Problem 1 (15 points) Consider the following two lines

$$L_1: \ x = 1 + t, \ y = 1 - t, \ z = 3t$$
$$L_2: \ x = 7 - 2s, \ y = 4 - s, \ z = 15 - 5s$$

(a) Find the point of intersection of L_1 and L_2.

L_1 and L_2 meet for values of t and s such that

$$1+t = 7-2s \quad (1)$$
$$1-t = 4-s \quad (2)$$
$$3t = 15-5s \quad (3)$$

(1) + (2): $\quad 2 = 11 - 3s \Rightarrow \boxed{s=3}$

$1-t = 4-s = 1 \Rightarrow \boxed{t=0}$

. $t=0$, $s=3$ solves (3). $\quad 3(0) = 15 - 5(3)$ ✓

Point of intersection: $(1+0, 1-0, 3(0)) = \underline{(1,1,0)}$

(b) Find cosine of the *acute* angle between L_1 and L_2.

Direction vectors of L_1 and L_2: $\vec{V}_1 = \langle 1, -1, 3\rangle \quad \vec{V}_2 = \langle -2, -1, -5\rangle$

$\vec{V}_1 \cdot \vec{V}_2 = 1(-2) + (-1)(-1) + 3(-5) = -16$

$|\vec{V}_1| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$

$|\vec{V}_2| = \sqrt{4+1+25} = \sqrt{30}$

$$\Rightarrow \boxed{\cos(\theta) = \frac{16}{\sqrt{11}\sqrt{30}}}$$

cosine of an acute angle is positive.

(c) Find the equation of the plane containing L_1 and L_2.

Normal vector $= \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -2 & -1 & -5 \end{vmatrix} = 8\hat{i} - \hat{j} - 3\hat{k}$

Point on the plane $= (1,1,0)$

Equation of the plane: $\quad 8(x-1) - (y-1) - 3(z-0) = 0$

$$\equiv \boxed{8x - y - 3z = 7}$$

Problem 2 (15 points)

(a) Prove that there is no vector $\vec{v}$ such that

$$\langle 1, 2, 1\rangle \times \vec{v} = \langle 3, 1, 5\rangle$$

Since $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, if there were such a $\vec{v}$, $\langle 3,1,5\rangle$ would be orthogonal to $\langle 1,2,1\rangle$ which is not true

Since $\langle 1,2,1\rangle \cdot \langle 3,1,5\rangle = 10 \neq 0$.

(b) Prove that the following three vectors are coplanar

$$\hat{\imath} + 5\hat{\jmath} - 2\hat{k} \qquad 3\hat{\imath} - \hat{\jmath} \qquad 5\hat{\imath} + 9\hat{\jmath} - 4\hat{k}$$

Volume of the parallelopiped formed by these vectors $= \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix}$

$= 1(4) - 5(-12) - 2(27 - (-5))$

$= 4 + 60 - 64 = 0$

Hence they are coplanar

(c) For a non-zero vector $\vec{v}$ and a vector $\vec{u}$ prove that $\mathrm{Proj}_{\vec{v}}(\vec{u}) \bullet \vec{v} = \vec{u} \bullet \vec{v}$.

$$\mathrm{Proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

$$\Rightarrow \mathrm{Proj}_{\vec{v}}(\vec{u})\cdot\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\cdot\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}|\vec{v}|^2$$

$$= \vec{u}\cdot\vec{v}$$

Problem 3 (20 points) Consider the line $L : \vec{r} = \langle t, -t, 2t \rangle$ and the plane $\mathbb{P} : x+y+2z = 8$.

(a) Find the point of intersection of L and $\mathbb{P}$.

L and $\mathbb{P}$ meet for the value of t such that $t+(-t)+2(2t)=8$

$\Rightarrow \boxed{t=2}$

Point of intersection $= (2,-2,4)$

(b) If $\vec{v}$ is the direction vector of L and $\vec{n}$ is the normal vector of $\mathbb{P}$, compute $\text{Orth}_{\vec{n}}(\vec{v})$. (recall $\text{Orth}_{\vec{v}}(\vec{u}) = \vec{u} - \text{Proj}_{\vec{v}}(\vec{u})$).

