9/3 - Fourier Anal.

Tao Vu - Additive Combinatorics

Notes: - Andrew Granville

- Sound's Notes (Soundararajan)

- Gowers

- Tao - Arithmetic Ramsey Theory (background); Ramsey Theory, Graham-Rothschild-Spencer

Overview:

Def: X is a set. An r-coloring of X is a map $\chi: X \to [r]$, where $[r] = \{1, 2, \ldots, r\}$. This induces a partition of X (the level sets), into r classes

Thm (Schur): If we r-color the natural numbers $\mathbb{N}$, then we have a monochromatic solution to the eqn $x + y = z$.

(ie, one of the classes of the partitions will have a sol'n)

This thm follows from

Thm: If we r-color the edges of the complete graph K_N, & N is sufficiently large in terms of r, then we have a monochromatic triangle (ie. a complete graph on 3-vertices).

- A graph is a set V of vertices and E of edges, where $e = xy$ for $x, y \in V$, $x \neq y$. The complete graph (on n vertices) is V w/ $\hat{E}$ $|V| = n$ & E is all possible edges.

Thm: (Weak Ramsey Thm): Same statement w/ k replacing 3, & $N > N_0(r, k)$.

The analogue of this for $\mathbb{N}$ replaces complete graphs by intervals of the form $[N]$.

- What does it mean to have a monochromatic copy of $[k]$ inside $[N]$?

The notion is "homothetic copy," which is something of the form $\{x + \lambda, x + 2\lambda, \ldots, x + k\lambda\}$ (translate of a dilation) - length k, gap size λ.

Thm (van der Waerden): If N is sufficiently large in terms of k & r, then any r-coloring of $[N]$ contains a monochromatic arithmetic progression of length k (i.e. a homothetic copy of $[k]$). [$k=2$: Pigeonhole Principle]

- Which one?

3-term a.p.'s (arith prog's) : [3 pts equally spaced.]

$\overset{\cdot}{x} \xrightarrow{h} \overset{\cdot}{y} \xrightarrow{h} \overset{\cdot}{z} \Rightarrow x+z=2y$, i.e. can partition $\mathbb{N}$ s.t. one set contains a sol'n to this lin. eqn - similar to prev. thm, but have $2y$.

- For $x+y=z$, odd's don't have sol'n, but odd #'s have density $1/2$.
- If coeff's don't sum to zero, then residue class of mod p that doesn't contain a sol'n. Prev. thm, $1+1-1\neq 0$. Here, $1+1-2=0$.

Density: Given $A\subseteq\mathbb{N}$, the natural density of A is $\lim_{N\to\infty}\frac{|A\cap[N]|}{N}$.
But this limit rarely exists. The upper density of A is $\bar{\delta}(A)=\limsup_{N\to\infty}\frac{|A\cap[N]|}{N}$, which always exists.

- If $\bar{\delta}(A)>0$, we'd expect a 3-term arith. prog.

Thm (Roth): If $A\subseteq\mathbb{N}$ is a set of positive upper density ($\bar{\delta}(A)>0$), then A contains a 3-term a.p.

3 proofs:

(1) Fourier Analysis
(2) Δ-removal lemma (pure combinatorics)
(3) Ergotic Theory. (Owen)

Thm (Szemeredi): Same as Roth, but for k-term a.p's.
All 3 proofs above have been generalized to prove this.
We'll do Gower's proof for $k=4$ (proof very long for arb. k)
(sumset of A (abel. gp) is $2A=A+A=\{a+a' \mid a,a'\in A\}$. $|A|^2$ sums. If $A+A$ contains approx same # of elt's as A, then $2A$ is an almost closed subset, i.e. a subset of a subgp of A. If $A=\mathbb{Z}$, subset of $k\mathbb{Z}$, i.e. an arith. prog. centered at 0.)

<u>Green-Tao Thm</u>: The primes contain arbitrarily long a.p.'s.

-proved using Gower's pf for 4-term a.p.'s.

If $A \subseteq$ Primes, there's a notion of <u>relative</u> upper <u>density</u> $\displaystyle \limsup_{N\to\infty} \frac{|A \cap [N]|}{\pi(N)}$

& can reduce it to Szemeredi Thm.

$\pi(N)$ ← # of primes $\le N$.

9/5 - Four. Anal.

Start w/ a linear eqn $l(x) = \sum_{i=1}^{n} a_i x_i = 0$, where $a_i \in \mathbb{Z}$, $a_i \neq 0$, & $x \in \mathbb{Z}^n$.

Given $A \subseteq [-N,N] \subseteq \mathbb{Z}$ & $|A| = \delta N$, how big do we think $\{x \in A^n \mid l(x) = 0\}$ is?

- Given a sol'n $x = (x_1, \dots, x_n) \in [-N,N]^n$ w/ $l(x) = 0$, how likely is it that $x \in A^n$? δ^n (likelihood of each $x_i \in A$ is δ)
- So E.V. of is $\delta^n \cdot (\#$ of sol'ns in $A^n) = \delta^n \cdot |\{x \in [-N,N]^n \mid l(x) = 0\}|$, where $|\{x \in [-N,N]^n \mid l(x)=0\}| \approx N^{n-1}$

Given A, we want to evaluate $\quad = \delta^n \cdot N^{n-1}$

$$\sum_{x \in \mathbb{Z}^n} 1_A(x_1) \cdots 1_A(x_n) 1_H(x)$$

, where $1_\cdot$ denotes a characteristic fcn & $H = \{x \in \mathbb{Z}^n \mid l(x) = 0\}$.

Recall:

(X, μ) is a σ-finite measure space.

- $L^p(X)$ norms: $\|f\|_{L^p(X)}^p = \int |f|^p \, d\mu$
- $L^p(X)$ is a Banach sp (i.e. complete)
- $L^2(X)$ is a Hilbert sp (w/ inner prod.)

$$\langle f, g \rangle_{L^2(X)} = \int f \bar{g} \, d\mu$$

- $|\langle f, g \rangle_{L^2(X)}| \leq \|f\|_{L^2(X)} \|g\|_{L^2(X)}$ (Cauchy-Schwartz)
- With $1 = \frac{1}{p_1} + \cdots + \frac{1}{p_n}$, $\int |f_1 \cdots f_n| \, d\mu \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(X)}$ (Hölder's Ineq)
- Log convexity of norms: If $s < t$ & $\frac{1}{p} = \frac{\theta}{s} + \frac{1-\theta}{t}$, $\theta \in (0,1)$ ($\Rightarrow p \in (s,t)$) then $\|f\|_{L^p(X)} \leq \|f\|_{L^s(X)}^{\theta} \cdot \|f\|_{L^t(X)}^{1-\theta}$

We need this on $\mathbb{Z}$ & on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

$\mathbb{Z}$ gets the counting measure:

$$\int_{\mathbb{Z}} \phi(x)\, d\mu(x) = \sum_{x\in\mathbb{Z}} \phi(x)$$

$\mathbb{T}$ gets the Lebesgue measure from $\mathbb{R}$.

- $e(z) = e^{2\pi i z}$
- We define the Fourier transform of $\phi: \mathbb{Z} \to \mathbb{C}$ by

$$\hat{\phi}(\xi) = \sum_{x\in\mathbb{Z}} \phi(x) e(x\xi), \quad \xi \in \mathbb{T}$$ (if it's real, then $e(x\xi)$ is period 1, so $\xi \in \mathbb{T}$)

- We have that $\phi \to \hat{\phi}$ is an isometry btwn $L^2(\mathbb{Z})$ & $L^2(\mathbb{T})$:
  - Plancherel: $\langle \phi, \psi \rangle_{L^2(\mathbb{Z})} = \langle \hat{\phi}, \hat{\psi} \rangle_{L^2(\mathbb{T})}$
- We have the Fourier Inversion formula..

$$\phi(y) = \int_{\mathbb{T}} \hat{\phi}(\xi) e(-y\xi)\, dy$$

  - This holds b/c $\int_0^1 e(x\xi)\, d\xi = \begin{cases} 1 & x = 0 \\ 0 & \text{else} \end{cases}$

Back to our sum.

$$T(\phi_1, \dots, \phi_n) = \sum_{x\in\mathbb{Z}^n} \phi_1(x_1) \cdots \phi_n(x_n) 1_H(x)$$. We want to introduce the Fourier tr.

$$1_H(x) = \int_0^1 e(\ell(x)\xi)\, d\xi$$

We can put these together: $= \sum_{x_1,\dots,x_n}$ sum splits b/c x_i are ind. for some choice of coeff's

$$T(\phi_1, \dots, \phi_n) = \int_0^1 \sum_{x\in\mathbb{Z}^n} \phi_1(x_1) \cdots \phi_n(x_n) e((a_1x_1 + \cdots + a_nx_n)\xi)\, d\xi$$

$$= \int_0^1 \hat{\phi}_1(a_1\xi) \cdots \hat{\phi}_n(a_n\xi)\, d\xi$$

<u>Prop</u>: Let $n \geq 3$, & $|\phi_1|, \dots, |\phi_n| \leq 1$, & $\phi_1, \dots, \phi_n$ are supported on $[-N, N]$. Then, $|T(\phi_1, \dots, \phi_n)| \lesssim \|\hat{\phi}_1\|_{L^\infty(\mathbb{T})} N^{n-2}$ where the implied constants never depend on N.

Assume this for the moment. For a given set $A \subseteq [-N, N]$, define the <u>balanced fcn of A</u> by $f_A = 1_A - \delta 1_{[-N,N]}$ where $|A| = \delta(2N+1)$.

density of A

$\sum f_A = 0$

Note that $f_A + \delta 1_{[-N,N]} = 1_A$.

So, look at

$T(1_A, \dots, 1_A) = T(f_A + \delta, \dots, f_A + \delta)$. We multiply this out, & we have the term (main) & a bunch of error terms

b/c multilinear expression

$$T(\delta \cdot 1_{[-N,N]}, \dots, \delta \cdot 1_{[-N,N]}) = \delta^N T(1_{[-N,N]}, \dots, 1_{[-N,N]})$$

There are $2^n - 1$ other terms, which all look like

$T(g_1, \dots, g_n)$, where each g_i is δ or f_A, & at least one is f_A.

If we have $\|\hat{f}_A\|_{L^\infty(\mathbb{T})} \lesssim \delta^n \cdot N$, then $T(1_A, \dots, 1_A) \gtrsim \delta^n N^{n-1}$

the smallest it is is gotten from subtr. the upper bound of the error terms.

We say A is ε-pseudorandom if $\|\hat{f}_A\|_{L^\infty(\mathbb{T})} \le \varepsilon N$

Pf of Prop:

$$\int \hat{\Phi}_1(a_1 \xi) \cdots \hat{\Phi}_n(a_n \xi)\, d\xi = T(\Phi_1, \dots, \Phi_n)$$

By Hölder (w/ $1 = \frac{1}{\infty} + \frac{1}{n-1} + \cdots + \frac{1}{n-1}$)

$$|T| \le \|\hat{\Phi}_1\|_{L^\infty(\mathbb{T})} \prod_{i=2}^{n} \|\hat{\Phi}_i\|_{L^{n-1}(\mathbb{T})}$$

$$\|\hat{\Phi}_i\|_{L^2} = \|\Phi\|_{L^2} \le N^{1/2}$$

$$\|\hat{\Phi}_i\|_{L^\infty} \le \sum |\Phi(x)| \le N$$

So we get $\|\hat{\Phi}_i\|_{L^{n-1}(\mathbb{T})} \le N^{1 - 1/(n-1)}$. $\square$

9/8 – F.A

Recall: $l(x)$ a linear form $\sum_{i=1}^{n} a_i x_i$ in the n variables $x=(x_1,\dots,x_n)$, nonzero integral coeffs. The set H is the collection of $x \in \mathbb{Z}^n$ w/ $l(x)=0$. We looked at the expression

$$T(\phi_1,\dots,\phi_n) = \sum_{x\in\mathbb{Z}^n} \phi_1(x_1)\cdots\phi_n(x_n)\mathbb{1}_H(x)$$

for $\phi_1,\dots,\phi_n : \underset{\subseteq \mathbb{Z}}{[-N,N]} \longrightarrow [-1,1] \subseteq \mathbb{R}$

We had a Fourier expression:

$$T(\phi_1,\dots,\phi_n) = \int_{\mathbb{T}} \hat{\phi}_1(a_1\xi)\cdots\hat{\phi}_n(a_n\xi)\,d\xi .$$

Prop: With the above notation & conditions, we have

(1) $|T(\phi_1,\dots,\phi_n)| \le \|\hat{\phi}_1\|_{L^\infty(\mathbb{T})} \left\{\prod_{i=2}^{n} \|\hat{\phi}_i\|_{L^{n-1}(\mathbb{T})}\right\} \sim N^{n-2}$

and (2) $\|\hat{\phi}_i\|_{L^{n-1}(\mathbb{T})} \le C_n N^{1-1/(n-1)}$ for each i.

Pf: (1) follows from Hölder's ineq. (generalized)

(2) follows from interpolating the L^p norms:

$$\|\hat{\phi}_i\|_{L^2}^2 = \sum_{x\in\mathbb{Z}} |\phi_i(x)|^2 \le 2N+1$$

$$\|\hat{\phi}_i\|_{L^\infty} \le \sum_{x\in\mathbb{Z}} |\phi_i(x)e(x\xi)| \le \sum_{x\in\mathbb{Z}} |\phi_i(x)| \le 2N+1$$

So interpolation gives $\|\hat{\phi}\|_{L^p} \le C_n N^{1-1/p}$ $\square$

For a set $A \subseteq [-N,N]$ w/ $\delta(2N+1)$ elements, we defined our balance fcn $f_A(x) = \mathbb{1}_A(x) - \delta\mathbb{1}_{[-N,N]}(x)$

Then, $T(\mathbb{1}_A,\dots,\mathbb{1}_A) = T(f_A+\delta\mathbb{1}_{[-N,N]},\dots,f_A+\delta\mathbb{1}_{[-N,N]})$

$$= \delta^n T(\mathbb{1}_{[-N,N]},\dots,\mathbb{1}_{[-N,N]}) + \underbrace{E}_{\text{error terms}}$$

– each E is some particular T, where at least one ϕ_i is f_A.

E is a sum of $2^n - 1$ terms, each of which can be bdd, by prop, namely, each one has bound

$$\|\hat{f}_A\|_{L^\infty(\mathbb{T})} \cdot C_n N^{n-2}$$

So $|E| \le C_n \|\hat{f}_A\|_{L^\infty(\mathbb{T})} N^{n-2}$.

Note: $T(\mathbb{1}_{[-N,N]}, \ldots, \mathbb{1}_{[-N,N]}) \ge C_{n,a_1,\ldots,a_n} N^{n-1}$

Which gives: $T(\mathbb{1}_A, \ldots, \mathbb{1}_A) \ge C_{n,a_1,\ldots,a_n} N^{n-1} \delta^n + C_n(\|\hat{f}_A\|_{L^\infty(\mathbb{T})} N^{n-2})$

A) (Pseudorandom case) If $\|\hat{f}_A\|_{L^\infty(\mathbb{T})}$ is at most $\frac{C_{n,a_1,\ldots,a_n}}{C_n} \cdot \frac{\delta^n}{2} \cdot N$,

then $T(\mathbb{1}_A, \ldots, \mathbb{1}_A) \ge \frac{C_{n,a_1,\ldots,a_n}}{2} \delta^n N^{n-1}$

(so many solutions coming from A, as $N \to \infty$)

B) ? (Case we need to worry about)

<u>Finite Fields</u>

$\mathbb{F}_p$ is the finite field w/ p elts, p prime.
$\mathbb{F}_p^d$ is the d-dim'l vector sp over $\mathbb{F}_p$.

The Fourier transform over $\mathbb{F}_p^d$ is:

$\phi : \mathbb{F}_p^d \to \mathbb{C}$

$$\hat{\phi}(\xi) = \mathop{\mathbb{E}}_{x \in \mathbb{F}_p^d} \phi(x) e\left(\frac{x \cdot \xi}{p}\right) \quad \text{where} \quad \mathop{\mathbb{E}}_{x \in \mathbb{F}_p^d} = \frac{1}{p^d} \sum_{x \in \mathbb{F}_p^d} \quad \text{(expectation notation)}$$

& $x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d$. $\left(e(z) = e^{2\pi i z}\right)$

The inversion formula is

$$\sum_{\xi \in \mathbb{F}_p^d} \hat{\phi}(\xi) e\left(\frac{-x \cdot \xi}{p}\right) = \phi(x)$$

Consider $A \subseteq \mathbb{F}_p^d$ w/ δp^d elts, & $f_A^{(x)} = \mathbb{1}_A(x) - \delta \mathbb{1}_{\mathbb{F}_p^d}(x)$.

So,

- $\mathbb{E}_{x \in \mathbb{F}_p^d} f_A(x) = 0$

- If we have $\|\hat{f}_A\|_{L^\infty(\mathbb{F}_p^d)} > \varepsilon$, then what do we get?

$\mathbb{F}_p^d$ w/ counting measure (not normalized)

$\rightsquigarrow$ one choice of ξ that makes our sum large, i.e.

$$\left| \mathbb{E}_{x \in \mathbb{F}_p^d} f_A(x) e(x \cdot \xi / p) \right| \geq \varepsilon$$

Since $x \cdot \xi \in \mathbb{F}_p$, we can split sum along those values. Set $W_t = \{x \mid x \cdot \xi = t\}$ for each $t \in \mathbb{F}_p$. The W_t are cosets of W_0, a true subsp. This gives:

p^{d-1} elts each

$$\left| \mathbb{E}_{t \in \mathbb{F}_p} \left(\mathbb{E}_{x \in W_t} f_A(x) \right) e(t/p) \right| > \varepsilon$$

So $\varepsilon < \mathbb{E}_{t \in \mathbb{F}_p} \left| \mathbb{E}_{x \in W_t} f_A(x) \right|$. Thus there is at least one t w/

$\left| \mathbb{E}_{x \in W_t} f_A(x) \right| \geq \varepsilon$ (b/c it's an average, of sorts)

If there were no absolute value, then $\mathbb{E}_{x \in W_t} (\mathbb{1}_A(x) - \delta \mathbb{1}(x)) \geq \varepsilon$, or

$\mathbb{E}_{x \in W_t} \mathbb{1}_A(x) \geq \varepsilon + \delta$. So this particular coset of a subsp has increased density, in comparison to W_0, of elts in A, by fixed ε.

