1. Suppose $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set in $\mathbb{R}^n$. Show that $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$ is also linearly independent.

2. Let M be the 3×3 matrix such that, for any $3 \times n$ matrix A, MA is the result of adding two times the first row of A to each of the second and third rows. Find M and M^{100}.

3. Assume A and B are square matrices of the same size. Use determinants to show that if AB is invertible, then A and B are both invertible.

4. Give an example of a matrix that is invertible but not diagonalizable.

5. Give an example of an $m \times n$ matrix A with $m > n$ and a left inverse of A, i.e., a matrix B such that $BA = I_n$. Can A also have a right inverse? Explain.

6. A matrix A is *symmetric* if $A = A^T$. If A is an $n \times n$ invertible symmetric matrix, is A^{-1} necessarily symmetric?

7. Are two row-equivalent matrices necessarily similar? (By row equivalent, I mean that one can be row-reduced to the other.) If so, prove it. If not, give a counterexample.

8. Compute the determinants of the following three matrices. Explain your answer. (Note: I do not expect you to do this directly for any of these matrices, though you can. Look for shortcuts!)

$$A = \begin{pmatrix} 7 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} -8 & 2 & 2 & 2 & 2 \\ -5 & 5 & 0 & 0 & 0 \\ -5 & 0 & 5 & 0 & 0 \\ -5 & 0 & 0 & 5 & 0 \\ -5 & 0 & 0 & 0 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 15 & 2 & 2 & 2 & 2 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

9. Suppose that A, B, C, D are invertible $n \times n$ matrices such that $A^T B C^T = D$. Express B^T in terms of A, C, and D.

10. Let V be the set of 4×4 matrices with all row- and column-sums equal to zero.

    (a) Show that V is a subspace of $M_{4,4}(\mathbb{R})$, the set of 4×4 matrices with entries in $\mathbb{R}$.

    (b) Find $\dim(V)$.

11. Consider the following four functions in $\mathcal{C}(\mathbb{R})$:

$$f(x) = \sin^2(x), \quad g(x) = \cos^2(x), \quad h(x) = \cos(2x), \quad k(x) = \sin(2x).$$

Are they linearly independent? If so, prove it. If not, express one of them as a linear combination of the others.

12. (a) Find numbers $w, x, y, z \in \mathbb{R}$ such that the characteristic polynomial of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ w & x & y & z \end{pmatrix}$$

is $(\lambda - 1)^4$.

(b) Is the resulting matrix diagonalizable? Why or why not?

13. Find a 3×3 matrix X such that

$$X^2 = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}.$$

How many such matrices are there?

14. Let $V = \operatorname{span}\{e^x, xe^x, x^2e^x\}$.

(a) Find the matrix of the linear transformation $D\colon V \to V$ defined by differentiation, with respect to the given basis of V.

(b) Find all function $f(x) \in V$ that are eigenvectors of D, with their corresponding eigenvalues.

15. Prove that if W_1, W_2 are subspaces of a vector space V, then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

16. Let $T : M_{2,3}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ be defined by

$$T\left(\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\right) = \begin{pmatrix} 2a - b & c + b \\ 0 & 0 \end{pmatrix}.$$

(a) Prove T is a linear transformation.

(b) Find $\dim(\ker(T))$ and $\dim(\mathrm{Image}(T))$.

(c) Is T injective? Is T surjective? Explain.

17. Let $\mathcal{A}, \mathcal{B}$ be two different bases for a vector space V. Prove that the change of basis matrix, $P_{\mathcal{A} \leftarrow \mathcal{B}}$ is invertible.

18. Prove that if $\text{Col}(A) = \text{Nul}(A)$, then A is a square matrix of even size.

19. Let A be an $n \times n$ matrix such that $\text{tr}(A^2) = -1$. Is A diagonalizable as a matrix in $M_{n\times n}(\mathbb{R})$? Why or why not?

20. An $n \times n$ matrix is called *nilpotent* if A^k is the zero matrix for some positive integer k. (For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent.)

    (a) If λ is an eigenvalue of a nilpotent matrix A, show that $\lambda = 0$.

    (b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix.

    (c) Let A be the matrix that represents $T\colon \mathcal{P}_5 \to \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $T(p) = \frac{dp}{dx}$. Without doing any computations, explain why A must be nilpotent.

21. If a matrix A is diagonalizable, show that for any scalar c, so is the matrix $A + cI$.

22. Let A be a square matrix. If the eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$ have distinct eigenvalues, show that these vectors are linearly independent.

23. Find a 2×2 real matrix A that has an eigenvalue $\lambda_1 = 1$ with eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and an eigenvalue $\lambda_2 = -1$ with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then compute the determinant of $A^1 0 + A$.

24. Two matrices, A and B, can be *simultaneously diagonalized* if there is an invertible matrix P that diagonalizes both of them. If A and B can be simultaneously diagonalized, show that $AB = BA$.

25. Let

$$A = \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & -6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{pmatrix}.$$

    Find bases for the nullspace, the column space, and the row space of A, and for the nullspace of A^T.