle 2, 2, 2 \rangle$ at $(1, 1, 1)$ and $\langle -2, 2, -2 \rangle$ at $(-1, 1, -1)$. For $(1, 1, 1)$, the cosine of the angle between the gradient and $\vec{r}\,'(1)$ is $\frac{2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3}{\sqrt{1+4+9}\sqrt{4+4+4}} = \sqrt{\frac{6}{7}}$. Drawing a picture as in the solution to the HW10 problem discussed in class to see that $\theta = \frac{\pi}{2} - \cos^{-1}(\sqrt{\frac{6}{7}})$, where θ is the angle of intersection, we get $\cos\theta = \sqrt{1 - (\sqrt{\frac{6}{7}})^2} = \frac{1}{\sqrt{7}}$. The cosine of the angle at $(-1, 1, -1)$ is $\frac{-2 \cdot 1 + 2 \cdot (-2) + (-2) \cdot 3}{\sqrt{1+4+9}\sqrt{4+4+4}} = -\sqrt{\frac{6}{7}}$ instead. In this case, the picture reveals again that $\cos\theta' = \frac{1}{\sqrt{7}}$, where θ' is the angle of intersection, since $\theta' = \cos^{-1}(-\sqrt{\frac{6}{7}}) - \frac{\pi}{2}$.