There are many techniques for finding a limit of the form

$$\lim_{(x,y)\to(a,b)} f(x,y).$$

When confronted with a limit problem that you don't immediately know how to solve, I suggest you try the following approaches, in the order listed. Note that there is usually not just one way to find a limit or show it does not exist, so different people may have use different (correct) methods to reach the same solution.

Most of these methods are basically trial and error: for instance you need to find the right paths to show the limit does not exist or find the right bounds to apply the Squeeze Theorem. That makes limit problems difficult! The best way to learn how to take limits is to try A LOT of problems and CHECK YOUR SOLUTIONS! If you are not checking your solutions, you may be making the same mistake again and again and you'll never know. The goal of this handout is to provide you with more problems with solutions to try. I strongly suggest you try each example on your own before looking at the solution provided.

Method 1: Direct substitution

Simply plug in (a,b) for (x,y), that is, find $f(a,b)$. If $f(a,b)$ is defined, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Note: Since this does not often work, it is easy to forget about this method. But it can save a lot of time and effort, so I suggest you always try it first!

Example:

$$\lim_{(x,y)\to(1,2)} \frac{xy^2}{x^2+y^2}.$$

Solution:

$$\lim_{(x,y)\to(1,2)} \frac{xy^2}{x^2+y^2} = \frac{1\cdot 4}{1+4} = \frac{4}{5}.$$

Method 2: Algebraic manipulation

Two of the most common techniques are factoring and multiplying by the conjugate of the denominator (or numerator) over itself.

Examples:

Determine the limit or show that it does not exist.

(i)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + x - y}{x - y}.$$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + x - y}{x - y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+1)}{x-y} = \lim_{(x,y)\to(0,0)} x + 1 = 1$$

(ii)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - y^2}{x - \sqrt{y}}$$

Solution:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2y - y^2}{x - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{x^2y - y^2}{x - \sqrt{y}} \cdot \frac{x + \sqrt{y}}{x + \sqrt{y}} \\ &= \lim_{(x,y)\to(0,0)} \frac{y(x^2 - y)(x + \sqrt{y})}{x^2 - y} \\ &= \lim_{(x,y)\to(0,0)} y(x - \sqrt{y}) \\ &= 0\end{aligned}$$

Method 3: Prove the limit DNE

Try to prove the limit does not exist by evaluating the limit along different paths to (a, b). If you find two paths to (a, b) which have different limits, then the limit does not exist. Most of our limits have are of the form $(x, y) \to (0, 0)$, in which case I suggest you try the following paths (in this order):

(a) $x = 0$ and $y = 0$

(b) $y = mx$

(c) $x = y^2$ and/or $y = x^2$ (especially if this allows you to simplify the algebra)

Examples:

Determine the limit or show that it does not exist.

(i)

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}.$$

Solution: We first evaluate the limit along the path $x = 0$ (this is the path that approaches $(0,0)$ along the y–axis):

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \bigg|_{x=0} \frac{x^4 - 4y^2}{x^2 + 2y^2} &= \lim_{y\to 0} \frac{4y^2}{2y^2} \\ &= \lim_{y\to 0} \frac{4}{2} \\ &= 2.\end{aligned}$$

We next evaluate the limit along the path $y = 0$ (approaching $(0,0)$ along the x–axis):

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \bigg|_{y=0} \frac{x^4 - 4y^2}{x^2 + 2y^2} &= \lim_{x\to 0} \frac{x^4}{x^2} \\ &= \lim_{x\to 0} x^2 \\ &= 0.\end{aligned}$$

Since $2 \neq 0$, the limit does not exist.

(ii)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

Solution: We first evaluate the limit along the path $y = x$:

$$\lim_{(x,y)\to(0,0)} \bigg|_{y=x} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^3}{x^4 + x^2} = \lim_{x\to 0} \frac{x^3}{x^2(x^2+1)} = \lim_{x\to 0} \frac{x}{x^2+1} = 0.$$

We next evaluate the limit along the path $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \bigg|_{y=x^2} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since $\frac{1}{2} \neq 0$, the limit does not exist.[1]

[1]For the first path, one could use $x = 0$, $y = 0$, or even $y = mx$ for any m instead; all of these will lead to a limit of 0. One indication that $y = x^2$ is a good path to try for this problem is that it makes the denominator just one term. For example, if the denominator was $x^6 + y^2$, then $y = x^3$ would be a good path to choose – this does not always work, but it does give you somewhere to start!

Method 4: Try the Squeeze Theorem

This can be tricky, as you need to find the right comparison functions. Most of the time, this works best if you first take the absolute value of the function, i.e., consider $|f(x,y)|$ instead. Remember that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists if and only if $\lim_{(x,y)\to(a,b)} |f(x,y)|$, and when they exist they are equal, so it's ok to consider the absolute value instead. The following are common inequalities used when applying the Squeeze Theorem.