We have $\mathbb{E}_{x \in \mathbb{F}_p^d} f_A(x) = 0 \implies \mathbb{E}_{t \in \mathbb{F}_p} \Big(\underbrace{\left| \mathbb{E}_{x \in W_t} f(x) \right|}_{|y|} + \underbrace{\mathbb{E}_{x \in W_t} f(x)}_{y} \Big) \geq \varepsilon$, &

$y + |y| = 2 \max\{0, y\}$. Since we get a positive value above, we can lose the abs. val. & add $\varepsilon/2$, from the factor of 2.

9/10 - F.A

Finite Field version of Roth's thm

We consider the eqn $x+y=2z$ (in 3 vars), where $x,y,z\in\mathbb{F}_p^d$ $(p>2)$. Over $\mathbb{F}_p^d$ this is one eqn. Over $\mathbb{F}_p$, this is d eqns (linear - in $3d$ variables).

We have

$$H=\{(x,y,z)\mid x+y=2z\},\quad |H|=p^{2d}$$

← $3d$ vars, H is codim d, b/c d eqns.

For $A\subset\mathbb{F}_p^d$, we have $|A|=\delta p^d$. We want to estimate

$$\mathop{\mathbb{E}}_{x,y,z\in\mathbb{F}_p^d}\mathbb{1}_A(x)\mathbb{1}_A(y)\mathbb{1}_A(z)\mathbb{1}_H(x,y,z)$$
$$=\left(\frac{1}{p^d}\right)^3\sum_{x,y,z\in\mathbb{F}_p^d}\mathbb{1}_A(x)\mathbb{1}_A(y)\mathbb{1}_A(z)\mathbb{1}_H(x,y,z)$$

↑ one normalizing elt for each of x,y,z

- to guess its size, replace $\mathbb{1}_A$ w/ its expectation, which is its density, so δ, & replace $\mathbb{1}_H$ w/ # of elts, p^{2d}

So we expect $\approx p^{-3d}\delta^3p^{2d}$ → we'll multiply by p^d (const.) to expect δ^3.

For general fcns ϕ_1,ϕ_2,ϕ_3 on $\mathbb{F}_p^d$, we have

$$\mathop{\mathbb{E}}_{x,y,z\in\mathbb{F}_p^d}\phi_1(x)\phi_2(y)\phi_3(z)(\mathbb{1}_H(x,y,z)\cdot p^d)$$
$$=\sum_{\xi\in\mathbb{F}_p^d}\hat{\phi}_1(\xi)\hat{\phi}_2(\xi)\hat{\phi}_3(-2\xi),$$

← coeffs from linear eqn

b/c $p^d\mathbb{1}_H(x,y,z)=\sum_{\xi\in\mathbb{F}_p^d}e\left[(x+y-2z)\cdot\frac{\xi}{p}\right]$ (as in integral case)

Given $A\subseteq\mathbb{F}_p^d$ w/ $|A|=\delta p^d$, we set $f_A=\mathbb{1}_A-\delta$. Then we have

$$\mathop{\mathbb{E}}_{x,y,z\in\mathbb{F}_p^d}\mathbb{1}_A(x)\mathbb{1}_A(y)\mathbb{1}_A(z)p^d\mathbb{1}_H(x,y,z)=\mathop{\mathbb{E}}_{x,y,z\in\mathbb{F}_p^d}(f_A+\delta)(x)(f_A+\delta)(y)(f_A+\delta)(z)p^d\mathbb{1}_H(x,y,z)$$

This gives our main term: $\underbrace{p^d\mathop{\mathbb{E}}_{x,y,z\in\mathbb{F}_p^d}}_{p^{-3d}}\delta^3\cdot\underbrace{\mathbb{1}_H(x,y,z)}_{p^{2d}}=\delta^3$

↑ just counting the pts in H

↑ from normalization

and error terms which involve a balanced fcn (f_A)

Consider one of these:

$$E_1 = p^d \cdot \mathop{\mathbb{E}}_{x,y,z \in \mathbb{F}_p^d} f_A(x) f_A(y)\, \delta \cdot \mathbb{1}_H(x,y,z)$$

$$E_1 = \sum_{\xi \in \mathbb{F}_p^d} \hat{f}_A(\xi)\, \hat{f}_A(\xi)\, \hat{\delta}(-2\xi)$$

Cauchy-Schwartz

$$|E_1| \le \underbrace{\sup_{\xi} |\hat{f}_A(\xi)|}_{L^\infty \text{ norm}} \cdot \Big(\sum_{\xi} |\hat{f}_A(\xi)|^2\Big)^{1/2} \Big(\underbrace{\sum_{\xi} |\hat{\delta}(-2\xi)|^2}_{(\delta^2)}\Big)$$

} Parseval

$(\delta^2) < \delta$ b/c $\delta < 1$

(δ^2) b/c δ const. fcn. & $\hat{\delta}$ has factor of $1/p^d$

$$\sum_{\xi} |\hat{f}_A(\xi)|^2 = \mathop{\mathbb{E}}_{x \in \mathbb{F}_p^d} |f_A(x)|^2$$
$$= \frac{1}{p^d} \cdot \left(|A|(1-\delta)^2 + |A^c|\,\delta^2\right)$$
$$= (1-\delta)\delta < \delta$$

So $|E_1| \le$ above $\le \sup_{\xi} |\hat{f}_A(\xi)| \cdot \delta$. Same for all 7 error terms.

Combining these gives

main term

$$\text{error} = \left| \mathop{\mathbb{E}}_{x,y,z \in \mathbb{F}_p^d} \mathbb{1}_A(x)\, \mathbb{1}_A(y)\, \mathbb{1}_A(z)\, \mathbb{1}_H(x,y,z) - \delta^3 \right| \le 7 \cdot \|\hat{f}\|_{L^\infty} \cdot \delta .$$

We want the error term to be less than the main term, which will happen if $\|\hat{f}_A\|_{L^\infty} \le \delta^2/14$, & then we have the # of 3-term arith. progressions in A:

$$\mathop{\mathbb{E}}_{x,y,z \in \mathbb{F}_p^d} \mathbb{1}_A(x)\, \mathbb{1}_A(y)\, \mathbb{1}_A(z)\, \mathbb{1}_H(x,y,z) \ge \delta^3/2$$

There are trivial sol'ns we don't care about - like $x=y=z \to$ the trivial progression. There's a trivial prog. for each pt in A. So if $\underbrace{\text{this}}_{\hookrightarrow \delta^3/2}$ is larger than $\underbrace{p^{-d}}_{\text{really } \delta p^{-d}}$ then there is a nontriv. prog. in A

really δp^{-d}, but the δ doesn't contribute.

If we have $\|\hat{f}_A\|_{L^\infty} \ge \delta^2/14$, we have a subsp W_0 w/ some coset $W_t = W_0 + t$ w/ $\dfrac{|A \cap W_t|}{|W_t|} \ge \delta + \dfrac{\delta^2}{28}$ (from last time), i.e. a coset w/ increased density - so let's look here for a sol'n.

We project $A \cap W_t$ onto W_0 & call this set A'.

$x+y=2z \Leftrightarrow (x+t)+(y+t)=2(z+t)$, i.e. translation invariance.

(Thus $\exists$ sol'n in $A \Leftrightarrow \exists$ sol'n in A')

So 3-A.P.'s in A' provide 3-A.P's in $A \cap W_t$ & so in A.

set: $A \subset \mathbb{F}_p^d \Rightarrow A' \subset \mathbb{F}_p^{d-1}$ (b/c subsp.), so we want to iterate this argument.

density: $\delta \Rightarrow \delta + c\delta^2$ $[c = 1/28]$

$$\Rightarrow A^{(j)} \subseteq \mathbb{F}_p^{d-j}$$

$$\text{density} \geq \delta + jc\delta^2$$

This stops in at most $\approx \delta^{-2}$ steps, b/c density ≤ 1 (if d suff. large in terms of δ).

If $d \approx \delta^{-2}$, then we get a 3-term A.P.

Let $p^d = N$ $\quad d \approx \log N$, so if $|A| \geq \frac{N}{(\log N)^{1/2}}$, we get 3-A.P.'s

can be improved to get rid of the $1/2$.

9/12 - F.A.

Today: Sketch out Roth's thm on $\mathbb{Z}$.

Over $\mathbb{F}_p^d$, we had 2 cases:

(1) Uniform case: $\|\hat{f}_A\|_\infty$ small
In this case, we had many 3-term A.P.'s. } already done on $\mathbb{Z}$

(2) Increased density on a coset (Density Increment)
(a large set)

Recall: Set $\ell(x) = \sum_{i=1}^n a_i x_i$, $a_i \in \mathbb{Z}\setminus\{0\}$, & $\sum_{i=1}^n a_i = 0$ (implies translation invariance of sol'ns)
Set $H = \{x \mid \ell(x) = 0\}$. We have from before that

$$\sum_{x\in[-N,N]^n} \mathbb{1}_A(x_1)\cdots\mathbb{1}_A(x_n)\mathbb{1}_H(x) = \delta^n \sum_{x\in[-N,N]^n} \mathbb{1}_H(x) + E,$$

$$|E| \le (2^n - 1)\|\hat{f}_A\|_{L^\infty(\mathbb{T})}\,\delta^{n-2} N^{n-2}$$ [If E small compared to main term, we have many sol'ns]

We have $\sum_{x\in[-N,N]^n} \mathbb{1}_H(x) \ge cN^{n-1}$, where c depends on $a_1,\dots,a_n$ & n.

If $\|\hat{f}_A\|_{L^\infty} \le \frac{c}{2}\delta^2 N$, we have many sol'ns as $N\to\infty$ [Uniform Case]

Otherwise? $\hat{f}_A(\xi) = \sum_{x\in\mathbb{Z}} \left(\mathbb{1}_A(x) - \delta\mathbb{1}_{[-N,N]}\right) e(x\xi)$

We expect to have a density increment on a set where $e(x\xi) \approx$ const.
Problem: We don't understand what these sets are (unlike in $\mathbb{F}_p$ case)

Lemma: Let $\xi\in\mathbb{T}$ & $\varepsilon>0$ be given. If $N \ge N_0(\varepsilon)$, then we have a partition of $[-N,N]$ as $P_1\cup\cdots\cup P_J\cup E$ where the P_i's are A.P.'s, $|P_i| = |P_j| \ge N^{1/4}$, & $|E| \le \varepsilon(2N+1)$ [the "error set"], & most importantly, for $x,y\in P_i$, $|e(x\xi) - e(y\xi)| < \varepsilon$, $\forall i \le J$.

Pf: Set $\varepsilon \ge N^{-1/12}$, $L = N^{1/3}$, $M = N^{1/4}$. We find integers $(a,q)=1$, $1 \le a \le q \le L$, s.t. $|\xi - a/q| \le \frac{1}{q\cdot L}$ (by Dirichlet's approx. thm)

Consider $P^* = \{0, q, 2q, \dots, q(M-1)\}$. On P^* we have (i.e. for $x \in P^*$)

$$e(x\xi) \underset{\text{b/c in } P^*}{=} e(qy\xi) = e(q\beta y), \text{ where } \beta = \xi - a/q, \text{ for some } y \le M$$

$q\xi = q\beta + \underbrace{a}_{\in\mathbb{Z}}$ in exponential → 1

So, $|e(x\xi)-1| = |e(yq\beta)-1| \leq \frac{M}{L}$ $\quad (q\beta < \frac{1}{L}$ up to int$)$

b/c $|e^x - 1| \leq x$ $\quad = N^{-1/12} \leq \varepsilon.$

Note: Our partition relies on our choice of ξ

We have $\bigcup_{0\leq s\leq q-1} (P^* + s) = [0, qM)$ $(\subseteq \mathbb{Z})$. This interval can tile $[-N,N]$,

[so this is split into 0 mod q, 1 mod q, etc. $\quad q \leq L$

with at most some exceptional set E w/ $|E| \leq qM. \leq N^{7/12} \leq \varepsilon^5 N.$

We can then let the P_i be translates of P^*

On the same translate, we have $|e((x+t)\xi) - e((y+t)\xi)| = |e(t\xi)(e(x\xi) - e(y\xi))|$
(i.e. $x+t, y+t \in P^* + t$)
$= |e(x\xi) - e(y\xi)|$

So character is effectively constant.

So we are done. $\square$

Density Increment

We assume $|A| = \delta(2N+1)$ & we have some ξ w/ $\left|\sum_{x\in[-N,N]} f_A(x)\, e(x\xi)\right| \geq \eta N$.

Since $\sum_{x\in[-N,N]} f_A(x) = 0$, we have $\left|\sum_{x\in[-N,N]} \left(f_A(x) e(x\xi) + f_A(x) \right)\right|$

$$\left|\sum_{i=1}^{J} \sum_{x\in P_i} f_A(x)\, e(x\xi)\right| \geq \eta N - \sum_{x\in E} |f_A(x)| \geq (\eta - \varepsilon)\cdot N$$, where ε as in lemma.

So $$\sum_{i=1}^{J} \left|\sum_{x\in P_i} f_A(x)\, e(x\xi)\right| \geq (\eta-\varepsilon)\cdot N$$

Now we add in $\sum_{i}^{J}\sum_{x\in P_i} f_A(x) + \underbrace{\sum_{x\in E} f_A(x)}_{\leq \varepsilon\cdot N} = 0$, so $\sum\sum f_A(x) \leq \varepsilon\cdot N$, & we get

$$\sum_i \left(\left|\sum_{x\in P_i} f_A(x)\, e(x\xi)\right| + \sum_{x\in P_i} f_A(x) \right) > (\eta - 2\varepsilon)\cdot N.$$ So $\exists$ some i w/

$$\sum_{x\in P_i} f_A(x)\, e(x\xi) - \left(\sum_{x\in P_i} f_A(x)\right) e(x_0\xi) \leq \varepsilon |P_i|$$

for any $x_0 \in P_i$, effectively constant, so can pull it out of $\sum$

So $$\sum_i \left|\sum_{x\in P_i} f_A(x) e(x\xi) + \sum_{x\in P_i} f(x)\right| \Rightarrow \sum_i \left(\left| e(x_0(i)\xi)\left(\sum_{x\in P_i} f_A(x)\right)\right| + \sum_{x\in P_i} f_A(x)\right) > (\eta - 3\varepsilon)N$$

As in $\mathbb{F}_p$ case, we have a gain of $\left(\frac{\eta - 3\varepsilon}{2}\right)|P_i|$, so $\exists$ some i w/ increased density.

9/15 - FA

Density increment step:

$A \subseteq [-N, N]$, $|A| = \delta(2N+1)$

$\|\hat{f}_A\|_{L^\infty(\mathbb{T})} \geq \eta N$.

Let ξ be some value w/ $|\hat{f}_A(\xi)| \geq \eta N$. We apply the decomposition $[-N,N] \longrightarrow (\bigcup_{j=1}^{J} P_j) \cup E$ with ε to be chosen later.

Recall: $|E| \leq \varepsilon N$, $|P_j| = |P_i| \geq N^{1/4}$, & $|e(x\xi) - e(y\xi)| \leq \varepsilon$ $\forall x, y \in P_i, \forall i$.

We have
$$\hat{f}_A(\xi) = \sum_{x \in [-N,N]} f_A(x) e(x\xi)$$
$$= \sum_{j=1}^{J} \Big(\sum_{x \in P_j} f_A(x) \Big) e(x^{(j)}\xi) \text{ , where } x^{(j)} \text{ is chosen arbitrarily in } P_j$$
$$+ \sum_{j=1}^{J} \Big(\sum_{x \in P_j} f_A(x) \big(e(x\xi) - e(x^{(j)}\xi) \big) \Big) \quad (1) \quad (\leq \varepsilon \cdot \#\text{ pts in } [-N,N])$$
$$+ \sum_{x \in E} f_A(x) e(x\xi) \qquad (2) \quad (\leq \text{size of } E)$$

(1) & (2) are each $\leq \varepsilon N'$ $(N' = 2N+1)$

This gives
$$\sum_{j=1}^{J} \Big| \sum_{x \in P_j} f_A(x) e(x^{(j)}\xi) \Big| \geq (\eta - 2\varepsilon) N'$$

Also, $\Big| \sum_j \sum_{x \in P_j} f_A(x) \Big| \leq \varepsilon N$ (b/c sum over whole int. is 0, & exceptional set is $\leq \varepsilon N$)

Putting this together, $\sum_{j=1}^{J} \Big(\Big| \sum_{x \in P_j} f_A(x) \Big| + \sum_{x \in P_j} f_A(x) \Big) \geq (\eta - 3\varepsilon) N'$

Pick $\varepsilon = \eta/6$, & we have $\sum_{x \in P_j} f_A(x) > \frac{\eta}{4} \frac{N'}{J}$ (b/c $\exists$ some j s.t. $\sum f_A(x) > 0$ & then lose factor of 2)

$\uparrow$ b/c average $[\,|y| + y = 2\max\{0, y\}\,]$

$\hookrightarrow$ like fin. field case

So, $\frac{N'}{J} \geq |P_j|$ (b/c $\frac{N' - |E|}{J} = |P_j|$) & so $\sum_{x \in P_j} \mathbb{1}_A(x) = (\delta + \frac{\eta}{4}) |P_j|$

RK $\cdot$ $N \geq \varepsilon^{-12}$ is needed to get $P_1, \dots, P_J$, so this holds if $N \geq (6/\eta)^{12}$

Trichotomy: One of these is true:

(1) $N \le c\delta^{-24}$

(2) A has $c\delta^n N^{n-1} - |A|$ non-trivial solutions to $l(x) = \sum_{i=1}^n a_i x_i = 0$ where $\sum_{i=1}^n a_i = 0$ & $n \geq 3$.