$\vec{v} = \langle 1,-1,2 \rangle \qquad \vec{n} = \langle 1, 1, 2 \rangle$

$$\text{Proj}_{\vec{n}}(\vec{v}) = \left(\frac{\vec{n}\cdot\vec{v}}{|\vec{n}|^2}\right)\vec{n} = \frac{4}{6}\langle 1,1,2 \rangle = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{4}{3} \right\rangle$$

$$\text{Orth}_{\vec{n}}(\vec{v}) = \langle 1,-1,2 \rangle - \left\langle \frac{2}{3}, \frac{2}{3}, \frac{4}{3} \right\rangle = \left\langle \frac{1}{3}, \frac{-5}{3}, \frac{2}{3} \right\rangle$$

(c) Write the symmetric equations of the line passing through the point of part (a) parallel to the vector of part (b) (called the shadow of L in $\mathbb{P}$).

Point on the line $= (2,-2,4)$

Direction vector = any vector parallel to $\left\langle \frac{1}{3}, \frac{-5}{3}, \frac{2}{3} \right\rangle$

$= \langle 1,-5,2 \rangle$

Equations :
$$\frac{x-2}{1} = \frac{y-(-2)}{-5} = \frac{z-4}{2}$$

Problem 4 (15 points) True/False. You do not have to justify your answer. Please answer just T or F to each of the following.

(a) $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

F

(b) If $\vec{v} \bullet \vec{v} = 0$ then $\vec{v} = \vec{0}$.

T

(c) There is a unique line parallel to a given plane and passing through a given point.

F

(d) $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \bullet \vec{b}$.

T

(e) A pair of planes in $\mathbb{R}^3$ is either parallel or meets in a unique line.

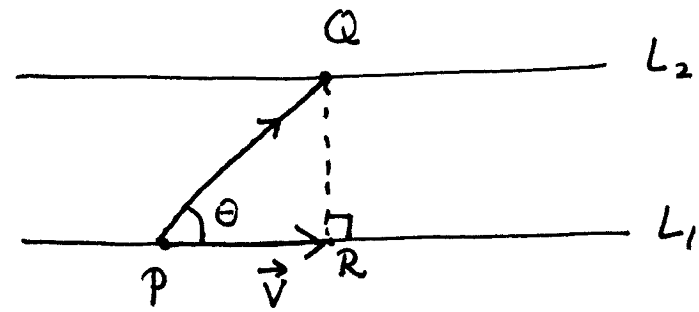
T

Problem 5 (20 points)

(a) Let L_1 and L_2 be two parallel lines with direction vector $\vec{v}$ and passing through points P and Q respectively. Prove that the distance between L_1 and L_2 is given by:

$$\text{Distance between } L_1 \text{ and } L_2 = \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}$$

$$\begin{aligned}
\text{Distance} &= \text{length of } \vec{QR} \\
&= |\vec{PQ}| \sin\theta \\
&= \frac{|\vec{PQ}| \cdot |\vec{v}| \sin\theta}{|\vec{v}|} \\
&= \frac{|\vec{PQ} \times \vec{v}|}{|\vec{v}|}
\end{aligned}$$



(b) Use the formula above to find the distance between

$$L_1: \ x = 5 + 2t, \ y = 3 - 2t, \ z = 1 + t$$

$$L_2: \frac{x-3}{2} = -\frac{y}{2} = z + 1$$

$\vec{v} = \langle 2, -2, 1 \rangle$ $\qquad P = (5, 3, 1) \qquad Q = (3, 0, -1)$

$$\vec{PQ} = \langle 3-5, 0-3, -1-1 \rangle = \langle -2, -3, -2 \rangle$$

$$\vec{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -3 & -2 \\ 2 & -2 & 1 \end{vmatrix} = -7\hat{i} - 2\hat{j} + 10\hat{k}$$

$$|\vec{PQ} \times \vec{v}| = \sqrt{49 + 4 + 100} = \sqrt{153}$$

$$|\vec{v}| = \sqrt{4 + 4 + 1} = 3$$

$$\text{Distance} = \frac{\sqrt{153}}{3} = \sqrt{17}$$

Problem 6 (15 points) Consider the equation $x^2 + 4y^2 = a + z^2$.

