Name: **Solutions to Practice Final**

1. Consider the line $\vec{r}(t) = \langle 3+2t,\ 2-t,\ 6\rangle$.

(a) Find symmetric equations for this line.

(b) Find the point where the first line $\vec{r}(t)$ intersects the surface $z = x + y^2$.

Solution.

(a) We have $x = 3+2t, y = 2-t, z = 6$ so solving for t we get the symmetric equations

$$\frac{x-3}{2} = 2 - y, \quad z = 6.$$

(b) We substitute $x = 3+2t, y = 2-t, z = 6$ into the equation $z = x + y^2$ and get

$$t^2 - 2t + 1 = 0,$$

whose only solution is $t = 1$. The point of intersection is therefore $\vec{r}(1) = (5, 1, 6)$.

2. Consider the two lines $\vec{r}(t) = \langle 1+t,\ 2+t,\ 3-t\rangle$ and $\vec{s}(t) = \langle t,\ -1+3t,\ 1+2t\rangle$.

(a) Find an equation for the plane that contains the two lines $\vec{r}(t)$ and $\vec{s}(t)$.

(b) Find the distance from this plane to the point $P = (1, 2, 4)$.

Solution.

(a) First we notice that $\vec{r}(0) = \vec{s}(1) = \langle 1, 2, 3\rangle$, so the point $P_0 = (1, 2, 3)$ lies on the plane. The vector $\vec{n}$ normal to the plane can be taken to be the cross product of the directions of the two lines, $\vec{d_1} = \langle 1, 1, -1\rangle, \vec{d_2} = \langle 1, 3, 2\rangle$, so

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \langle 5, -3, 2\rangle.$$

Therefore the plane has equation

$$5(x-1) - 3(y-2) + 2(z-3) = 0,$$

or in other words $5x - 3y + 2z = 5$.

(b) The distance is given by

$$\frac{\left|\overrightarrow{P_0P} \cdot \vec{n}\right|}{||\vec{n}||} = \frac{2}{\sqrt{38}}.$$

3. Find the distance between the plane $x+2y-z=4$ and the plane $-3x-6y+3z=9$. Then find the parametric equation of the line that passes through the point $(1,1,1)$ and is perpendicular to both planes.

Solution. We divide by -3 and rewrite the second plane as

$$x+2y-z=-3,$$

which now has the same normal vector $\vec{n}=\langle 1,2,-1\rangle$ as the first plane. The point $P_0=(0,0,3)$ lies on the second plane, so it suffices to compute the distance from this point to the first plane (since the planes are parallel). Applying the distance formula, we get

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{||\vec{n}||}=\frac{|-3-4|}{\sqrt{1+4+1}}=\frac{7}{\sqrt{6}}.$$

The line has direction $\vec{n}$, so it can be parametrized by

$$x=1+t, \quad y=1+2t, \quad z=1-t.$$

4.

(a) Identify the surfaces $4y^2+z^2-x-16y-4z+10=0$ and $2x^2-y^2+z^2+4x-4y-6z+7=0$.

Solution.

(a) We complete the squares in the first surface

$$-x+4(y-2)^2+(z-2)^2=10,$$

and see that it is an elliptic paraboloid. We do the same for the second surface

$$2(x+1)^2-(y+2)^2+(z-3)^2=0,$$

and see that it is a cone.

5. Consider the curve

$$\vec{r}(t)=\langle 12t,\ 8t^{3/2},\ 3t^2\rangle,$$

with $0\leqslant t\leqslant 1$.

(a) Find the velocity $\vec{r}'(1)$ and acceleration $\vec{r}''(1)$ at time $t=1$.

(b) Find the total length of the curve.

Solution.

(a) The velocity at any time $t>0$ is equal to

$$\vec{r}'(t)=\left\langle 12, 12t^{1/2}, 6t\right\rangle,$$

so at time $t=1$ this equals $\vec{r}'(1)=\langle 12,12,6\rangle$. The acceleration at any time $t>0$ is equal to

$$\vec{r}''(t)=\left\langle 0, 6t^{-1/2}, 6\right\rangle,$$

so at time $t=1$ this equals $\vec{r}''(1)=\langle 0,6,6\rangle$.

(b) To compute the length we first compute

$$||\vec{r}\,'(t)|| = \sqrt{144 + 144t + 36t^2} = 6\sqrt{4 + 4t + t^2} = 6(2+t).$$

The total length of the curve is then equal to

$$\int_0^1 ||\vec{r}\,'(t)||\, dt = 6\int_0^1 2 + t\, dt = 6\left(2t + \frac{t^2}{2}\right)|_0^1 = 15.$$

6. A surface in $\mathbb{R}^3$ is described by the equation

$$e^{xyz} = z\ln(xy) + e^2.$$

(a) Find an equation for the tangent plane to the surface at the point $(1,1,2)$.