(3) There is a progression P of length at least $N^{1/4}$ where A has density $\delta + c\delta^2$ on P.

Assume (1) doesn't happen. So have (2) or (3). If (3), then focus on that progression & iterate: $A \to A \cap P$ & rescale P to $[-N', N']$, & the image of $A \cap P$ is our new set A'. Repeat. $\hookrightarrow N' = \frac{N^{1/4}}{2}$

We increase the density at each step, until (at worst) the density is 1 (& then done, b/c just finding sol'ns in interval). Thus, this can happen at most $c\delta^{-2}$ times. But we shrink the set considerably each time, so need N signicantly large:

As long as $\left(\frac{N}{2}\right)^{(1/4)^{c\delta^{-2}}} \geq c\delta^{-24}$, i.e. $\delta \geq \frac{c}{\sqrt{\log\log N}}$, then there is a nontrivial solution to $l(x)$ in A.

Roth: If $A \subseteq [N]$ has at least $\frac{cN}{\sqrt{\log\log N}}$ elements, where N is large enough, then A contains a nontrivial 3-term A.P..

Improvements (to δ)

(1) (Heath-Brown, Szemeredi) $\delta \geq \frac{c}{(\log N)^c}$

(2) (Bourgain) $\delta \geq c\left(\frac{\log\log N}{\log N}\right)^{1/2}$

(3) (— " —) $\delta \geq c \cdot \frac{(\log\log N)^2}{(\log N)^{2/3}}$

(4) (Sanders) $\delta \geq \frac{c \cdot (\log\log N)^5}{\log N}$

* In finite field case, $N = p^d$,
$\delta \geq c \cdot \frac{1}{\log N}$. (Meshulam)

For $p=3$ (Bateman-Katz)
$\delta \geq c \cdot \frac{1}{\log N^{1+\varepsilon}}$ [Special case $p=3$]

The other direction: Is there a large set w/ no A.P.'s?

- Random construction gives $A \subseteq [N]$ w/ $\sim N^{2/3}$ elts.

Consider $[0,M]^n$. Then $\exists$ one $r \in [0, nM^2]$ w/

$$|S_r| := \left|\left\{ x \in [0,M]^n \,\middle|\, \sum_{i=1}^n x_i^2 = r \right\}\right| \geq \frac{M^{n-2}}{n} \quad \text{(sphere of rad. } \sqrt{r}\text{)}$$

We map $S_r \xrightarrow{\Phi} [0,N]$ via $\Phi(x) = \Phi(x_1, \dots, x_n)$ = integer whose i^{th} digit in base L is x_i for each i, i.e.

$$\Phi(x) = \sum_{i=1}^n x_i L^{i-1}$$

Then $\Phi(S_r) \subseteq [0, L^n]$. Pick L to be $3M$, & we have

(i) Φ is injective

(ii) $\Phi(x) + \Phi(y) = 2\Phi(z) \Rightarrow x + y = 2z \Rightarrow x = y = z$ (b/c sphere has no arith. prog's, b/c sphere int. line at ≤ 2 pts)

How big is this set? We have $N = (3M)^n$ & $M = \frac{N^{1/n}}{3}$

$\Rightarrow$ the density is $\dfrac{N^{1-2/n}}{3^{n-2} \cdot n \cdot N} \Rightarrow e\left(-\frac{2}{n}\log(N)\right) - (\log(3))(n-2) + \cdots$

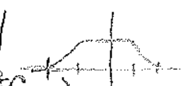
Want to pick n s.t. this is large $\rightarrow$ pick $n \simeq \sqrt{\log N}$

$\Rightarrow \; \geq e(-D\sqrt{\log(N)}) \geq N^{-\varepsilon} \;\; \forall \varepsilon$ & $\leq \dfrac{1}{\log(N)^A}$

9/17 - FA

- For Roth, we considered $\ell(x) = x_1 + x_3 - 2x_2 = 0$ (Vander Corput solves this over primes) [homogeneous & coeff's sum to 0] this was only nec. for incr. density argument?
- $\ell(x) = x_1 + x_2 + x_3 = \lambda$. for suff. large odd λ, Vinogradov solves this for x_1, x_2, x_3 prime. → i.e. $\exists$ 3-term AP's in the primes
- $\Lambda(x) = \begin{cases} \log(x), & x \text{ prime} \\ 0, & \text{else} \end{cases}$; $\Lambda_N = \Lambda \cdot \mathbb{1}_{[N]}$

Lemma (Vinogradov): For a given c, $\exists C$ s.t. we have:
Let Ψ be a smooth fcn on $\mathbb{R}$, $= 1$ on $[-1,1]$, & w/ $\operatorname{supp}(\Psi) = [-2,2]$, nonnegative. Set $\Psi_N(\xi) = \Psi(N \log(N)^{C} \xi)$. big



Define $F(a,q) = \frac{1}{\phi(q)} \sum_{s \in U_q} e(as/q)$, & $g_N(\xi) = \sum_{q \le (\log N)^C} \sum_{a \in U_q} F(a,q) \cdot \Psi_N(\xi - a/q)\, \underbrace{\widehat{\mathbb{1}}_{[N]}(\xi - a/q)}_{\text{Fourier transf}}$
Where ϕ is the totient fcn. &
U_q is the multiplicative gp of $\mathbb{Z}/q\mathbb{Z}$,
& we let $U_1 = \{0\}$.
Then,
$$\|\widehat{\Lambda}_N - g_N\|_{L^\infty(\mathbb{T})} \le \frac{N}{\log(N)^{c}} \leftarrow \text{little}$$
Pf: exercise!

Rk: $\cdot\ F(a,q) \approx \frac{\mu(q)}{\phi(q)}$ Mobius fcn of q, as is $\underbrace{\Psi_N(\xi - a/q)}_{\text{supp. only close to rat'ls w/ small denom, rel to } N.}$
- $\widehat{\mathbb{1}}_{[N]}(\xi - a/q)$ just a geom. seq.

We want to estimate $\sum_{x \in [N]^3} \Lambda(x_1)\Lambda(x_2)\Lambda(x_3) \mathbb{1}_{H_\lambda}(x) = R(\lambda, N)$,
where $H_\lambda = \{(x_1,x_2,x_3) \mid x_1 + x_2 + x_3 = \lambda\}$. $R(\lambda,N)$ is a weighted count of how many ways we can write λ as the sum of 3 primes.

$$R(x,N) = \int_{\mathbb{T}} \left(\hat{\Lambda}_N(\xi)\right)^3 e(-x\xi)\,d\xi$$

Write $\hat{\Lambda}_N = g_N + h_N$, ← plays role of balance fcn; just $\hat{\Lambda}_N - g_N$.

& substitute this in $R(x,N)$ to get main term w/ all g_N's & error terms w/ at least one h_N.

Main term: $\int g_N^3(\xi)\, e(-x\xi)\,d\xi$. To bound our error terms, we need L^p bounds for $\hat{\Lambda}_N$ & g_N (which gives one for h_N).

$$\|\hat{\Lambda}_N\|_{L^p(\mathbb{T})} = \log(N)\cdot \Big\| \underbrace{\left(\frac{\hat{\Lambda}_N}{\log(N)}\right)}_{\text{bdd}: \le 1} \Big\|_{L^p(\mathbb{T})} \le \log(N)\cdot \underbrace{N^{1-1/p}}_{\text{from interpolation}}$$

For g_N, we have to find both the L^∞ & L^2 norms, in order to interpolate. $\|g_N\|_{L^\infty} \le N$, b/c the terms of sum are disjointly supported:

<u>Rk]</u> The fcns $\Psi_N(\xi - a/q)$ have disjoint support.

Because: $|\xi - a/q| \le \frac{\log(N)^c}{N}$, & if 2 overlap w/ ξ in overlap, $\frac{a_1}{q_1} - \frac{a_2}{q_2} + \xi - \xi$, $q_1 q_2 \ge \frac{N}{\log(N)^c}$, one big.

$\Rightarrow$ sum is $\le \max\{L^\infty\text{-norms of each piece}\}$

$\le \{1, 1, N\}$

This immediately gives $\|g_N\|_{L^\infty} \le N$. Also these fcns are orthogonal in L^2

$$\text{For } \|g_N\|_{L^2}^2 = \int |g_N|^2 = \underbrace{\sum_q{}' \sum_a{}' |F(a,q)|^2}_{} \underbrace{\int_{\mathbb{T}} |\Psi_N(\xi - a/q)|^2 \left(\mathbb{1}_{[-N,N]}(\xi - a/q)\right)^2 d\xi}_{\text{not dep. on } a,q}$$

$$= \sum_{q \le (\log(N))^c}{}' \frac{1}{\phi(q)} \underset{\substack{\uparrow \\ \text{from } (*)}}{\lesssim} \log(N)^2 \qquad \le N \text{ b/c of } L^2\text{-norm of } \mathbb{1}$$

$$(*)\ \phi(q) \ge \frac{q}{\log(\log(q))}$$

Interpolation yields $\|g_N\|_{L^p(\mathbb{T})} \leq \log(N) N^{1-1/p}$.

Thus, $R(\lambda, N) = \int (g_N(\xi))^3 e(-\lambda\xi)\, d\xi + E(N)$, where

$E(N) \leq N^2 \cdot \frac{(\log(N))^2}{\log(N)^c}$. If pick $c > 2$, error term < main term

b/c supp of terms of g_N disj

$$\int (g_N(\xi))^3 e(-\lambda\xi)\, d\xi = \sum_q \sum_a (F(a,q))^3 \int (\Psi_N(\xi - a/q))^3 (\hat{\mathbb{1}}_{[N]}(\xi - a/q))^3 e(-\lambda\xi)\, d\xi$$

$(\tau = \xi - a/q)$

$$\underset{\approx}{\downarrow} \underbrace{\sum_q \sum_a (F(a,q))^3 e(-\lambda a/q)}_{\text{singular series}} \underbrace{\int_{|\tau| \leq N^{-1}\log(N)^c} (\hat{\mathbb{1}}_{[N]}(\tau))^3 e(-\lambda\tau)}_{(\#\text{ of sol'ns in } [N]^3) + O(N/\log(N)^c)}$$

geom. series concentrated very close to origin

$= \prod_p \mu_p$

\# of sol'ns (mod p) in $\mathcal{U}_p$

9/19 - FA

Consider a "surface" in $\mathbb{Z}^n$ (i.e. zero set of poly's), called S.
Set $S_n = S \cap [N]^n$. We want to estimate

$$\mathop{\mathbb{E}}_{x \in S_n} f_1(x_1) \cdots f_n(x_n), \text{ where } \mathop{\mathbb{E}}_{x\in S_n} = \frac{1}{|S_n|}\sum_{x \in S_N} \quad (f_i \geq 0 \text{ \& bdd above by } 1)$$

as we did in the setup for Roth. We'd like a particular norm that bounds (or "controls") these averages,

i.e. $$\mathop{\mathbb{E}}_{x\in S^n} f_1(x_1)\cdots f_n(x_n) \leq \min_i \|f_i\|_K :$$

In Roth's thm, $\|f\|_K = \|\hat{f}\|_{L^\infty(\mathbb{T})} N^{-1}$. Worked for a single eqn w/ at least 3 vars. Also with Vinogradov.
If we can come up w/ this bound, we can repeat Roth verbatim & come up w/ sol'ns in our set. Sim, can also repeat Vinogradov almost verbatim & get sol'n in primes.

Q: For a sys. of lin. eqns (homogeneous, lin. ind), can you find conditions on the sys. which makes S_N controlled by $\|\hat{\cdot}\|_{L^\infty(\mathbb{T})}$?
It is known that

$$\begin{aligned} x_1 + x_3 &= 2x_2 \\ x_2 + x_4 &= 2x_3 \end{aligned} \quad , \text{ an A.P. of length } 4,$$

is <u>not</u> controlled by $\|\hat{\cdot}\|_{L^\infty(\mathbb{T})}$.

Also, $$\begin{aligned} x_1 - x_2 + x_3 &= 4 \\ x_3 &= 2 \end{aligned}$$ reduces to Twin Primes conjecture, so <u>not</u> controlled by $\|\hat{\cdot}\|_{L^\infty(\mathbb{T})}$.

What other types of norms can we introduce so we can do these? i.e., control the averages on these surfaces?

Let S_N be def. by $x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2 + x_6^2 + x_7^2$.
Write $\mathbb{1}_{S_N}(x) = \int_{\mathbb{T}} e((x_1^2 + \cdots - x_7^2)\xi)\,d\xi$, as before, & we see sums of form

$$\sum_{x_1} f(x_1) e(x_1^2 \xi).$$

We define a norm as follows:

Set $P_2 = \{ p(x) \mid p \text{ a poly. in } x,\ \deg p \le 2, \text{ real coeffs} \}$

Define $\|f\|_{u^2(N)} = \sup_{\phi \in P_2} \left| N^{-1} \langle f, e(\phi) \rangle_{L^2([N])} \right|$

$\left(\sum_{x \in [N]} f(x) e(\phi(x)) \le \| 1_{[N]} f \|_{L^1} \text{, so finite} \right)$

<u>Notes</u>: • $\|f\|_{u^2(N)} = \| \widehat{1_{[N]} f} \|_{L^\infty(\mathbb{T})}$, i.e. over P_1

• $\|f\|_{u^2(N)} \le \|f\|_{u^3(N)}$, so get incr. seq. of norms.
$\uparrow$ b/c $P_2 \supseteq P_1$

• $\|\cdot\|_{u^3(N)}$ should control surfaces defined by quadratic eqns; & does, if the surface is "nice".

• Vinogradov's approximation

$\Lambda_N = G_N + H_N$, where $\|H_N\|_{u^2(N)} \lesssim \log(N)^{-c}$,

generalizes to the $\|\cdot\|_{u^d}$ norms $\forall d$.

But we're not interested in higher deg polys. We're interested in controlling $\begin{array}{l} x_1 + x_3 = 2x_2 \\ x_2 + x_4 = 2x_3 \end{array}$. To get a norm controlling this,

we'll embed the problem in $\mathbb{Z}/N\mathbb{Z}$, b/c it has a gp structure, while $[N]$ does not. Let $Z_N = \mathbb{Z}/N\mathbb{Z}$, & consider the averages in Z_N.

$T = \mathbb{E}_{x,h \in Z_N} f_1(x) f_2(x+h) f_3(x+2h)$, for a 3-term AP, w/ $|f_i| \le 1\ \forall i$

$\uparrow$ probability measure, so good for Cauchy-Schwartz.

$$T^2 = \Big(\mathbb{E}_x \Big(\mathbb{E}_h \underbrace{f(x)}_{\substack{\text{can pull} \\ \text{out b/c doesn't depend} \\ \text{on } h,\ \&\ \le 1}} f(x+h) f(x+2h) \Big) \Big)^2 \le \mathbb{E}_x \mathbb{E}_{h,h'} f(x+h) f(x+h') f(x+2h) f(x+2h')$$

$\uparrow$ from square, b/c get 2 sums

$$\overset{h' = h+k}{=} \mathbb{E}_x \mathbb{E}_{h,k} f(x+h) f(x+h+k) f(x+2h) f(x+2h+2k)$$

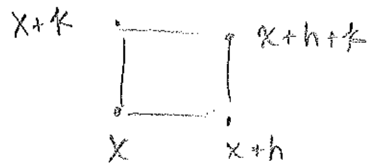
$$\overset{x \to x-h}{=} \mathbb{E}_x \mathbb{E}_{h,k} f(x) f(x+k) f(x+h) f(x+h+2k)$$

<u>Repeat</u>: $T^4 \leq \mathbb{E}_{x,h} f(x) f(x+h) \left(\mathbb{E}_{k} f(x+k) f(x+h+2k) \right)^2$

(after substitutions) $\leq \mathbb{E}_{x,h,k} f(x) f(x+h) f(x+k) f(x+k+h)$

$:\equiv \|f\|^4_{U^2(\mathbb{Z}_N)}$. This is actually a norm: b/c

$= \|\hat{f}\|^4_{L^4(\hat{\mathbb{Z}}_N)}$ (we may check this later)



} so like summing fcn over rectangles. Sometimes called averages over cubes.

<u>Note</u>: $\|f\|_{U^2(\mathbb{Z}_N)} = \| e(\phi) f \|_{U^2(\mathbb{Z}_N)}$, for ϕ a linear poly, as

b/c actually complex conjugation in formula.

$e(\phi(x+h+k) - \phi(x+h) - \phi(x+k) + \phi(x))$

is a 2^{nd} difference of a linear poly, so $=0$.

