Name: ______________________________

Instructor: Shrenik Shah

MATH V1201, Section 12 - Final

December 21, 2015 (170 minutes)

This examination booklet contains 9 problems, plus an extra credit problem. There are 13 sheets of paper including the front cover and formula sheet at the back. This is a closed book exam. Do all of your work on the pages of this exam booklet. Show all your computations and justify/explain your answers (except for the true/false problems). Cross out anything you do not want graded. Calculators are NOT allowed.

You have 170 minutes to complete the exam. Do not begin until instructed to do so. When time is up, stop working and close your test booklet. Cell phones, headphones, laptops and other electronic devices are not allowed.

Problem	Possible score	Your score
1	5	
2	10	
3	10	
4	10	
5	15	
6	10	
7	10	
8	15	
9	15	
EC	(+10)	
Total	100(+10)	

Grades will be posted on CourseWorks.

1. (5 pts.) Find all the fourth roots of the complex number i.

2. Circle the word True or False to indicate your answer. (No explanation needed.)

(a) (2 pts.) Suppose $f(x, y)$ is a function of two variables that is differentiable at the point $(1, 1)$. If the gradient $\nabla f(1, 1)$ is not zero, there are exactly two direction unit vectors $\vec{u}$ such that $D_{\vec{u}}f(1, 1) = 0$.

True   False

(b) (2 pts.) Suppose $\vec{r}(t)$ is a smooth parametrization of a curve and that $|\vec{r}(t)| = 1$ for all t. Then $\vec{r}(t)$ and $\vec{r}\,'(t)$ are always orthogonal to one another for each value of t.

True   False

(c) (2 pts.) Every continuous function on the domain $D = \{(x, y) \in \mathbf{R}^2 | x^2 + y^2 < 1\}$ attains an absolute maximum on D.

True   False

(d) (2 pts.) Let $f(x, y)$ be a differentiable function on the entire xy-plane. Then $f(x, y)$ must have at least one critical point.

True   False

(e) (2 pts.) The binormal component of acceleration is always 0.

True   False

3.

(a) (5 pts.) Find an equation for the plane with x-intercept 2, y-intercept 3, and z-intercept 4.

(b) (5 pts.) Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS, where

$$P = (2, 0, 3), Q = (2, 3, 4), R = (3, 1, 5), \text{ and } S = (4, 4, 3).$$

4. (5 pts.) Calculate the following limits or show that they do not exist.

(a) (5 pts.)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2}$$

(b) (5 pts.)

$$\lim_{(x,y)\to(0,0)} \frac{y^4\cos^2 x}{2x^4+y^4}$$

5.

(a) (10 pts.) Suppose that the acceleration of an object is given by $\vec{a}(t) = \langle -\cos t, -\sin t \rangle$ and that $\vec{r}(\frac{\pi}{2}) = \langle 0, 1 \rangle$ and $\vec{r}(\pi) = \langle -1, 0 \rangle$, where $\vec{r}(t)$ denotes the position vector of the object. Find $\vec{r}(t)$.

(b) (5 pts.) What is the osculating place to the curve $\vec{r}(t) = \langle 1, t, t^2 \rangle$ when $t = 2$?

Hint: It is possible to avoid having to do much calculation.

6. (10 pts.) Find the distance between the planes $-2x - 4y - 4z = -18$ and $x + 2y + 2z = 0$. (You may use any formulas from class without proving them.)

7. (10 pts.) Bob is at the origin, $(0, 0)$, in the xy-coordinate plane, and the temperature $T(x, y)$ is a differentiable function of x and y. The directional derivative of T at the origin in the direction of the point $(1, \sqrt{3})$ is 1 degree Celsius per second. The directional derivative at the origin in the direction of $(-1, 0)$ is -1 degree Celsius per second. What direction should Bob walk so that the temperature will increase as quickly as possible? (Give your answer in the form of a unit direction vector.) What is the rate of increase?

8. (15 pts.) Let D be the region bounded by $x \geq 0$, $y \geq 0$, and $y + x^2 \leq 1$. Find the absolute minimum and absolute maximum of $f(x, y) = x^2 - x + y^2 - y$ on D. Specify the points where this minimum/maximum is obtained. Also, classify all the critical points of $f(x, y)$.

Hint: Two of the values you find will be close to one another. If you aren't sure which is smaller, just provide both (no need to simplify) and you will receive full credit.

9. (15 pts.) Suppose that a rectangular cardboard box has a long diagonal (i.e. corner to furthest opposite corner) of length 1 unit. Find the maximum possible surface area of the box, making sure to show all your work.

Hint: If using Lagrange multipliers, try summing the components of the gradient equation.

Extra credit (10 pts.)
The smooth curve defined by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ intersects the sphere $x^2 + y^2 + z^2$ at two points. What are the cosines of the angles of intersection at these points?

Some useful formulas

Feel free to tear this sheet off from your exam booklet.

Arc length and curvature

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}, \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \text{ and } \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \tag{1}$$

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}, \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}} \tag{2}$$

Motion in space

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) \tag{3}$$

$$\vec{v}(t) = v(t)\vec{T}(t) \tag{4}$$

$$\vec{a}(t) = v'(t)\vec{T}(t) + \kappa(t)v^2(t)\vec{N}(t) = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|}\vec{T}(t) + \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}\vec{N}(t) \tag{5}$$

Trigonometry

$$\frac{d}{dx}(\tan x) = \sec^2 x, \frac{d}{dx}(\csc x) = -\csc x \cot x, \frac{d}{dx}(\sec x) = \sec x \tan x, \frac{d}{dx}(\cot x) = -\csc^2 x \tag{6}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \tag{7}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}, \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}, \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2} \tag{8}$$