(b) Find a parametrization for the line that passes through $(1,1,2)$ and is perpendicular to the tangent plane to the surface (i.e. the normal line).

(c) Find the point where the normal line intersects the xz-plane.

Solution.

(a) We can write the equation of the surface as a level surface $F(x,y,z) = 0$ with $F(x,y,z) = e^{xyz} - z\ln(xy) - e^2$. Then we know that the tangent plane at $(1,1,2)$ has normal vector $\nabla F(1,1,2)$. We compute the partial derivatives

$$\frac{\partial F}{\partial x}(x,y,z) = yze^{xyz} - \frac{z}{x}, \quad \frac{\partial F}{\partial y}(x,y,z) = xze^{xyz} - \frac{z}{y}, \quad \frac{\partial F}{\partial z}(x,y,z) = xye^{xyz} - \ln(xy),$$

and so

$$\nabla F(1,1,2) = \langle 2e^2 - 2, 2e^2 - 2, e^2\rangle.$$

Therefore the tangent plane has equation

$$(2e^2 - 2)(x-1) + (2e^2 - 2)(y-1) + e^2(z-2) = 0.$$

(b) The normal line passes through $(1,1,2)$ with direction $\nabla F(1,1,2) = \langle 2e^2 - 2, 2e^2 - 2, e^2\rangle$, and so it can be parametrized by

$$x = 1 + (2e^2 - 2)t, \quad y = 1 + (2e^2 - 2)t, \quad z = 2 + e^2 t.$$

(c) To find the intersection of the normal line with the xz-plane we just set $y = 0$ and get $t = -\frac{1}{2e^2-2}$, so the intersection point is

$$\left(0,\ 0,\ 2 - \frac{e^2}{2e^2 - 2}\right).$$

7. Consider the surface S in $\mathbb{R}^3$ given by the equation $z = e^x\cos(xy^2)$.

(a) Find an equation for the tangent plane to S at the point $(0,0,1)$.

(b) Use linear approximation to estimate the value of $e^{0.001} \cdot \cos(0.001 \cdot (0.01)^2)$.

Solution.

(a) First we compute the two partial derivatives

$$\frac{\partial z}{\partial x}(x,y) = e^x \cos(xy^2) - y^2 e^x \sin(xy^2), \quad \frac{\partial z}{\partial y}(x,y) = -2xye^x \sin(xy^2).$$

If we compute these at the point $(x, y) = (0, 0)$ we get

$$\frac{\partial z}{\partial x}(0,0) = 1, \quad \frac{\partial z}{\partial y}(0,0) = 0.$$

The equation for the tangent plane to S at $(0, 0, 1)$ is then

$$z - 1 = x.$$

(b) The linearization of $f(x, y) = e^x \cos(xy^2)$ at $(x, y) = (0, 0)$ is

$$L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0) \cdot x + \frac{\partial f}{\partial y}(0,0) \cdot y = 1 + x,$$

therefore the value

$$f(0.001, 0.01) \approx L(0.001, 0.01) = 1 + 0.001 = 1.001.$$

This is not too far from the value we actually want, $f(0.001, 0.01)$.

8. Find a point on the surface $z = -\frac{e^x}{2} + 3y^2$ where the tangent plane is parallel to the plane $x - y + 2z = 3$.

Solution. We can rewrite the equation of the plane as $z = \frac{3}{2} - \frac{x}{2} + \frac{y}{2}$. In order for the tangent plane to be parallel to this line, we need their direction vectors to be parallel. Therefore, we need to find all points (a, b) such that

$$\frac{\partial z}{\partial x}(a,b) = -\frac{1}{2}, \quad \frac{\partial z}{\partial y}(a,b) = \frac{1}{2}.$$

Computing the partials, we find

$$\frac{\partial z}{\partial x}(x,y) = -\frac{e^x}{2}, \quad \frac{\partial z}{\partial y}(x,y) = 6y,$$

which gives us the two equations

$$-\frac{e^x}{2} = -\frac{1}{2}, \quad 6y = \frac{1}{2}.$$

Therefore, we must have $x = 0, y = \frac{1}{12}$, and this gives us the point $(0, \frac{1}{12}, -\frac{23}{48})$.

9. Find the absolute maximum and minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$.

Solution. Call the constraint $g(x, y) = 1$ with $g(x, y) = x^4 + y^4$. Then we have

$$\nabla f = \langle 2x,\ 2y\rangle,$$

$$\nabla g = \langle 4x^3,\ 4y^3\rangle,$$

and the Lagrange multiplier equations are

$$2x = 4\lambda x^3, \quad 2y = 4\lambda y^3.$$

In the first equation, we have the solution $x = 0$. In this case, since $x^4 + y^4 = 1$, we get $y = \pm 1$, so we have two candidate points $(0, 1)$ and $(0, -1)$. At either one, the value of the function $f(x, y)$ is 1. If $x \neq 0$, we can divide by x^3 and get $2\lambda = \frac{1}{x^2}$.