- We can repeat this process to control norms of AP's of longer lengths.

$$T = \mathbb{E}_{x,h \in \mathbb{Z}_N} f_1(x) f_2(x+h) f_3(x+2h)$$

Cauchy-Schwarz

$$T^2 \leq \mathbb{E}_{x,h,k} f_2(x+h) f_2(x+h+k) f_3(x+2h) f_3(x+2h+2k)$$

$$= \mathbb{E}_{x,h,k} f_2(x) f_2(x+k) \; f_3(x+h) f_3(x+h+2k)$$

Cauchy-Schwartz

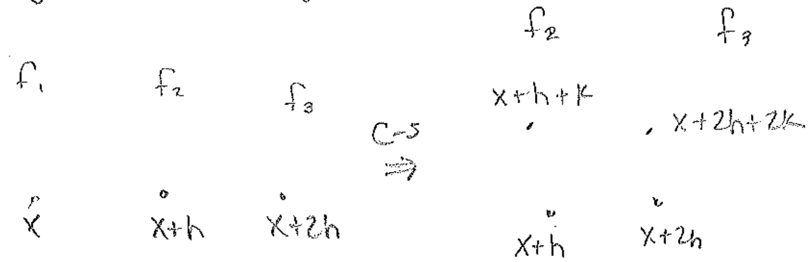
$$T^4 \leq \mathbb{E}_{x,h,k,l} f_3(x+h) f_3(x+h+2k) f_3(x+h+l) f_3(x+h+l+2k)$$

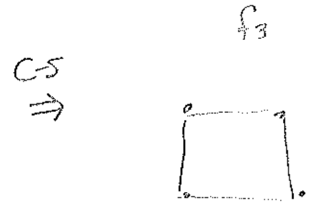
$$= \mathbb{E}_{x,k,l} f_3(x) f_3(x+2k) f_3(x+l) f_3(x+l+2k)$$

Make N odd, & then $2k \to k$.

$$T^4 \leq \mathbb{E}_{x,h,k} f_3(x) f(x+h) f_3(x+k) f_3(x+k+h) \;:=\; \|f\|^4_{U^2(\mathbb{Z}_N)}$$

Note: When you do C-S, you end up doubling one of the variables





(averaging over "cube" or parallelogram)

→ so averages over the ▱ controls the ave. over . . .

→ If had one more pt (4-term AP's), would do C-S one more time & 3-D cube would control the 4-term AP.

Note: If $f_i = \mathbb{1}_A, \forall i$, then this would count the # of 3-term AP's in A.

Gowers on Roth: If the balanced fcn of the set A $(\mathbb{1}_A - \delta)$ is small in the $U^2(\mathbb{Z}_n)$ norm, we will have the expected # of 3-term APs. $(\approx \delta^3 N^2)$

It is easy to check that $\|f\|_{U^2(\mathbb{Z}_N)} = \|\hat{f}\|_{L^4(\hat{\mathbb{Z}}_N)}$. Also easy to see that

$\|\hat{f}\|_{L^\infty(\hat{\mathbb{Z}}_N)} \leq \|f\|_{U^2(\mathbb{Z}_N)} \leq \|\hat{f}\|^2_{L^\infty(\hat{\mathbb{Z}}_N)}$

- If large in U^2 norm, large in L^∞ norm, so can use a density increment argument as in proof of Roth.

So;

If $\|f\|_{U^2(\mathbb{Z}_N)} \geq \varepsilon$, then $|\langle f, e(\phi)\rangle_{L^2(\mathbb{Z}_N)}| \geq \varepsilon^{1/2}$, for some linear fcn ϕ. w/ real coeffs.

$\uparrow \|\hat{f}\|_{L^\infty(\hat{\mathbb{Z}}_N)}$

This is an inverse statement for the U^2 norm.

i.e., for longer AP's

So if we want to do this w/o relying on Fourier transforms, we need to replace this inverse statement w/ smth - extra step for the Gowers norm. This gives a density increment argument.

<u>For k-term APs</u>:

(1) (Generalized von Neumann)

$$|\mathbb{E}\, f_1(x) f_2(x+h) \dots f_k(x+(k-1)h)| \leq \min_i \|f_i\|_{U^{k-1}(\mathbb{Z}_N)}$$

(2) Inverse theorem for U^k norms

(3) Density increment that follows from (2).

* don't need $\mathbb{Z}_N$, just gp w/ odd char.

Gowers Norm

($\overline{}$ = cx. conj)

The U^2 norm: $\|f\|^4_{U^2} = \mathbb{E}_{x,h,k} f(x)\overline{f(x+h)}\overline{f(x+k)} f(x+h+k)$ for any f.

Define $\Delta_h f(x) := \overline{f(x+h)} f(x)$, the <u>multiplicative derivative</u>,

& $\Delta_{h_1, \dots, h_m} f(x) := \Delta_{h_1}(\Delta_{h_2, \dots, h_m} f(x))$. [permuting the h_i's does not change the value]

So $\|f\|^4_{U^2} = \mathbb{E}_{x,h,k} \Delta_{h,k} f(x)$. Define inductively:

$\|f\|^{2^k}_{U^k} := \mathbb{E}_h \|\Delta_h f\|^{2^{k-1}}_{U^{k-1}}$. E.g. $\|f\|^8_{U^3} = \mathbb{E}_{x,h_1,h_2,h_3} f(x)\overline{f(x+h_1)}\,\overline{f(x+h_2)}\,\overline{f(x+h_3)}$
$f(x+h_1+h_2) f(x+h_2+h_3)$
$f(x+h_1+h_3)\overline{f(x+h_1+h_2+h_3)}$

(inputs form a combinatorial cube)

In general,

$$\|f\|_{U^k}^{2^k} = \mathbb{E}_{x,h_1,\dots,h_k} \Delta_{h_1,\dots,h_k} f(x) .$$

Also,

$$\left(\|f\|_{U^{k-1}}^{2^{k-1}}\right)^2 = \left(\mathbb{E}_{h_1,\dots,h_{k-1}}\mathbb{E}_{x} \Delta_{h_1,\dots,h_{k-1}} f(x)\right)^2 \quad \text{C-S w/ } x\text{-sum \& } 2^{nd} \text{ fcn} = 1$$

$$\le \mathbb{E}_{h_1,\dots,h_{k-1}} \mathbb{E}_{x} \mathbb{E}_{h_k} \left(\Delta_{h_1,\dots,h_{k-1}} f(x)\right)\left(\overline{\Delta_{h_1,\dots,h_{k-1}} f(x+h_k)}\right)$$

$$= \|f\|_{U^k}^{2^k}$$

$\Rightarrow$ $\|f\|_{U^{k-1}} \le \|f\|_{U^k}$, i.e, these form an increasing seq. of norms.

The U^1 norm would be

$\mathbb{E}_{x,h} f(x)\overline{f(x+h)}$, & these can be summed independently.

(U^1 norm of set $= \delta^2$)

This is actually a seminorm - b/c we could have a nonzero fcn w/ average $= 0$.

But we have $\|f\|_{U^1} \le \|f\|_{U^2} \le \cdots \le \|f\|_{U^k} \le \cdots$

since this is a norm, $\|f\|_{U^k}$ cannot be a seminorm for $k > 2$.

Thus, all we need to check to show $\|f\|_{U^k}$ is a norm is the Δ-inequality, for $k > 2$.

The Gowers-Cauchy-Schwartz ineq. :

$\mathcal{C}^{|\omega|} \Rightarrow$ cx conjugate if $|\omega| = 1$ & not if $|\omega| = 0$.

Define

$$\langle (f_\omega)_{\omega\in\{0,1\}^k}\rangle_{U^k(\mathbb{Z}_N)} = \mathbb{E}_{x,h_1,\dots,h_k} \prod_{\omega} \mathcal{C}^{|\omega|} f(x+\omega\cdot h)$$

where $\omega\cdot d = \omega_1 d_1 + \cdots + \omega_k d_k$, & $|\omega| = \sum_i \omega_i \pmod 2$

9/24 - FA

Gowers norms ($\|f\|_{U^k(\mathbb{Z}_n)}$)

Gowers Cauchy-Schwarz ineq - gives the Δ ineq.

$$\|f\|_{U^k(\mathbb{Z}_n)}^{2^k} = \mathop{\mathbb{E}}_{x,h_1,\dots,h_k} f(x)f(x+h_1)\cdots f(x+h_1+\cdots+h_k)$$

$$= \mathop{\mathbb{E}}_{x,h_1,\dots,h_k} \Delta_{h_1,\dots,h_k} f(x)$$

This is nonnegative:

$$|\mathop{\mathbb{E}}_{x} f(x)|^2 = \mathop{\mathbb{E}}_{h,x} \Delta_h f(x) \geq 0 \qquad [\Delta_h f(x) = \overline{f(x+h)}\, f(x)]$$

Square $\nearrow$: $\overset{C\text{-}S}{\leq} \mathop{\mathbb{E}}_{h_1,h_2,x} \Delta_{h_1,h_2} f(x)$, etc.

Recall, we have $\|f\|_{U^1} \leq \|f\|_{U^2} \leq \cdots \leq \|f\|_{U^k} \leq \cdots$

Δ-Inequality (we'll show it for U^2): w/ $\|f_i\|_{U^2} \leq 1$ (by nomalizing by the U^2 norm).

Gowers C-S : $\langle f_1, f_2, \dots, f_4\rangle_{U^2} := \mathop{\mathbb{E}}_{x,h_1,h_2} f_1(x) f_2(x+h_1) f_3(x+h_2) f_4(x+h_1+h_2)$

We want

$$|\langle f_1,f_2,f_3,f_4\rangle_{U^2}| \leq \prod_{i}^{4} \|f_i\|_{U^2} \quad \text{(similar to C-S for 2 fcns)}$$

This follows from

$$\langle f_1,\dots,f_4\rangle \leq 1 \text{ if } \|f_i\|_{U^2} \leq 1 \quad \forall i$$

$$|\langle f_1,\dots,f_4\rangle_{U^2}|^2 \overset{\text{C-S in } h_1:}{\leq} \mathop{\mathbb{E}}_{x,h_1,h_2} \mathop{\mathbb{E}}_{h_3} \Delta_{h_3} f_2(x+h_1)\, \Delta_{h_3} f_4(x+h_1+h_2)$$

h_1 sum an average, so it's redundant

$$= \mathop{\mathbb{E}}_{x,h_2} \mathop{\mathbb{E}}_{h_3} \Delta_{h_3} f_3(x)\, \Delta_{h_3} f_4(x+h_2).$$

$$|\langle f_1,\dots,f_4\rangle_{U^2}|^4 \overset{\text{C-S on } h_3}{\leq} \mathop{\mathbb{E}}_{x,h_2,h_3,h_4} \Delta_{h_4}\Delta_{h_3} f_4(x+h_2) = \mathop{\mathbb{E}}_{x,h_3,h_4} \Delta_{h_4}\Delta_{h_3} f_4(x) = \|f_4\|_{U^2}^4$$

We can do this for any f_i.

Consider

$$\|f+g\|_{U^2}^4 = \langle f+g, f+g, f+g, f+g \rangle_{U^2}$$
$$= \langle f,f,f,f\rangle_{U^2} + \langle f,f,f,g\rangle_{U^2} + \cdots + \langle g,g,g,f\rangle_{U^2} + \langle g,g,g,g\rangle_{U^2}$$
$$\leq \|f\|_{U^2}^4 + \|f\|_{U^2}\|f\|_{U^2}\|f\|_{U^2}\|g\|_{U^2} + \cdots + \|g\|_{U^2}^4$$
$$= (\|f\|_{U^2} + \|g\|_{U^2})^4 \checkmark$$

Thus $\|\cdot\|_{U^2}$ is actually a norm. This Δ-ineq. works identically for higher norms.

<u>The Generalized von Neumann Inequality</u>: Let $|f_1|, \ldots, |f_k| \leq 1$. We have, if $(N, (k-1)!) = 1$,

$$\left| \mathop{\mathbb{E}}_{x,h \in \mathbb{Z}_N} f_1(x) f_2(x+h) \cdots f_k(x+(k-1)h) \right| \leq \|f_i\|_{U^k(\mathbb{Z}_N)} \quad \text{for each } i.$$

<u>Pf</u>: We have

$$\left| \mathop{\mathbb{E}}_{x,h} f_1(x) \cdots f_k(x+(k-1)h) \right|^2 \overset{\text{c-s in } h}{\leq} \mathop{\mathbb{E}}_{x,h_1,h_2} \Delta_{h_2} f_2(x+h_1) \cdots \Delta_{h_2} f_k(x+(k-1)h_1)$$

$$= \mathop{\mathbb{E}}_{x,h_1,h_2} \Delta_{h_2} f_2(x) \cdots \Delta_{h_2} f_k(x+(k-2)h_1)$$

$$= \mathop{\mathbb{E}}_{h_2} \left(\mathop{\mathbb{E}}_{x,h_1} \Delta_{h_2} f_2(x) \Delta_{h_2} f_3(x+h) \cdots \Delta_{h_2} f_k(x+(k-2)h) \right)$$

$$\leq \underbrace{\mathop{\mathbb{E}}_{h_2} \|\Delta_{h_2} f_i\|_{U^{k-1}}}_{\text{induction hyp.}}$$

C-S 2^{k-1} times

$$T^2 \leq \mathop{\mathbb{E}}_{h_2} \|\Delta_{h_2} f_i\|_{U^{k-1}}$$
$$\vdots$$
$$T^{2^k} \leq \mathop{\mathbb{E}}_{h_2} \|\Delta_{h_2} f_i\|_{U^{k-1}}^{2^{k-1}} \Rightarrow T^{2^k} \leq \|f_i\|_{U^k}^{2^k} \Rightarrow T \leq \|f_i\|_{U^k}$$

Let D_h be the additive derivative:

$$D_h f(x) = f(x+h) - f(x)$$

Note: $\Delta_h e(\Phi(x)) = e(D_h \Phi(x))$, so

$\Delta_{h_1,\dots,h_k} e(\Phi(x)) = 1$ if Φ a poly w/ deg $\leq k$, for every value of x & h.

So our phase polys have Gowers norm 1.

9/26 - FA

- $\Delta_h \overbrace{e(\phi)}^{\text{phase poly}} = e(D_h \phi)$
  - $\|\phi\|_{U^k} = 1$ if ϕ a poly of deg $< k$, where a polynomial is any fcn for which $D_{h_1,\ldots,h_k}\Phi(x) \equiv 0$ $\forall x$. (def. of poly in finite abel. gp)

- For U^2, we have:
  - if $\|f\|_{U^2} > \varepsilon$, then $|\langle f, e(\phi)\rangle| > c(\varepsilon)$ for some linear poly ϕ.

- For U^3, we have:
  - if $\|f\|_{U^3} > \varepsilon$, then $|\langle f, e(\phi)\rangle| > c(\varepsilon)$ for some quadratic poly ϕ.

  → not true for a <u>real</u> quad. poly ϕ, only for some ϕ w/ our def. of poly above. (called a generalized poly.)

<u>Thm</u>: (Szemeredi) If $A \subseteq \mathbb{Z}$ w/ positive upper density, then A contains arb. long non-triv. AP's.

<u>Thm</u>: (Furstenberg) Let $k \geq 1$ be an integer, & $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. Given $E \subset X$ w/ $\mu(E) > 0$, there exists $r > 0$, an integer, s.t. $\mu(E \cap T^r E \cap \cdots T^{(k-1)r} E) > 0$.

<u>Note</u>: The powers of T are the A.P.

<u>Def</u>: A <u>measure-preserving system</u> is a pair $(X, \mathcal{B}, \mu)$ a probability space & $T : (X, \mathcal{B}, \mu) \circlearrowleft$ an isomorphism, meaning T is invertible; T & T^{-1} are measurable; & $\mu(T^n E) = \mu(E)$ $\forall E$ & n.

(Furstenberg) $\Leftrightarrow$ (Szemeredi)

Furstenberg Correspondence Principle gives the equality of these 2 thms.

<u>Multiple recurrence</u>: $E \cap T^r E \cap \cdots \cap T^{(k-1)r} E$

<u>Poincaré Recurrence</u>: Let $(X, \mathcal{B}, \mu, T)$ be a m.p.s, $E \subseteq X$ w/ $\mu(E) > 0$. Then

$$\limsup_{n \to \infty} \mu(E \cap T^n E) \geq \mu(E)^2$$

<u>Pf</u>:

$$\int \sum_{n=1}^{N} \mathbb{1}_{T^n(E)}\, d\mu = N \cdot \mu(E)$$

By C-S,
$$N^2 \mu(E)^2 \leq \int \Big(\sum_{n=1}^{N} \mathbb{1}_{T^n(E)} \Big)^2 d\mu$$
$$= \int \sum_{n,m=1}^{N} \mathbb{1}_{T^n E} \mathbb{1}_{T^m E}\, d\mu$$
$$= \sum_{n,m=1}^{N} \mu(T^n(E) \cap T^m(E))$$
$$= \sum_{n,m=1}^{N} \mu(T^{n-m}(E) \cap E) \quad \text{b/c meas-preserving}$$
$$\leq \Big[\sup_{1 \leq n \leq N} \mu(T^n(E) \cap E) \Big] \cdot N^2 + \underbrace{o(N^2)}_{\text{where } m=n,\ N\mu(E)}$$

- divide by N^2 & take limsups. □

Let $A \subseteq \mathbb{Z}$. Consider the set X of sequences $(x_i)_{i \in \mathbb{Z}}$ where $x_i = \begin{cases} 0 \\ 1 \end{cases}$. $A \longmapsto \mathbb{1}_A$ puts A "inside X." There is an operator S on X which takes (x_i) to (y_i), where $y_{i+1} = x_i$.

9/29 - FA (Owen Sizemore - notes)

Def: Let (X, Ω, μ) be a measure space. It is diffuse (non-atomic) if $\forall B \in \Omega$ s.t. $\mu(B) > 0$, then $\exists C \subseteq B$ s.t. $0 < \mu(C) < \mu(B)$.

ex: δ-measure non-atomic, Lebesgue meas. $=\lambda$ is.

Thm: Let (X, Ω, μ) be a diffuse probability space (i.e. $\mu(X) = 1$). Then $(X, \Omega, \mu) \cong ([0,1], \mathcal{B}, \lambda)$

$\mathcal{B}$ = Borel sets.

[So the gp of iso's of this space is huge]

Cor: $\mathrm{Aut}(X, \mu)$ is huge!!

$\mathrm{Aut}(X,\mu)$ = measure-preserving self maps

Def: Let Γ be a (countable) (discrete) gp. An action σ is a homomorphism $\sigma : \Gamma \to \mathrm{Aut}(X, \mu)$ $(\Gamma \curvearrowright X)$.