(a) Sketch x and z traces of the surface, for $a = 1$.

x-traces

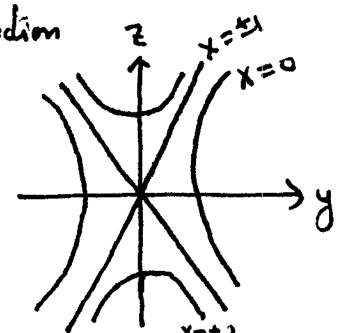
$x = 0$: $4y^2 - z^2 = 1$

hyperbola opening in y-direction

$x = \pm 1$: $4y^2 - z^2 = 0$

pair of lines

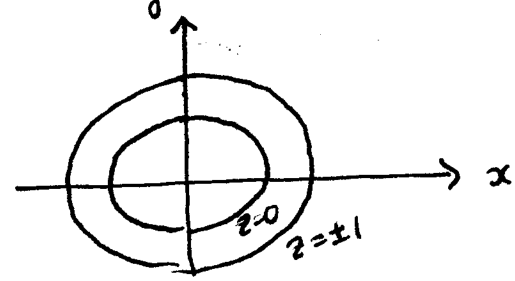
$x = \pm 2$: $z^2 - 4y^2 = 3$



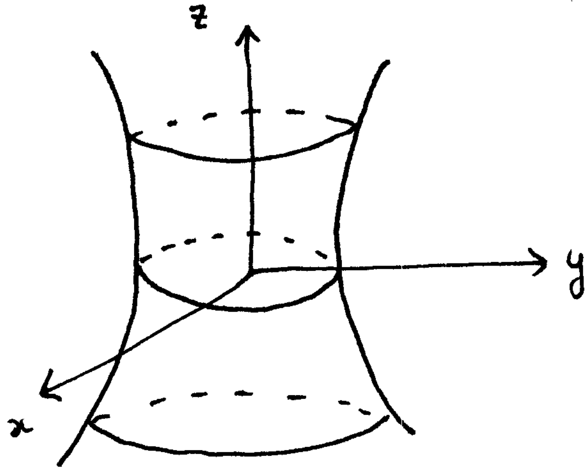
z-traces: $x^2 + 4y^2 = 1$ for $z = 0$

$x^2 + 4y^2 = 2$ for $z = \pm 1$

(ellipses)

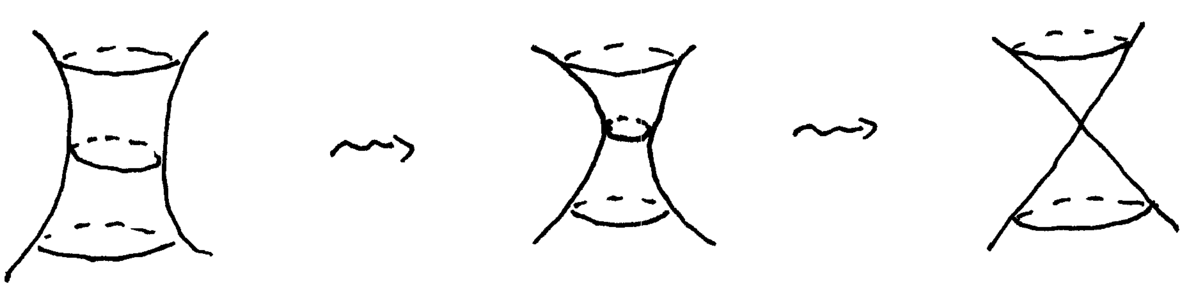


(b) Use the traces of part (a) to sketch the surface (for $a = 1$).



(c) **Bonus: 5 points** Describe how the surface changes when $a \to 0^+$.

The waist of the hourglass shrinks to a point until the hyperboloid degenerates to a cone



$a = 0$

$a = 1$