In the second equation, we have the solution $y = 0$. In this case, since $x^4 + y^4 = 1$, we get $x = \pm 1$, so we have two candidate points $(1, 0)$ and $(-1, 0)$. At either one, the value of the function $f(x, y)$ is 1. If $y \neq 0$, we can divide by y^3 and get $2\lambda = \frac{1}{y^2}$.

If neither x nor y equals zero, we therefore get

$$\frac{1}{x^2} = 2\lambda = \frac{1}{y^2},$$

so $x = \pm y$. We plug this in the constraint $x^4 + y^4 = 1$ and get $x^4 = \frac{1}{2}$ so $x = \pm y = \pm \frac{1}{\sqrt[4]{2}}$. This gives four candidate points

$$\left(\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right), \quad \left(\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}\right), \quad \left(-\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}\right), \quad \left(-\frac{1}{\sqrt[4]{2}}, -\frac{1}{\sqrt[4]{2}}\right).$$

At any of these points, the value of the function $f(x, y)$ is $\sqrt{2}$. Therefore we get that the maximum is $\sqrt{2}$, achieved at these 4 points, and the minimum is 1, achieved at the other 4 points $(1, 0), (-1, 0), (0, 1), (0, -1)$.

10. Find the absolute maximum and minimum values of $f(x, y) = x^2 + xy$ on the region D in $\mathbb{R}^2$ that is enclosed by the parabolas $y = x^2 - 1$ and $y = 1 - x^2$. Be sure to specify all the points where the maximum/minimum are attained.

Solution. First we look for critical points inside D, where $\nabla f = \vec{0}$. We have $\nabla f = \langle 2x + y, x\rangle$, so at a critical point we must have $x = y = 0$. Therefore, the only critical point inside D is the origin, where the value of f is 0.

Next we consider the function f on the boundary. Note that the two parabolas meet at the points $(-1, 0)$ and $(1, 0)$.

On the piece of parabola $y = x^2 - 1$, where $-1 \leqslant x \leqslant 1$, the function $f(x, y) = x^2 + xy$ equals $x^2 + x^3 - x$, with $-1 \leqslant x \leqslant 1$. The first derivative of this, $2x + 3x^2 - 1$, vanishes only at $x = -1$ and $x = \frac{1}{3}$, and looking at the second derivative, $6x + 2$, shows that $x = -1$ is a local maximum and $x = \frac{1}{3}$ a local minimum. The corresponding values for y can be obtained from the equation $y = x^2 - 1$, and are $y = 0$ and $y = -\frac{8}{9}$ respectively. The value of f at these points is $f(-1, 0) = 1$ and $f\left(\frac{1}{3}, -\frac{8}{9}\right) = -\frac{5}{27}$. We also need to consider the value of the function at the other endpoint $f(1, 0) = 1$.

On the other piece of parabola $y = 1 - x^2$, where $-1 \leqslant x \leqslant 1$, the function $f(x, y) = x^2 + xy$ equals $x^2 - x^3 + x$, with $-1 \leqslant x \leqslant 1$. The first derivative of this, $2x - 3x^2 + 1$, vanishes only at

$x = 1$ and $x = -\frac{1}{3}$, and looking at the second derivative, $-6x + 2$, shows that $x = 1$ is a local maximum and $x = -\frac{1}{3}$ a local minimum. The corresponding values for y can be obtained from the equation $y = 1 - x^2$, and are $y = 0$ and $y = \frac{8}{9}$ respectively. The value of f at these points is $f(1, 0) = 1$ and $f\left(-\frac{1}{3}, \frac{8}{9}\right) = -\frac{5}{27}$. We also need to consider the value of the function at the other endpoint $f(-1, 0) = 1$.

By comparing the values of f at all the points we found, we see that the absolute minimum value of the function is $-\frac{5}{27}$, attained at $\left(\frac{1}{3}, -\frac{8}{9}\right)$ and $\left(-\frac{1}{3}, \frac{8}{9}\right)$, and the absolute maximum value of the function is 1, attained at $(1, 0)$ and $(-1, 0)$.

11. Suppose the sum of three positive real numbers is 9. What is the maximum possible value of their product?

Solution. Call the three positive numbers x, y, z. We want to maximize $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x + y + z = 9$. We have

$$\nabla f = \langle yz,\ xz,\ xy\rangle,$$

$$\nabla g = \langle 1,\ 1,\ 1\rangle,$$

The Lagrange multiplier equations are

$$yz = \lambda, \quad xz = \lambda, \quad xy = \lambda,$$

and so $yz = xz = xy$. Since these are positive numbers, we get $x = y = z$. Their sum is 9, so $x = y = z = 3$, and the maximum possible value of the product is $3 \cdot 3 \cdot 3 = 27$.