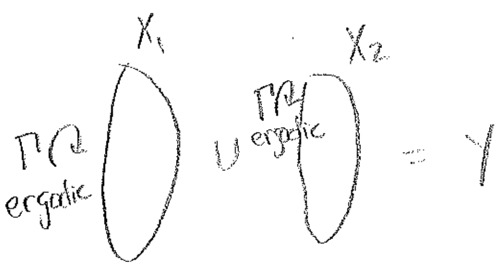
Bernoulli Shift: $(\nu(0) = \nu(1) = \frac{1}{2})$

- Let $Y = \{0,1\}$, ν = counting measure, $X = \prod_\Gamma Y$, $\mu = \prod_\Gamma \nu$

$\Gamma \overset{\sigma}{\curvearrowright} X$ by $\sigma_g((y)_{\gamma \in \Gamma}) = (y)_{g\gamma}$. This is a measure-pres automorphism of X, so $\Gamma \leq \mathrm{Aut}(X, \mu)$

Compact Actions: K is cpt, $X = K$, μ = Haar probability measure. $X \curvearrowright X$ by $\sigma_g(h) = gh$, so cpt gps $\leq \mathrm{Aut}(X, \mu)$

Def: Let $\Gamma \overset{\sigma}{\curvearrowright} X$. We say σ is ergodic if $A \subset X$ s.t. $\sigma_g(A) = A$ $\forall g \in \Gamma$, then $\mu(A) = 0$ or $\mu(A) = 1$.

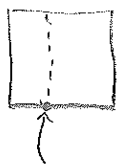


$\Gamma \curvearrowright Y$, but not ergodic.

Can we decompose Y into ergodic pieces? In general, no.

Thm: $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$. There exist $Y \subseteq X$, a map $y \longmapsto \nu_y$ that associates to every $y \in Y$ a Γ-invariant ergodic probability measure ν_y on X s.t. $\forall f: X \to \mathbb{C}$, the map $y \longmapsto \int_X f \, d\nu_y$ is measurable & $\forall f \in L^1(X, \mu)$, $\int_Y \left(\int_X f(x) \, d\nu_y \right) d\mu(y) = \int_X f \, d\mu(x)$.

Idea:

 $= \begin{array}{c} \Gamma \curvearrowright \\ X_1 \times X_2 \end{array}$ s.t. $\Gamma \curvearrowright X_1$ trivial, $\Gamma \curvearrowright X_2$ ergodic

$Y = X_1 \times \{0\}$

- thm says $\int$ over fibers = integrating the square.

the map puts a measure vertical slice, which have an ergodic action on them. (not. nec. same one, though)

→ "sums" of ergodic pieces.

Koopman Representation

Let $\Gamma \curvearrowright (X, \mu)$. Define $\pi: \Gamma \to \mathcal{U}(L^2(X, \mu))$ [i.e. unitary operators on $L^2(X, \mu)$] by $\pi_g(f)(x) = f(\sigma_g^{-1}(x))$. This is a representation, the Koopman Rep.

Note: $\mathbb{C} \cdot \mathbb{1} \subseteq L^2(X, \mu)$ are invariant.
const. fcns

The reduced Koopman Rep. $\pi_0: \Gamma \to \mathcal{U}(L^2(X, \mu) \ominus \mathbb{C} \cdot \mathbb{1})$ (orthogonal complement)

$$\pi_0 = \pi_g \big|_{L^2(X, \mu) \ominus \mathbb{C} \mathbb{1}} .$$

- Since σ_g preserves measure, it preserves $\int$ & so inner prod's, & it's invertible, so it is, in fact, unitary.

This embeds $\mathrm{Aut}(X, \mu) \subset \mathcal{U}(\mathcal{H})$ ← operators on $\mathcal{H}$, a Banach sp

Hilbert sp → here $L^2(X, \mu)$.

this has a topology (a lot of them)

On $\mathcal{U}(\mathcal{H})$ we have the strong operator topology, in which $U_n \to U$ if $\forall \xi \in \mathcal{H}$, $\| U_n \xi - U \xi \|_2 \to 0$
← norm in Hilbert sp.

We get a topology on $\mathrm{Aut}(X, \mu)$: $T_n \to T$ if $\forall A \subseteq X$ $\mu(T_n A \, \Delta \, TA) \to 0$.

Def: Let $\Gamma \overset{\sigma}{\curvearrowright} (X,\mu)$. We say σ is mixing if $\forall\, A,B \subseteq X$,

$$\lim_{\substack{g\to\infty \\ (g\in\Gamma)}} |\mu(A\cap\sigma_g(B)) - \mu(A)\mu(B)| = 0.$$

Y

10/1 - F.A.

Def: Let $\Gamma \curvearrowright^{\sigma} (X,\mu)$. We say σ is

(1) mixing if $\forall A,B \subset X$, $\lim_{g \to \infty} |\mu(A \cap \sigma_g(B)) - \mu(A)\mu(B)| = 0$.
i.e. $\forall \infty$ seq's of discrete elts in Γ

(2) weak mixing if

$\exists g_n \in \Gamma$ s.t. $\forall A,B \subset X$, $\lim_{n\to\infty} |\mu(A \cap \sigma_{g_n}(B)) - \mu(A)\mu(B)| = 0$.

(3) compact if $\overline{\sigma(\Gamma)} \subset \mathrm{Aut}(X,\mu)$ is compact.

Ex: • Bernoulli: $\Gamma \curvearrowright (\prod_\Gamma Y, \mu)$ is mixing.

1st prove for open sets:

$$\ldots \ \underline{Y} \ \overbrace{\underline{Y_{i_1}} \ \underline{Y_{i_2}} \ \underline{Y_{i_3}}}^{\text{open in } Y} \ \underline{Y} \ \ldots = A$$

$$B = \ \ldots \ \underline{X} \ \underline{X_{i_1}} \ \underline{X_{i_2}} \ \underline{X_{i_3}} \ \underline{X} \ \ldots$$

Γ acts by shifting, so after a finite #, the difference is actually 0.

• Generalized Bernoulli: Let $\Gamma \curvearrowright I$, a countable set. Then, $\Gamma \curvearrowright^{\sigma} (\prod_I Y, \mu)$. Γ's action on I determines if the action is mixing or wk mixing: If $I = \Gamma/N$, w/ $|N| = \infty$ & $|\Gamma : N| = \infty$, then it's weak mixing. Not mixing, b/c if go in direction of N, σ doesn't do anything.

• $\Gamma \hookrightarrow G$, G a cpt gp. Then $\Gamma \curvearrowright (G, \text{haar})$ is cpt

Prop: Mixing $\underset{\text{clear}}{\Rightarrow}$ wk mixing $\Rightarrow$ ergodic.

Pf: If $A \subset X$ w/ $\mu(A) \neq 0,1$ & $\sigma_g(A) = A$, then $\mu(A \cap \sigma_g(A)) = \mu(A)$
$\neq \mu(A)^2 \Rightarrow$ not wk mixing

Thm: $\Gamma \hookrightarrow G$ (G cpt), then $\Gamma \curvearrowright (G, \text{haar})$ is ergodic if Γ is dense in G.

Pf: Assume not. Then $\exists A \subset G$ w/ $\sigma_g(A) = A$ & if $B = G \setminus A$, $\sigma_g(B) = B$. $\exists$ nested open sets $A \subset O_n$, $B \subset U_n$ w/ $\mu(O_n \setminus A)$, $\mu(U_n \setminus B) < \varepsilon/2^n$

Let $\Gamma = \{g_1, g_2, \dots\}$, $x \in G$, $O \subseteq G$ open. We want to move x into O.

$\exists g \in \underset{\text{open} \Rightarrow \text{nontriv. int w/ } \Gamma \text{ b/c } \Gamma \text{ dense}}{\underline{Ox^{-1}}} \Rightarrow gx \in O \Rightarrow$ every orbit is dense.

Thus, $\forall n\ \forall x$, $\exists i_n$ s.t. $g_{i_n} x \in O_n$ (b/c O_n's are open)

$\Rightarrow B \subseteq \bigcup_{g_i \in \Gamma} g_i(O_n \setminus A)$ ← b/c elts of B never land in A.

↑ i.e. pull back all O_n's

<u>Note</u>: $g_1 O_2 \subset g_1 O_1 \Rightarrow B \subset g_1(O_2 \setminus A) \cup g_2(O_2 \setminus A) \cup \dots$
$\subset g_1(O_1 \setminus A) \cup g_2(O_2 \setminus A) \cup \dots$

Sim, $B \subset g_1(O_3 \setminus A) \cup g_2(O_3 \setminus A) \cup \dots$
$\subset g_1(O_1 \setminus A) \cup g_2(O_2 \setminus A) \cup \dots$

$\Rightarrow$ Taking intersection, $B \subset \bigcup g_i(O_i \setminus A) \Rightarrow \mu(B) < \sum \varepsilon/2^i$ ↯. □

Prop: If $\Gamma \curvearrowright^{\sigma} (X, \mu)$ is weak mixing, then it is not compact.

Pf: $|\mu(A \cap \sigma_g(B)) - \mu(A)\mu(B)|$. If σ is cpt, $\exists g_n$ w/ $\sigma_{g_n} \to \Phi$, Φ some elt of $\mathrm{Aut}(X, \mu)$. Let $A = \Phi(B)$. Then

$\mu(\Phi(B) \cap \sigma_{g_n}(B)) \to \mu(\Phi(B) \cap \Phi(B)) = \mu(\Phi(B)) \neq \mu(\Phi(B))^2$, for an approp. choice of B.

□

Representation Theory Analogs

Def: Let $\pi: \Gamma \to \mathcal{U}(H)$, $\xi \in H$. We say ξ is <u>almost periodic</u> if $\overline{\pi_\Gamma(\xi)} \subset H$ is cpt (i.e. orbit of ξ under action of Γ — you may not get back to where you started, but you get close)

Thm: Let $\Gamma \curvearrowright^{\sigma} (X, \mu)$ be an action, π the Koopman rep. on $L^2(X)$. σ is cpt $\Leftrightarrow$ all $\xi \in L^2(X)$ are almost periodic.

<u>Lemma</u>: Let (M,d) be a cpt metric sp. Then $\mathrm{Isom}(M,d)$ w/ top. of ptwse-convergence is cpt.

<u>Pf</u>: $\mathrm{Isom}(M,d) \subseteq M^M$, which is cpt by Tychanof. So all we need to show is that we have a closed subsp.

Let $\phi \in \overline{\mathrm{Isom}(M,d)}$ w/ $f_n \to \phi$. Then $\forall x, y \in M$,

$d(\phi(x), \phi(y)) = \lim d(f_n(x), f_n(y)) = d(x,y) \Rightarrow \phi \in \mathrm{Isom}(M,d)$. $\square$

10/3 - FA

<u>Def</u>: Let π be a <u>unitary</u> rep. of Γ on H, a Hilbert sp. $\xi \in H$ is <u>almost periodic</u> if $\overline{\pi_\Gamma(\xi)} \subset H$ is cpt.

<u>Def</u>: π is <u>cpt</u> if all $\xi \in H$ are almost periodic.

<u>Thm</u>: $\Gamma \curvearrowright (X,\mu)$ is cpt $\iff$ $\pi : \Gamma \to \mathcal{U}(\underbrace{L^2(X,\mu)}_{=H})$ is cpt

<u>Pf</u>: ($\Rightarrow$) σ cpt $\Rightarrow$ $\overline{\pi_\Gamma} \subseteq \underline{\mathcal{U}(L^2(X,\mu))\ \text{cpt}}$ $\Rightarrow$ $\overline{\pi_\Gamma(\xi)}$ cpt $\forall \xi$.

($\Leftarrow$) Pick $\xi_1 \in H$. Then $H_{\xi_1} = \overline{\operatorname{span}\{\pi_\Gamma(\xi_1)\}}$. By assump. $\overline{\pi_\Gamma(\xi_1)}$ is cpt.
$\uparrow$ Hilbert subsp.
Pick $\xi_2 \in H \setminus H_{\xi_1}$ & repeat. Note, H_{ξ_1} are inv. under Γ b/c they are orbits
Apply Zorn's lemma to get $H = \bigoplus_{\xi \in \mathcal{J}} H_\xi$. If $g \in \Gamma$, then

$$\pi_g = \begin{pmatrix} \pi_g|_{H_{\xi_1}} & & 0 \\ & \pi_g|_{H_{\xi_2}} & \\ 0 & & \ddots \end{pmatrix}$$

We can embed $\pi_\Gamma \hookrightarrow \underbrace{\prod_i \operatorname{Isom}(\overline{\pi_\Gamma(\xi_i)})}_{}$
cpt (b/c Isom. of cpt sp's cpt & prod. of cpt is cpt)

$\Rightarrow \overline{\pi_\Gamma}$ cpt.

$\square$

<u>Thm</u> (Peter-Weyl) Let π be a cpt rep. of Γ on H. Then $\exists$ $H_i \subset H$ w/ $\dim H_i < \infty$, $\pi_g(H_i) \subset H_i$ $\forall g, i$, & $H = \bigoplus_i H_i$.

<u>Sketch of pf</u>: Assume Γ cpt (ok b/c Γ embeds into unitary gp w/ precpt image). Let $T(\xi) = \int_\Gamma \pi_g(\xi)\, d\mu(g)$. $T(\xi)$ is a cpt operator, & use spectral thm for cpt operators to decompose H as the eigensp's of T.

<u>Ex</u>: Let $\mathbb{Z} \curvearrowright (S^1, \text{haar})$ by $n(e^{i\theta}) = e^{inp}e^{i\theta} = e^{i(np+\theta)}$ w/ $p \in \mathbb{R} \setminus \mathbb{Q}$. This is a cpt action. Where do we get the f.d. subsp's that are invariant?

<u>Schur's Lemma</u>: $\pi : \Gamma \to \mathcal{U}(H)$, & $H_1 \subseteq H$ w/ $\pi_g(H_1) \subseteq H_1$
$\iff$ $\pi_g P_{H_1} = P_{H_1} \pi_g$ $\forall g$, where P_{H_1} is projection onto H_1.

Cor: If Γ is abel. & π is irred. (i.e. only invar. subsp's are 0 & whole thing), then H is 1-dim'l.

(b/c abel $\Rightarrow$ commutes w/ everything, but irred $\Rightarrow$ no irred subsp's)

So, $L^2(S^1) = \overline{\text{span}\{e^{ik\theta}\}}$, & $e^{ik\theta}$ are eigenvectors.

Def: Let $\pi: \Gamma \to \mathcal{U}(H)$ be a rep.

(1) π is mixing if $\forall \xi, \eta \in H$, $\forall g \in \Gamma$

$$\lim_{g\to\infty} \langle \pi_g(\xi), \eta \rangle = 0 \quad \text{(i.e. asymptotic orthogonality)}$$

(2) π is weak mixing if $\forall \xi, \eta \in H$, $\varepsilon > 0$, $\exists g \in \Gamma$ with $\langle \pi_g(\xi), \eta \rangle < \varepsilon$

(3) π is ergodic if $\nexists\, \xi \neq 0$ s.t. $\pi_g(\xi) = \xi$ $\forall g$.

Thm: $\Gamma \overset{\sigma}{\curvearrowright} (X, \mu)$ is mixing (w. mixing, ergodic) $\Leftrightarrow$ $\pi: \Gamma \to \mathcal{U}(\underbrace{L^2(X,\mu) \ominus \mathbb{C}}_{\text{remove constant fcns.}})$ is mixing (w. mix, erg.)

Pf: Ergodic: no inv. fcns $\Rightarrow$ no inv. characteristic fcns $\Rightarrow$ no inv. sets. In reverse, no inv. sets $\Rightarrow$ no inv. χ_A $\Rightarrow$ build up to no inv. fcns.

Weak mixing: Let $A, B \subseteq X$. $f_A = \chi_A - \mu(A)$, $f_B = \chi_B - \mu(B)$.

Then $\langle \pi_g(f_B), f_A \rangle = \langle \chi_{gB} - \mu(B), \chi_A - \mu(A) \rangle$

$$= \int \chi_{gB}\chi_A - \mu(B)\chi_A - \mu(A)\chi_{gB} + \mu(B)\mu(A)$$

$$= |\mu(\sigma_g(B) \cap A) - \mu(A)\mu(B)|.$$

Thm: Let $\pi: \Gamma \to \mathcal{U}(H)$ be a rep. Then π is w. mixing $\Leftrightarrow$ π has no f.d. subreps.

<u>Thm</u> (Koopman - Von Neumann): Let $\pi : \Gamma \to \mathcal{U}(H)$ be a rep.
Then $\exists$ H_c & $H_{wm} \subset H$ s.t. $H = H_c \oplus H_{wm}$, w/ $\pi_g(H_c) \subset H_c$ &
$\pi_g(H_{wm}) \subset H_{wm}$, $\pi|_{H_c}$ is cpt, & $\pi|_{H_{wm}}$ is weak mixing.

<u>Pf</u>: By Zorn's lemma, $\exists$ H_c, which is all the f.d. parts. Then
↳ i.e. $\oplus$ f.d. subreps.

$H_{wm} = H \ominus H_c$.

↳ just check your rep for f.d. subreps. If $\exists$, put it in H_c & check again. continue until you've built your H_c, & everything left is H_{wm}.

10/6 - FA

Koopman von Neumann: Γ a gp. π a unitary rep. of Γ, $\exists$ a unique subsp $K \leq \mathcal{H}$, our Hilbert sp, s.t. $\pi|_K$ is cpt & $\pi|_{K^\perp}$ is weak mixing. So $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$

In the case $\Gamma = \mathbb{Z}$, we have that TFAE:

$\mathbb{Z} \curvearrowright^{\sigma} (X,\mu)$ & $\sigma(1) = T$, our meas-pres. transf.

(1) σ is weak-mixing

(2) $\lim\limits_{N\to\infty} \mathbb{E}_{x\in[N]} |\mu(A \cap T^{-x}(B)) - \mu(A)\mu(B)| = 0$

Def: $\mathbb{X} = (X, \mathcal{B}_X, \mu, T)$, & $\mathbb{Y} = (Y, \mathcal{B}_Y, \nu, S)$ are meas-pres.