(a) $-1 \le \sin\theta, \cos\theta \le 1$, or $0 \le |\sin\theta|, |\cos\theta| \le 1$

(b) $0 \le \frac{|y|}{\sqrt{x^2+y^2}} \le 1$[2]

(c) $0 \le \frac{x^2}{x^2+y^2} \le 1$, $0 \le \frac{x^2+y^2}{x^2+2y^2} \le 1$, etc.[3]

Be careful with your inequalities. Note, for example, that $y \not\le y^2$ when $0 < y < 1$, so it is NOT TRUE that $0 \le \frac{|y|}{x^2+y^2} \le 1$ (you can check that this inequality is false by plugging in $y = 1/2$ and $x = 0$, for example). It is also NOT TRUE that $0 \le \frac{y^2}{y^2+x^3} \le 1$ (try $y = 1$, $x = -\frac{1}{2}$).

You can NEVER use the Squeeze Theorem to conclude that a limit does not exist. If you try to use the Squeeze Theorem and the limits of the right-hand and left-hand side of the inequality are NOT equal, then this simply means that your application of the Squeeze Theorem was inconclusive. You can try to bound $f(x,y)$ in a different way and see if the Squeeze Theorem will be conclusive, or you can try a different method.

Examples:

Determine the limit or show that it does not exist.

(i)

$$\lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right)$$

Solution: Use that $-1 \le \sin\left(\frac{1}{x^2+|y|}\right) \le 1$. Then,

$$\lim_{(x,y)\to(0,0)} -x^4 \le \lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right) \le \lim_{(x,y)\to(0,0)} x^4$$
$$0 \le \lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right) \le 0,$$

[2] This is true because $|y| = \sqrt{y^2} \le \sqrt{x^2+y^2}$.

[3] These are true, because (for the first one) $x^2 \le x^2 + y^2$ (because $y^2 \ge 0$). Similarly, for the second, $y^2 \le 2y^2$ because both sides of the inequality are non-negative (this is really important–note that $y \not\le 2y$ when y is negative) and $x^2 \ge 0$, so $x^2 + y^2 \le x^2 + 2y^2$.

and so

$$\lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right) = 0$$

by the Squeeze Theorem.

(ii)

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2}$$

Solution: First take the absolute value, and then use that $0 \leq \frac{x^2}{x^2+y^2} \leq 1$:

$$\left|\frac{4x^2y}{x^2+y^2}\right| = 4|y| \cdot \frac{x^2}{x^2+y^2},$$

and

$$\begin{aligned}\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} 4|y| \cdot \frac{x^2}{x^2+y^2} &\leq \lim_{(x,y)\to(0,0)} 4|y| \\ 0 \leq \lim_{(x,y)\to(0,0)} 4|y| \cdot \frac{x^2}{x^2+y^2} &\leq 0.\end{aligned}$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4x^2y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \left|\frac{4x^2y}{x^2+y^2}\right| = 0$$

by the Squeeze Theorem.

Method 5: Try polar coordinates

Notice that $(x, y) \to (0, 0)$ in Cartesian coordinates is equivalent to $r \to 0$ in polar coordinates. This method is particularly useful if $f(x, y)$ has terms like $x^2 + y^2$, or if the numerator and/or denominator of $f(x, y)$ is not a polynomial.

Examples:

Determine the limit or show that it does not exist.

(i)

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

Solution:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} &= \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} \\
&= \lim_{r\to 0} \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2} \\
&= \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) \\
&= 0
\end{aligned}$$

For the final step, we can use the Squeeze Theorem: $-2 \le \cos^3\theta + \sin^3\theta \le 2$, so

$$\begin{aligned}
\lim_{r\to 0} -2r \le \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) \le \lim_{r\to 0} 2r \\
0 \le \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) \le 0,
\end{aligned}$$

and therefore $\lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$.

(ii)

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1.$$ [4]

(iii)

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

Solution:

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0} r^2\ln(r^2) = \lim_{r\to 0} 2r^2 \ln r$$

$$= \lim_{r\to 0} \frac{2\ln r}{r^{-2}} \overset{\text{L'H}}{=} \lim_{r\to 0} \frac{\frac{2}{r}}{-2r^{-3}} = \lim_{r\to 0} -r^2 = 0.$$

Note that we could use L'Hôpital's Rule because $\frac{2\ln r}{r^{-2}}$ is indeterminate of the form $\frac{\infty}{\infty}$.

[4] Recall that $\lim_{t\to 0} \frac{\sin t}{t} = 1$.