(1) $\mathbb{Y}$ is a factor of $\mathbb{X}$ if $\exists$ $X' \subseteq X$ & $Y' \subseteq Y$ w/ $\mu(X') = 1$, $\nu(Y') = 1$, and $\exists$ map $\phi : X' \to Y'$ s.t. $\phi \circ T(x) = S \circ \phi(x)$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "\phi"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

(2) If ϕ is invertible, then this is an isomorphism.

(3) If (1) holds, $\mathbb{X}$ is an extension of $\mathbb{Y}$.

Fact: Factors of $\mathbb{X}$ correspond to sub-σ-algebras of $\mathcal{B}_X$. The set $A = \phi^{-1}(\mathcal{B}_Y)$ is a sub-σ-alg. of $\mathcal{B}_X$, which is invariant under T: $T^{-1}(A) = A$.

The decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{wm}$ will give us a special factor. $T: X \to X \Rightarrow Tf(x) := f(Tx)$, so T is an operator on L^2 (unitary), & so spectrum should be a subgp of $S^1 \subseteq \mathbb{C}$. The span of the eigenfcns of T is the set $\mathcal{H}_c$. This gives us a sub-σ-alg. $\mathcal{B}_T \subseteq \mathcal{B}_X$ which is the coarsest σ-alg on which all fcns in span of e-fcns are measurable.

The factor defined by this is known as the Kronecker (or cpt) factor. This is always isomorphic to a gp rotation on a cpt abel. gp.

G, $\alpha \in G$, $Tx = x + \alpha$., like rotation on a torus.

ex: $S^1 \times S^1$, set $T(x,y) = (x+\alpha, x+y)$

In this case, the Kronecker factor is $(S^1, \mathcal{B}, \mu, S)$ where $S(x) = x + \alpha$.

Lebesgue meas.

In Roth's thm, we are concerned with the average

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{x=1}^{N} \sum_{h=1}^{N} f(T^x y) f(T^{x+h} y) f(T^{x+2h} y)$$

We will decompose f (before we did it as f = balance + δ) here as $f = f_c + f_{wm}$, & to prove Roth, we need to show that our limit is

$$= \lim_{N\to\infty} \frac{1}{N^2} \sum_{x=1}^{N} \sum_{h=1}^{N} f_c(T^x y) f_c(T^{x+h} y) f_c(T^{x+2h} y).$$

This is essentially a projection of f onto a subsp of our Hilbert sp.

For other averages (e.g. 4-term arith. progressions), we cannot use the Kronecker factor – we have to use a diff one!!

10/8 - FA

Pf (of Ergodic Roth) [Notes by Magyar on this proof]
Let (X, μ) be a measure sp, T an invertible meas-pres. transformation, $A \subseteq X$ w/ $\mu(A) > 0$. WTS: $\exists$ an n s.t. $\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0$. T induces a unitary operator, U, on $L^2(X)$

(i.e. $Uf(x) = f(T^{-1}x)$)

Note: • $\mu(A \cap T^{-n}A \cap T^{-2n}A) = \int_X \mathbb{1}_A(x)\, U^n \mathbb{1}_A(x)\, U^{2n} \mathbb{1}_A(x)\, d\mu(x)$

• Also, we can show

(*) $\liminf_N N^{-1} \sum_{n=1}^{N} \int \mathbb{1}_A\, U^n \mathbb{1}_A\, U^{2n} \mathbb{1}_A\, d\mu > 0$ (b/c if ave > 0, some term must be > 0)

• We use $\mathbb{1}_A = f_1 + f_2$, where $f_1 \in \mathcal{H}_c$ & $f_2 \in \mathcal{H}_{wm}$.

• We'll look at the ave. of the $\int$ of prod. of 3 arbitrary general fcns,

Case I: One of the 3 fcns is weak mixing.

Lemma (van der Corput): Let $\{u_n\}$ be a bdd seq. in a Hilbert sp. If $\lim_{H \to \infty} H^{-1} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| N^{-1} \sum_{n=1}^{N} \langle u_{n+h}, u_n \rangle \right| = 0$, then

$$\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} u_n = 0.$$

Pf: See Owen's notes. $\square$

Lemma: Let $g_1, g_2 \in \mathcal{H}$, where, say, $g_1 \in \mathcal{H}_{wm}$. Then,

$$\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} U^n g_1\, U^{2n} g_2 = 0$$

Pf: Let $u_n = U^n g_1\, U^{2n} g_2$. Since U is unitary (i.e. norm-pres), This seq is bdd. Consider the inner products:

$$\langle u_{n+h}, u_n \rangle = \int U^{n+h} g_1\, U^{2n+2h} g_2\, U^n \overline{g_1}\, U^{2n} \overline{g_2}$$

$$= \int U^h g_1\, U^{n+2h} g_2\, \overline{g_1}\, U^n \overline{g_2}$$

U meas.-pres, so can take away one U^n.

Now we have, considering only those terms that depend on n,

$$N^{-1}\sum_{n=1}^{N} U^n(\overline{g_2}\, U^{2h} g_2) \xrightarrow[\text{thm}]{\substack{\text{std}\\ \text{ergodic}}} \langle g_2, U^{2h} g_2\rangle$$

Thus,

$$N^{-1}\sum_{n=1}^{N} \langle u_{n+h}, u_n\rangle \longrightarrow \langle g_2, U^{2h} g_2\rangle \int U^h g_1 \overline{g_1} = \underbrace{\langle g_2, U^{2h} g_2\rangle}_{\underset{C\text{-}S}{\le} \|g_2\|_2^2 \text{ b/c } U \text{ meas-pres.}} \cdot \langle g_1, U^h g_1\rangle$$

To apply van der Corput, we need to show ave. over h's $\xrightarrow[H\to\infty]{} 0$.

Because $g_1 \in \mathcal{H}_{wm}$, we have

$$\left|\lim_{H\to\infty} H^{-1}\sum_{h=1}^{H} \langle g_1, U^h g_1\rangle\langle g_2, U^{2h} g_2\rangle\right| \le \lim_{H\to\infty} \|g_2\|_2^2 H^{-1}\sum_{h=1}^{H} |\langle g_1, U^h g_1\rangle| = 0$$

Recall, $g \in \mathcal{H}_{wm} \Rightarrow \lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} \left|\int f U^n g\right| = 0 \quad \forall f \in L^2$

(This is an alternate def. of wm)

Thus, by van der Corput, we're done. □

Def: Let $A: a_1 < a_2 < \cdots < a_n < \cdots$ be a strictly increasing seq of integers. The sequence A is syndetic if $\forall i$

$a_{i+1} - a_i \le M$ for some $M \in \mathbb{Z}$.

Lemma: Let $f \in \mathcal{H}_c$. Then for any $\varepsilon > 0$, the set

$S = \{n \in \mathbb{Z} \mid \|f - U^n f\| < \varepsilon\}$ is syndetic.

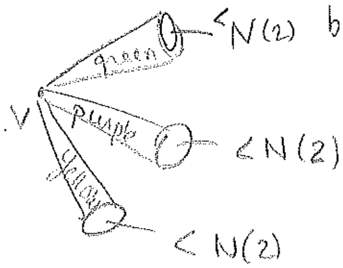
Pf: As $f \in \mathcal{H}_c$, the orbit closure $\overline{\{U^n f \mid n \in \mathbb{Z}\}}$ is cpt. There is a finite set $F \subset \mathbb{Z}$ s.t. the balls $\{B(U^n f, \varepsilon) \mid n \in F\}$ cover the orbit closure. Fix $m \in \mathbb{Z}$ arbitrary. We NTS $m = a + b$, where $a \in F$ & $b \in S$. By this construction $T^m f$ is in some ball, say $B(U^a f, \varepsilon)$. Then, $\|U^m f - U^a f\|$. But U is unitary, so $\|U^{m-a} f - f\| < \varepsilon$, so $m - a \in S$. □

10/13 - FA

Schur's Thm: Given an integer $r \in \mathbb{N}$, we have that every partition of $\mathbb{N}$ into r classes (i.e. color classes), $\exists$ one class w/ x, y, z s.t. $x + y = z$. [called a monochromatic sol'n to this eqn]

Lemma: Given an $r \in \mathbb{N}$, $\exists$ an $N(r)$ s.t. every partition of the edge set of K_N (the complete graph w/ N vertices) in r colors contains a monochromatic triangle, provided $N > N(r)$.

Pf: Induct on r:



b/c only purple & yellow in here...

- We can choose $rN(r-1)+1 = N(r)$

Lemma $\Rightarrow$ Schur's Thm: Color the first N integers w/ r colors. We associate a complete graph to this by letting the vertex set to be $[N]$. To get an induced coloring on our graph, color the edges of K_N, say ij, by $|i-j|$. By lemma, we have monochromatic Δ & this corresponds to a triple $i<j<k$ where $j-i$, $k-i$, $k-j$ are mono in $[N]$, & $(k-j)+(j-i) = k-i$. $\square$

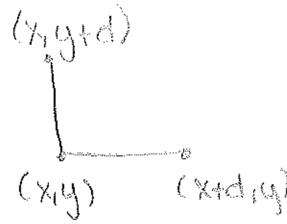
Q: Can we replace this lemma w/ a density-style lemma that will imply Roth's thm? Yes!

Triangle-Removal Lemma: Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. every graph on N vertices w/ at most δN^3 Δ's can be made Δ-free by removing at most εN^2 edges.

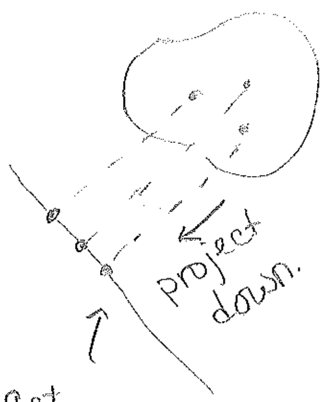
* This implies smth stronger than Roth's thm, a 2-dim'l version.

Thm (2-D Roth): Let $A \subseteq \mathbb{Z}_N^2$ w/ δN^2 elts. If N is large enough ($N > N_0(\delta)$), then $\exists$ a triple $\{(x,y), (x+d,y), (x,y+d)\} \subseteq A$, where $d \neq 0$.

These triples are called "2D corners", i.e. vertices of a rt Δ.



<u>2D Roth $\Rightarrow$ Roth</u>: Idea:



3 equally spaced pts on line.

<u>Prop</u>: 2D Roth $\Rightarrow$ Roth.

<u>Pf</u>: Let $A' \subseteq \mathbb{Z}_N$ w/ δN elts.

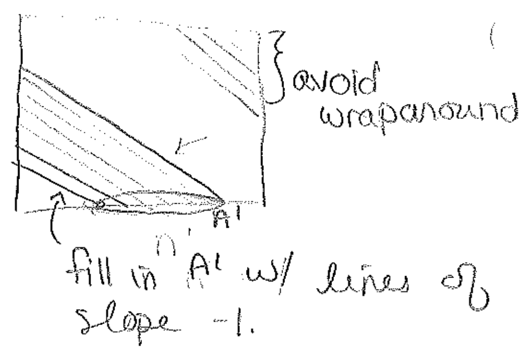
Let A be the set $\{(x,y) : x-y \in A'\}$

Then A has δN^2 elts. Thus we have a 2D corner if N suff. large, i.e.

$(x,y) \in A$, $(x+d,y) \in A$, & $(x,y+d) \in A$.

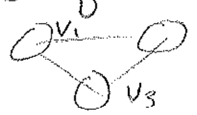
$\Rightarrow x-y \in A'$, $x+d-y \in A'$, $x-y-d \in A'$.

This is a 3-term AP in A'. (for some $d \neq 0$) $\square$



Given $A \subseteq [N]$, Take $A' \subset (4N, 5N]$, & $A'' \subseteq \mathbb{Z}_{9N}$. Then a 3-AP in $\mathbb{Z}_{9N}$ gives a 3-AP in $(4N, 5N]$, b/c won't have any wraparound.

<u>Pf</u> (2D Roth): Let $A \subseteq \mathbb{Z}_N^2$ be given. We construct a tripartite graph G (i.e. one w/ 3 sets of vertices & all edges go btwn distinct types of vertices:

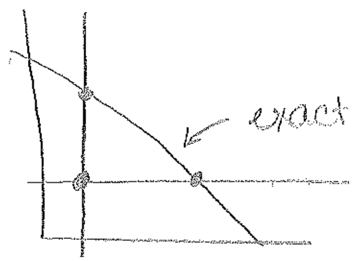


). w/ vertex sets

$V_1 = \{\text{vertical lines in } \mathbb{Z}_N^2\}$

$V_2 = \{\text{horiz. lines in } \mathbb{Z}_N^2\}$

$V_3 = \{\text{lines of slope } -1 \text{ in } \mathbb{Z}_N^2\}$



and edge set $E(V_1,V_2)$ defined by $v_1 v_2 \in E(V_1,V_2) \Leftrightarrow v_1 \cap v_2 \in A$ (i.e. the int. pt of the 2 lines lies in A). Similarly define $E(V_2,V_3)$ & $E(V_1,V_3)$. Any Δ in the graph must contain a vertex from each set (b/c tripartite), & this contains trivial Δ's, i.e. $\divideontimes$ $d=0$. We NTS it is not only trivial Δ's in our set.

If we have no non-degenerate Δ in G, then we have δN^2 Δ's in G, or $(\delta/N)N^3$. This quantity δ/N goes to 0 w/ N. To make G Δ-free, we need to remove at least one edge for each pt in A, b/c the degenerate Δ's are edge-disjoint (i.e. share no common edge), or δN^2 edges. [Note $\exists$ Δ for each pt of A]. Pick $\varepsilon = \delta/2$. We want a contradiction w/ Δ-removal lemma. For N suff. large, we should be able to remove $\leq \delta/2\, N^2$ edges to get Δ-free, but $\delta/N \to 0$ as $N \to \infty$, & we always need to remove one edge for each pt.

10/15 - FA

Thm (Roth, again): Let $\delta > 0$ be given. Then $\exists$ constant $c(\delta) > 0$ s.t. for every set $A \subseteq [N]$ w/ $|A| = \delta N$, we have

$$\mathop{\mathbb{E}}_{x,h \in [N]} 1_A(x) 1_A(x+h) 1_A(x+2h) \geq c(\delta).$$

i.e, A contains not only one 3-AP, but a positive proportion of all possible 3-APs (i.e. a pos. proportion of the expected #).

- Varnavides originally gave this version of Roth's thm.

Thm (Functional version of Roth): Let $f: [N] \to [0,1] (\subseteq \mathbb{R})$ have $\mathop{\mathbb{E}}_{x \in [N]} f(x) \geq \delta$ for some given $\delta > 0$. Then $\exists$ a constant $c(\delta) > 0$ w/

$$\mathop{\mathbb{E}}_{x,h \in [N]} f(x) f(x+h) f(x+2h) \geq c(\delta).$$

- These 2 thms are equivalent.
  - (2) $\Rightarrow$ (1) by letting $f = 1_A$.
  - (1) $\Rightarrow$ (2) by letting $A = \{x \mid f(x) > \delta/2\}$, which will have positive density.
- We can also replace $[N]$ w/ $\mathbb{Z}/N\mathbb{Z}$ & get the equivalent statements.

$\boxed{\text{Goal}}$: 4-term progressions. We'd like to follow our original density-increment argument.

Sumsets

Let G be an abel. gp, & let $A, B \subseteq G$. The sumset of A & B is the set $A + B = \{a + b \mid a \in A, b \in B\}$.

Ex: $A \subseteq \mathbb{Z}$. Let $|A| = n$. How small can $A+A$ be? $2n-1$.

- $|A| = n$ & A is an a.p. $\Longleftrightarrow$ $|A+A| = 2n-1$.
- Let us order the set A: $a_1 < a_2 < \cdots < a_n$. Then $a_1 + a_1 < a_1 + a_2 < \cdots < a_1 + a_n < a_2 + a_n \cdots < a_n + a_n$, which has $2n-1$ elts. Thus $2n-1$ is a lower bound for $|A+A|$. To show that $= \Rightarrow A$ is an a.p, show that $a_i - a_{i-1}$ is constant.

How large can $|A+A|$ be? If all sums are distinct, then $|A+A| = \binom{n}{2}$. Ex: $A = \{1, 2, \ldots, 2^{n-1}\}$

The <u>product set</u> of A is $A\cdot A = \{a_1 \cdot a_2 \mid a_1, a_2 \in A\}$

<u>Note</u>: prod. set of $A = \{1, 2, \ldots, 2^{n-1}\}$ is small.

prod set of $A = \{1, 2, \ldots, n\}$ is large

<u>Q</u>: Is there a set where both $|A+A|$ & $|A\cdot A|$ are small?

<u>Conjecture</u> (Erdös - Szemeredi): $|A+A| + |A\cdot A| \geq n^{2-\varepsilon}$

- E-Sz proved $n^{1+\varepsilon}$
- Solymosi: $n^{4/3-\varepsilon}$

<u>Question</u>: Given some constant c, can we classify sets $A \subseteq \mathbb{Z}$, $|A| = n$ w/ $|A+A| \leq cn$?

<u>Answer</u>: (Freiman's Thm, or Freiman - Ruzsa Thm): Essentially, dense subsets of typical things, like union of 2 a.p.'s.

<u>ex</u>: $A = \{1, \ldots, n\} \cup \{101, \ldots, 100+n\}$

<u>Def</u>: A <u>generalized arithmetic progression</u> is a set of the form $A = P_1 + P_2 + \cdots + P_d$, where the P_i are a.p's. [d-fold sumset of an a.p.]

Alternatively, if $g_1, \ldots, g_d$ are the gap sizes of $P_1, \ldots, P_d$, then $A = \{x_0 + i_1 g_1 + \cdots + i_d g_d \mid 1 \leq i_1 \leq k_1, \ldots, i_d \leq k_d\}$, where k_i is the length of the a.p. P_i. [think of g_i as independent vectors in $\mathbb{Z}^d$, so this has a cube structure] This is the image of a cube

$$[k_1] \times [k_2] \times \cdots \times [k_d]$$

The quantity d is the <u>dimension</u> of A. The size of A is $|A|$, which has a trivial bound $\prod_{i=1}^{d} k_i$.

10/20 - FA

Let $U, V, W \subseteq \mathbb{Z}$ be finite subsets. Then $|U|\cdot|V-W| \leq |U+V|\cdot|U+W|$.
The "Δ-Ineq" for set addition.

Pf (Plünnecke $\Rightarrow$ Plünnecke Ruzsa). Let A be a set, $|A+A| \leq C|A|$.
By the prop. from last time, we have $A'' \subseteq A' \subseteq A$ (wlog $l > k$) with
$|A' + kA| \leq C^k |A|$ and $|A'' + lA| \leq C^l |A|$. Then

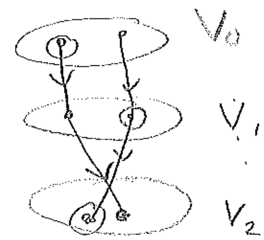
$$|\cancel{A''}||kA - lA| \leq |A'' + kA||A''+lA| \leq |A'+kA||A''+lA|$$
$$\leq C^{k+l}|A'||\cancel{A''}|$$
$$\Rightarrow |kA - lA| \leq C^{k+l}|A'|.$$

Pf (of Plünnecke, following an argument of G. Petridis) □

G is a Plünnecke graph of level h, w/ vertex set $V_0 \cup \cdots \cup V_h$.
Set $\Delta = (D_h(G))^{1/h}$. We are going to weight the vertices of G by $w(v) = \Delta^{-i}$ if $v \in V_i$.

Def: A separating set $S \subseteq V(G)$ is a set which contains a vertex from every directed path of max'l length.
S is called minimal

$$\sum_{v \in S} w(v) \quad \text{is minimized over all sep. sets.}$$



Lemma: (Petridis) Let $C > 0$ be a real # & G a level h Plün. graph w/ vertex weights $w(v) = C^{-i}$ for $v \in V_i$. Then $\exists$ a minimal separating set $S \subseteq V_0 \cup V_h$.

Cor: Let G be as above, where $C = \Delta$. The weight of a minimal separating set is $|V_0|$.

Pf: Apply Petridis lemma. Let $S_0 \cup S_h$ be the minimal sep. set. We know $Im^{(h)}(S_0^c) \geq D_h(G)|S_0^c|$ (by def. of magnification, $D_h(G)$), & then $w(S_0 \cup S_h) = w(S_0) + w(S_h) = |S_0| + (D_h(G))^{-1}|S_h|$

$$\geq |S_0| + |S_0^c| \quad (\text{b/c } Im^{(h)}(S_0^c) \subseteq S_h)$$
$$= |V_0|$$
□

Petridis Lemma $\Rightarrow$ Plünnecke: Let $Z \subseteq V_0$ w/ G weighted as before. $Z^c \cup \mathrm{Im}^{(i)}(Z)$ is a separating set. So we have:

$$|V_0| \leq w(Z^c) + w(\mathrm{Im}^{(i)}(Z))$$
$$\leq |V_0| - |Z| + \Delta^{-i}|\mathrm{Im}^{(i)}(Z)|$$

Rearrange & take minimum over all Z.
Then we have:

$(D_i(G))^{1/i} \geq \Delta$. This is equivalent to our original statement $D_1 \geq D_2^{1/2} \geq \cdots \geq D_h^{1/h}$ by looking at subgraphs & repeating the argument.

$\square$

Rk] Let S be a minimal sep. set. Then $\forall Z \subseteq S$,
$w(\mathrm{Im}(Z)) \geq w(Z)$ & $w(\mathrm{Im}^{-1}(Z)) \geq w(Z)$, where $\mathrm{Im}^{-1}(Z)$ is the set of vertices w/ neighbors in Z.

Lemma: $C > 0$ real, G Plünneke of level 2. Suppose $\forall S \subseteq V_1$, we have $|\underbrace{\mathrm{Im}(S)}_{\in V_2}| \geq C|S|$ & $|\underbrace{\mathrm{Im}^{-1}(S)}_{\in V_0}| \geq C^{-1}|S|$. Set $X_i \subseteq V_1$ w/ incoming degree i, & $Y_i \subseteq V_2$ w/ incoming degree i. Then $C|X_i| = |Y_i|$.
Set $X_i' \subseteq V_1$ with outgoing degree i & $Y_i' \subseteq V_0$ w/ outgoing degree i. Then $C^{-1}|X_i'| = |Y_i'|$.

Pf: The X_i partition V_1. Partition V_2 by:

$T_k = \mathrm{Im}(X_k)$, where k = max degree in G.
$T_{k-1} = \mathrm{Im}(X_{k-1}) \setminus T_k$.
$\vdots$
$T_1 = \mathrm{Im}(X_1) \setminus (T_k \cup \cdots \cup T_2)$

We have $\mathrm{Im}(X_j \cup \cdots \cup X_k) = T_j \cup \cdots \cup T_k$ $\forall j$. This gives
$\sum_{i=j}^{k} t_i \geq C \sum_{i=j}^{k} x_i$ where $|X_i| = x_i$ & $|T_j| = t_j$. Then

$$\sum_{i=1}^{k} i t_i \geq C \underbrace{\sum_{i=1}^{k} i x_i}_{\#\text{ edges btwn } V_1 \text{ \& } V_2}$$

By Plün. prop's & def. of T_i, the in-degree $d^-(v) \geq i$ $\forall v \in T_i$. Then
$|E(V_0,V_1)| = \sum_{i=1}^{k} |E(V_0,X_i)| = \sum_{i=1}^{k} i x_i \leq C^{-1} \sum_{i=1}^{k} i t_i \leq C^{-1} \sum_{i=1}^{k} E(X_i,T_i) \leq C^{-1} |E(V_1,V_2)|$.

By partioning V_1 & then using the induced partition on V_0, a similar argument gives

$$|E(V_1,V_2)| \leq C|E(V_0,V_1)|.$$

So, $|E(V_1,V_2)| = C|E(V_0,V_1)|$, by putting 2 ineq's together. Then by comparing at each step, we get the equality $C|X_i| = |Y_i|$. $\square$

Lemma: Let $c>0$, G a Plünnecke graph of level 2. The weights on G are given by $w(v)=c^{-i}$ if $v\in V_i$. If V_1 is a separating set of minimal weight, then so is V_0.

Pf: If $S\subseteq V_1$, we have $S^c\cup Im(S)$ & $S^c\cup Im^{-1}(S)$ are separating sets. The minimality of V_1 gives:

$$|Im(S)|\geq c|S| \quad \& \quad |Im^{-1}(S)|\geq c^{-1}|S|.$$

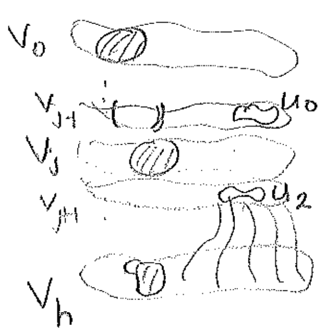
Then
$$\omega(V_1)=c^{-1}|V_1|=c^{-1}\sum_{i=1}^{k}|X_i'|$$
partition of V_1; $X_i=\{$elts w/ incoming degree $i\}$

$$=\sum_{i=1}^{k}|Y_i|=|V_0|=\omega(V_0).$$

Thus, V_0 is of minimal wt, as well.

Pf (Petridis' Lemma):

[**Recall**: Lemma says: we have sep set in $V_0\cup V_h$ of min. wt.]



Let S be a sep. set of minimal wt, & let $j=\max_{i<h}\{i\mid V_i\cap S\neq\emptyset\}$.

Let U_0 be the elts in V_{j-1} w/ no paths from $S\cap V_0,\dots,S\cap V_{j-2}$, & not in $V_{j-1}\cap S$.

Let $U_2\subseteq V_{j+1}$ be the collection of elts with image in V_h outside $V_h\cap S$.

We have that every path from U_0 to U_2 must pass through $V_j\cap S$, i.e. $(U_0, V_j\cap S, U_2)$ is Plünnecke. So apply previous lemma, & we have that U_0 is minimal (b/c $V_j\cap S$ is minimal).

Thus $U_0\cup\left(\bigcup_{i=0}^{j-1} S\cap V_i\right)\cup(S\cap V_h)$ is a minimal separating set. $\square$

Continue pushing back up to V_0.

Outline of Freiman's Thm:

(1) Let $|A+A| \le C|A|$. Then $|8A-8A| \le C^{16}|A|$.

(2) Ruzsa's Model Lemma: $\exists A' \subseteq A$, $|A'| \ge \frac{|A|}{8}$. $\exists B \subseteq \mathbb{Z}_N$, N prime in $[C^{16}|A|, 2C^{16}|A|]$; (Note: there is always a prime btwn x & $2x$) w/ $B \cong_8 A'$ (B & A are Freiman isomorphic of level 8).

(3) Bogolyubov's Lemma: $2B-2B$ contains a Bohr set $B(\Gamma, \varepsilon)$ of dimension $d = |\Gamma| \le cC^{32}$

(4) (Application of Minkowski's 2^{nd})
$2B-2B$ contains a proper (← unique sums) generalized arithmetic progression, Q, (GAP) of dim d and size $|Q| \ge e^{-cd\log(d)}|A|$

(5) $2A'-2A' \cong_2 2B-2B$, so Q pulls back to a proper GAP of same size & dimension in $2A'-2A' \subseteq 2A-2A$. The dimension is $\lesssim C^{32}$ & size $\gtrsim e^{-cC^{32}}|A|$.

(6) Get a GAP with $A \subseteq$ GAP. [we'll do this today]:
We have $|A+2A-2A| \le C^5|A|$ (by Plünnecke-Ruzsa). We want to find $X \subseteq A$ w/ $|X| \lesssim_c 1$, w/ $A \subseteq X+Q-Q$ (Ruzsa Covering Lemma)

- $Q-Q$ still a GAP of same dimension.
- X small, so should be contained in a GAP of dim $\le |X|$ & size $2^{|X|}$
- Thus $X+Q-Q$ a GAP of dim $\le \dim(X) + \dim(Q-Q) = \underbrace{\dim(X) + \dim(Q)}_{\text{all bdd in terms of } C}$

$\text{size}(X+Q-Q) \le |Q-Q| \le \underbrace{|X|}_{\lesssim_c 1} \cdot \underbrace{2^d|Q|}_{\text{mult of } C \text{ \& } |Q| \text{ bdd by } C \text{ \& } |A|.}$

Thus, this GAP has dim bdd by C & size bdd by $d, C,$ & $|A|$.

Ruzsa's Covering Lemma: Let S, T be subsets of an abel. gp G. If $|S+T| \le K|S|$, $\exists X \subseteq T$ w/ $|X| \le K$ s.t. $T \subseteq S - S + X$.

[i.e., we can cover T by a small # of translates of $S-S$]

Pf: Pick a max'l set $X \subseteq T$ s.t. the sets $S+x$ are pairwise disjoint for $x \in X$. Then $\left|\bigcup_x (S+x)\right| \le |S+T| \le k|S|$

$$\underset{|S|\cdot|X|}{\overset{\shortparallel}{}} \Rightarrow |X| \le k.$$

Also, as X is max'l, then $\forall t \in T$ $(t+S) \cap (S+x) \ne \emptyset$ for some $x \in X$.
Then $t \in S - S + x \subseteq S - S + X$ $\forall t \in T$.

Ruzsa's Model Lemma: Let $A \subseteq \mathbb{Z}$ of size n w/ $|A+A| \leq Cn$. Given $k \geq 2$, $\exists$ a subset of A, A', s.t. $|A'| \geq n/k$, & $A' \cong_k B$ for some $B \subseteq \mathbb{Z}/m\mathbb{Z}$, as long as $m \geq C^{2k} n$.

Def: (Freiman k-homomorphism) Let $A \subseteq G$ & $B \subseteq H$ where G, H are abel. gps. If $\phi : A \to B$ satisfies:

If $x_1 + x_2 + \cdots + x_k = y_1 + \cdots + y_k \Rightarrow \phi(x_1) + \cdots + \phi(x_k) = \phi(y_1) + \cdots + \phi(y_k)$,

then ϕ is a k-homomorphism. If ϕ^{-1} exists and is a k-hom, then ϕ is a k-isomorphism, & we write $A \cong_k B$.

Basic Facts:

(1) If $\phi : G \to H$ is a gp hom, then ϕ a k-hom. $\forall k$.

(2) If ϕ^{-1} exists, then ϕ is not nec. a k-iso

ex: $\{0,1\} \subseteq \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z}$

(3) Composition of k-homs gives a k-hom.

(4) If $\phi(x_1) + \cdots + \phi(x_k) = \phi(y_1) + \cdots + \phi(y_k) \Rightarrow x_1 + \cdots + x_k = y_1 + \cdots + y_k$, in addition to ϕ being a k-hom, then ϕ is a k-iso.

(5) ϕ a k-hom $\Rightarrow$ ϕ a k'-hom if $k' < k$

Exercises:

① If A is a GAP in G, & $B \cong_2 A$, then B is a GAP of the same dimension. [induct on dimension]

② If $A \cong_8 B$, then $2A - 2A \cong_2 2B - 2B$.

Pf (Ruzsa's Model Lemma): We consider the maps

$$\mathbb{Z} \xrightarrow{\phi_1} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\phi_2(q)} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\phi_3} \mathbb{Z} \xrightarrow{\phi_4} \mathbb{Z}/m\mathbb{Z}$$

- ϕ_1 = reduction mod p, w/ p s.t. ϕ_1 is 1-1 on A, & p prime. $\Rightarrow \phi_1$ a k-hom $\forall k$.
- ϕ_4 is reduction mod m.
- ϕ_2 is mult. by some $q \in (\mathbb{Z}/p\mathbb{Z})^*$.
- ϕ_3 takes $\mathbb{Z}/p\mathbb{Z}$ into $[0, \ldots, p-1]$.

[More or less, $x \longmapsto (qx \bmod p) \bmod m$.]

• $\Phi_1, \Phi_2(q)$, & Φ_4 are gp homs.

Φ_3 is a Freiman k-iso if we restrict to an interval of the form
$I_j = \left(\frac{j-1}{k}p, \frac{j}{k}p\right]$.

For each q, we have that for $S_j = \{x \in A \mid \Phi_2 \circ \Phi_1(x) \in I_j\}$, $\exists$ one j, say $j(q)$ s.t. $|S_{j(q)}| \geq n/k$ (by averaging)

The map $\Phi = \Phi_4 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1$ is a k-hom. when restricted to $S_{j(q)}$.

We need a q s.t. $\Phi(x_1) + \cdots + \Phi(x_k) = \Phi(y_1) + \cdots + \Phi(y_k) \Rightarrow x_1 + \cdots + x_k = y_1 + \cdots + y_k$, b/c then Φ is a k-iso, as desired. If this fails for some q, then some $s = x_1 + \cdots + x_k - y_1 - \cdots - y_k \neq 0$ s.t. $qs \pmod p = 0 \pmod m$.

Let q range through $(\mathbb{Z}/p\mathbb{Z})^*$. For each s, qs runs through $(\mathbb{Z}/p\mathbb{Z})^*$, & in particular, there are at most $\frac{p-1}{m}$ bad choices of q (b/c if $m \mid q$, $qs \bmod m = 0$). Note: $s \in kA - kA$, & $|kA - kA| \leq C^{2k} n$, so there are at most $C^{2k} n$ choices of s. If $m > C^{2k} n$, then $s \equiv 0 \pmod m$. Thus the total number of bad choices for q, is $C^{2k} n \cdot \frac{p-1}{m}$. If $C^{2k} n \cdot \frac{p-1}{m} < p-1$, (& is if $m > C^{2k} n$) then $\exists$ an acceptable value of q. Pick this value, q_0. Then $A' = S_{j(q_0)}$. (by Plunnecke)

$\square$

Def: In $\mathbb{Z}/N\mathbb{Z}$, we define the Bohr set $B(\Gamma, \varepsilon)$ by

$$\{x \mid |e(\tfrac{x\xi}{N}) - 1| < \varepsilon \quad \forall \xi \in \Gamma\}$$

OR $\|x\xi/N\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon$, i.e. distance to closest integer to $x\xi/N$ is $< \varepsilon$.

for $\Gamma \subseteq \widehat{\mathbb{Z}/N\mathbb{Z}} \cong \mathbb{Z}/N\mathbb{Z}$ [$\widehat{\ }$ implies Potraygin? dual, so diff. as meas. sp's]
= gp of characters of $\mathbb{Z}/N\mathbb{Z}$
$\hookrightarrow e(x\xi/N)$

$|\Gamma|$ = dimension (rank) of $B(\Gamma, \varepsilon)$, &
ε is the radius.

Ex: Consider $\mathbb{Z}_p^n$. We have that $B(\Gamma, 0) \leq \mathbb{Z}_p^n$ is a subsp of codimension $|\Gamma|$. (b/c get $|\Gamma|$ linear eqns $x \cdot \xi = 0$) i.e. $\{x \mid x \cdot \xi = 0 \quad \forall \xi \in \Gamma\}$

Note: $B(\Gamma, \varepsilon)$ are like approximate subsp's

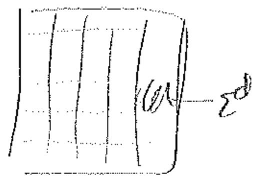
10/27 - FA

Def: Let $\Gamma \subseteq \mathbb{Z}/n\mathbb{Z}$, $|\Gamma| = d$, $\Gamma = \{\xi_1, \dots, \xi_d\}$ $(\xi_i \neq 0)$. For $\varepsilon > 0$, we have the Bohr Set $B(\Gamma, \varepsilon) = \{x \mid \|\frac{x \cdot \xi_1}{N}\|, \dots, \|\frac{x \cdot \xi_d}{N}\| < \varepsilon\}$, where $\|x\|$ is the distance to the nearest integer. The dim. (rank) of $B(\Gamma, \varepsilon)$ is $|\Gamma| = d$.

Then $e(\frac{x \cdot \xi}{N}) \approx 1$.

Fact 1: $|B(\Gamma, \varepsilon)| \gtrsim \varepsilon^d N$

Pf: Break up into cubes of size ε^d (Similar to pf of Dirichlet)



Fact 2: We have $B(\Gamma, N^{-1/d}) \neq \emptyset$ really, want smth non-0 in there, i.e. ≥ 2. If $x \in B(\Gamma, N^{-1/d})$, then $\|\frac{jx\xi}{N}\| < \varepsilon$ if $j \leq \varepsilon N^{1/d}$.

I.e., Bohr sets sets contain AP's of very large size, roughly $N^{1/d}$.

Bogolyubov's Lemma: Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$ w/ $|A| = \delta N$. Then $2A - 2A$ contains a Bohr set of dimension at most $4\delta^{-2}$.

Pf: Set $\Gamma = \{\xi \mid |\widehat{\mathbb{1}}_A(\xi)| > \delta^2/2\}$

By Planch, we have $\delta^2 = \sum_{\xi \in \widehat{\mathbb{Z}/N\mathbb{Z}}} |\widehat{\mathbb{1}}_A(\xi)|^2 \geq |\Gamma| \delta^4/4$.

So NTS $B(\Gamma) \subseteq 2A - 2A$: Look at L^4 norm on $\wedge$ side.

Let $b \in 2A - 2A$. Then $\sum_\xi |\widehat{\mathbb{1}}_A(\xi)|^4 e(-b\xi/N) > 0$:

$$\sum_\xi \left| \mathbb{E}_{x \in \mathbb{Z}/N\mathbb{Z}} \mathbb{1}_A(x) e\left(\frac{x\xi}{N}\right) \right|^4 e(-b\xi/N)$$

$$= \sum_\xi \mathbb{E}_{x_1} \mathbb{E}_{x_2} \mathbb{E}_{x_3} \mathbb{E}_{x_4} \mathbb{1}_A(x_1) \cdots \mathbb{1}_A(x_4) \cdot e((x_1 + x_2 - x_3 - x_4 - b)\xi/N)$$

$$= N^{-3} \cdot \#\{(x_1, \dots, x_4) \mid x_1 + x_2 - x_3 - x_4 - b = 0\} \geq 0$$

$\mathbb{E}_x = \frac{1}{N} \sum_x$

$\left(\sum_\xi = 0 \text{ unless } x_1 + x_2 - x_3 - x_4 - b = 0 \right)$

Let $B(\Gamma, 1/4)$. First, note that if $x \in B(\Gamma, 1/4)$, then $\text{Re}(e(x\xi/N)) \geq 0$. We now consider:

$$(*)\quad \sum_{\xi=0} + \sum_{\xi\in\Gamma} + \sum_{\xi\notin\Gamma} \left(|\hat{1}_A(\xi)|^4 e(\tfrac{+b\xi}{N})\right) = \left(\sum + \sum + \sum\right)\left(|\hat{1}_A(\xi)|^4 \mathrm{Re}(e(b\xi/N))\right)$$

Fourier transform at 0 ⇒ just density of set A

- For every b, $|\hat{1}_A(0)|^4 = \delta^4$.
- If $\xi \in \Gamma$, $b \in B(\Gamma, 1/4)$, then $\sum_{\xi\in\Gamma} > 0$.
- If $\xi \notin \Gamma$, then $\sum_{\xi\notin\Gamma}$ is bdd:

$$\sum_{\xi\notin\Gamma} |\hat{1}_A(\xi)|^4 \le \underbrace{\delta^4/4 \cdot \delta^2}_{=\delta^6/4}, \text{ since outside } \Gamma,\ |\hat{1}_A(\xi)| \le \delta^2/2.$$

& get 2 extra terms from Plancherelle.

doesn't cancel out the δ^4, which is our main term.

Thus, $(*) > 0$ if $b \in B(\Gamma, 1/4)$, so $B(\Gamma, 1/4) \subseteq 2A - 2A$ □

Rk] We have a Bohr set inside $2A-2A$, so we have a very long AP ($\approx N^{\delta^2}$) inside $2A-2A$, & all we need is that A is dense.

What if we wanted to do the same w/ $A-A$? The best you can do, i.e. the longest AP, is $\le e(c\log(N))^{2/3}$.

In $\mathbb{R}^k$, a lattice, Λ, is a discrete subgp (which implies f.g.). Assume the rank of the lattice is full. [i.e., full rank = dim k]. The fundamental parellelogram is $\mathbb{R}^k/\Lambda$, & $\mathrm{vol}(\Lambda) = \mathrm{vol}(\mathbb{R}^k/\Lambda)$.

Minkowski's 1st Thm: Let C be a centrally symmetric convex body (i.e. $-C = C$). If $\mathrm{Vol}(C) \ge 2^k \mathrm{Vol}(\Lambda)$, then $\Lambda \cap C \ne \emptyset$.

Given C a convex body & Λ a lattice, define the successive minima λ_j to be the minimum value of λ s.t. λC (dilates) contains j linearly independent elements from Λ.

Minkowski's 2nd Thm: Then, $\prod_{j=1}^{k} \lambda_j \le 2^k \cdot \mathrm{Vol}(\Lambda)/\mathrm{Vol}(C)$

Prop: $B(\Gamma, \delta)$ contains a proper GAP (i.e. distinct sums) of dim d, & size at least $(\delta/d)^d N$, for N prime.

10/29 - FA

<u>Prop</u>: Let $B(\Gamma, \varepsilon)$ be a Bohr set in $\mathbb{Z}/N\mathbb{Z}$, of dim k (i.e., $\Gamma = \{\xi_1, \dots, \xi_k\}$. Then $B(\Gamma, \varepsilon)$ contains a GAP of dim k & size at least $(\varepsilon/k)^k \cdot N$. Moreover, the GAP is proper.

<u>Pf</u>: Let Λ be the lattice generated by $N\mathbb{Z}^k$ & $(\xi_1, \dots, \xi_k) \subseteq \mathbb{Z}^k$. Λ has full rank & $\mathrm{Vol}(\Lambda) = N^{k-1}$ under the assumption that N is prime. Let $C = \{x \in \mathbb{Z}^k \mid |x_i| < 1\}$, a convex set. Then, by Minkowski's 2nd thm, prod. of successive minima is at most $\underbrace{2^{\dim} \cdot \mathrm{Vol}(C)}_{\text{CHECK}}$?, so:

$$\prod_{i=1}^{k} \lambda_i \le N^{k-1}, \text{ since } \mathrm{Vol}(C) = 2^k$$

& we have basis vectors $\underline{b}_1, \dots, \underline{b}_k$, where $b_i \in \lambda_i C$.

Consider linear combinations $\sum_{j=1}^{k} n_j \underline{b}_j$ where $|n_j| \le \frac{\varepsilon N}{k\lambda_i}$.

Then $\left|\sum_{j=1}^{k} n_j \underline{b}_j\right| \le \varepsilon N$. For each $\underline{b}_j$, we can write this as $\underline{b}_j \bmod N = (b_j\xi_1, \dots, b_j\xi_j) \bmod N$ [b/c b_j is a comb of $N\mathbb{Z}^k$ & $(\xi_1, \dots, \xi_k)$, so taking mod N leaves b_j as lin. comb. of ξ_i's]

Thus, $\left|\sum_{i=1}^{k} b_i n_i\right| \pmod N \le \varepsilon N$. This means that $\sum_{i=1}^{k} b_i n_i \pmod N \in B(\Gamma, \varepsilon)$, since $\frac{1}{N} \sum b_i n_i \pmod N \le \varepsilon$. Thus dim of $B(\Gamma, \varepsilon)$ is k, & the collection of these $\sum_{i=1}^{k} b_i n_i$ is our GAP, also of dim k. We need to show this GAP is proper, & then its size will be $\prod \frac{\varepsilon N}{k\lambda_i}$:

If $\sum b_i n_i = \sum b_i n_i'$, then $\sum \underline{b}_i n_i \equiv \underline{b_i} n_i' \pmod N$, and as each of the coordinates of the $\sum \underline{b}_i n_i$'s are at most εN, also $=$ in $\mathbb{Z}^N$. But the linear independence of b_i show that $n_i = n_i'$ $\forall i$. Thus this is proper, as all sums have to be distinct. Finally, the size is at least $(\varepsilon N)^k / k^k \cdot \frac{1}{N^{k-1}}$.

∎

If we followed along properly with the constants:

$$|A+A| \le C|A| \Rightarrow A \subseteq \text{GAP of dim} \le C^{34} \text{ \& size} \le e^{C^{34}} \cdot N.$$

Chang's Refinement: $\dim \le C^2 \log C$ & size $\le e^{kC^2 \log C} \cdot N$.

Def: Let G be a bipartite graph on the vertex set (V_1, V_2) w/ $|V_i| = N$. Assume G has αN^2 edges. For any sets $A \subseteq V_1$ & $B \subseteq V_2$, the edge
$0 \le \alpha \le 1$, edge density

density $\rho(A,B) = \dfrac{\#(E(A,B))}{|A|\,|B|}$, so we expect $\rho(A,B) = \alpha$.

G is said to be ε-regular if $|\rho(A,B) - \alpha| < \varepsilon$ $\forall A, B \subseteq V_1, V_2$, resp, w/ $|A|, |B| \ge \varepsilon N$.

- essentially a measure of how random a graph can be w/ parameter ε. Note: only truly random graph is a complete graph

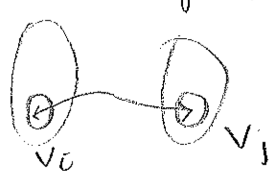
Szemeredi Regularity Lemma:

For every graph G on $N \ge N(\varepsilon)$ vertices, there is a partition of V, the vertex set of G, into boundedly many parts

$V = V_0 \cup \left(\bigcup_{i=1}^{K} V_i \right)$, where $K \le K(\varepsilon)$, $|V_0| \le \varepsilon N$, & all but εK^2 of the bipartite subgraphs on (V_i, V_j) are ε-regular, & $|V_i| = |V_j|$.

What we will do is a weaker, one pair version:

G a bipartite graph, find one such pair (V_i, V_j).



10/31-FA

G is a bipartite graph on the vertex set (V_1, V_2). The edge density of G is

$$\frac{|E(G)|}{|V_1| \cdot |V_2|} \quad \text{i.e., } \frac{\#\text{ of edges}}{\#\text{ of possible edges}}.$$

Def: G is ε-regular if $\varepsilon \in [0,1]$ & we have $|\rho(X,Y) - \rho(V_1,V_2)| < \varepsilon$ as long as $|X| > \varepsilon|V_1|$ & $|Y| > \varepsilon|V_2|$, where $\rho(X,Y) =$ edge density on induced subgraph (X,Y), where $X \subseteq V_1$ & $Y \subseteq V_2$.

$\rightarrow$ G is approximating a random graph.

Prop (Weak Regularity Lemma): Let $\varepsilon \in (0,1)$ be given, & also let G be a bipartite graph on (V_1, V_2) w/ edge density ρ. There is a constant $C = C(\varepsilon)$ s.t. we have some $X \subseteq V_1$, $Y \subseteq V_2$, $|X| \geq c|V_1|$, $|Y| > c|V_2|$, & the graph induced on (X,Y) is ε-regular. Moreover, $\rho(X,Y) \geq \rho$.

Pf: (1) If G is ε-regular, done. Else, we have $X_1, Y_1 \subseteq V_1, V_2$, resp, w/ $|\rho(X_1, Y_1) - \rho| \geq \varepsilon$.

Set $X_2 = X_1^c$, $Y_2 = Y_1^c$, & $\lambda_{ij} = \dfrac{|X_i||Y_j|}{|V_1||V_2|}$. $(i=1,2, j=1,2)$

Note that $\sum \lambda_{ij} = 1$. Then $\sum \lambda_{ij}\rho = \rho$, $^{(\dagger)}\sum \lambda_{ij}(\rho(X_i, Y_j) - \rho) = 0$

Also, $^{(*)}\sum \lambda_{ij}|\rho(X_i, Y_j) - \rho| > \varepsilon^3$

$(\dagger) + (*) = \sum 2(\lambda_{ij}(\rho(X_i, Y_j) - \rho))_+ > \varepsilon^3$ i.e. the positive part

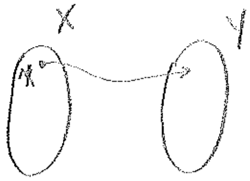
Thus we have a density increment of $\varepsilon^3/8$, b/c $\lambda_{ij} < 1$.

So, either it's ε-regular, or we have a density increment, so we can iterate. Set $V_1^{(2)} = X_1$, $V_2^{(2)} = Y_1$, & repeat. This can only continue finitely many times, $8\varepsilon^{-3}$ steps. Take $C(\varepsilon) = \left(\frac{1}{\varepsilon}\right)^{8\varepsilon^{-3}}$. $\square$

Observation: Let G be a graph w/ m edges. Then $\exists X, Y \subseteq V(G) = V$ s.t. $X \cup Y = V$ & $X \cap Y = \phi$, & the induced bipartite graph on (X,Y) has $m/2$ edges.

$\rightarrow$ i.e. getting a bipartite approx of original graph.

Pf: Take any partition of V, say $X \cup Y$.



- if can move v to Y & increase # of edges, i.e. $|N_X(x)| > |N_Y(x)|$.
- continue.

But there's a bdd # of edges, total, so this has to stop, & then $|N_X(x)| = |N_Y(x)|$ $\forall x$ & $\sum_x |N_X(x)| + |N_Y(x)| = V$.

D

Recall: The Ramsey # $r(G)$ of a graph G is the min. # N s.t. every coloring of the edges of the complete graph K_N has a monochromatic copy of G.

Burr-Erdös Conjecture: There is a special family of graphs, D, s.t. $\forall G \in D$, $r(G) \leq c|G|$, where $|G|$ = # of vertices of G, uniformly in D (i.e. one const. for entire family).

Thm: If $D = \{G \mid \text{max. degree of a vertex in } G \leq d\}$, then the conjecture holds.

- you can prove this thm for $D' = \{G \text{ bipartite} \mid \text{——} \}$ from our weak regularity lemma.

11/3 - FA

Let Z be a finite abelian gp. An <u>additive quadruple</u> in Z is a quadruple $(a,b,c,d) \in Z^4$ s.t. $a+b = c+d$.

<u>Recall</u>: For a fcn f on Z, the Gowers U^2 norm is given by

$$\mathbb{E}_{x,h,k} f(x)f(x+h)f(x+k)f(x+h+k) = N^{-3}\sum_{x+y=z+w} f(x)f(y)f(z)f(w), \quad N = |Z|.$$

We're adding over additive quadruples $(x)+(x+h+k) = (x+h)+(x+k)$

How many quadruples are there? N^3 (4 vars, but lose one degree of freedom from $x+y=z+w$)

<u>Goal Thm</u>: (Balog-Szemeredi) Let A be a subgp of Z s.t. $\Gamma_4(A) := \#\{(a,b,c,d) \mid a-d = c-b\} \geq \delta|A|^3$ for some given $\delta \in (0,1)$. Then there exists a subset $B \subseteq A$ s.t. $|B| \geq c(\delta)|A|$ and

$|B-B| \leq \frac{1}{c(\delta)} \cdot |B|$.

If $|A| > \delta N$, then

$$\delta^4 \leq \left(\mathbb{E}_{x\in Z} \mathbb{1}_A(x)\right)^4 \leq \|\mathbb{1}_A\|^4_{U^2(Z)}$$

<u>Lemma</u>: Let $0 < \varepsilon < \delta/4$, & G be an ε-regular bipartite graph on the vertex set (X,Y), and $|E(G)| > \delta|X|\cdot|Y|$. Then $\exists\, B \subseteq X$ w/ $|B| \geq (1-\varepsilon)|X|$, & $\forall a,b \in B$,

$P_4(a,b) := \#\{(c_2,c_3,c_4) \in Y\times X\times Y \mid (ac_2),(c_2c_3),(c_3,c_4),(c_4,b) \text{ are edges of } G\}$

$\geq (\delta^4/16)|X|\cdot|Y|^2$.

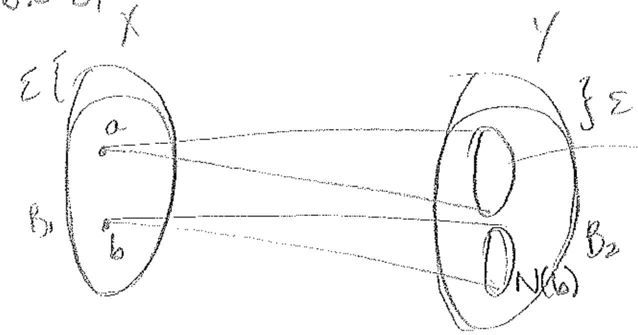
<u>Note</u>: We expect about $\delta^4|Y|^2|X|$ b/c δ is the density of the edges, (i.e. $a \in X$ ctd to $\sim\delta|Y|$ pts, etc).

<u>Pf</u>: Let $B_1 \subseteq X$ be s.t. $x \in B_1$ has at least $(\delta-\varepsilon)|Y|$ neighbors. Since B_1^c has elts w/ $< (\delta-\varepsilon)|Y|$ neighbors, the edge density of the pair (B_1^c, Y) is at most $\frac{(\delta-\varepsilon)|B_1^c||Y|}{|B_1^c||Y|} = \rho(B_1^c,Y)$

So, $|\rho(B_1^c,Y) - \delta| \geq \varepsilon$, & thus $|B_1^c| < \varepsilon|X|$, by the ε-regularity of G, & so $|B_1| \geq (1-\varepsilon)|X|$

We also have a set $B_2 \subseteq Y$ w/ elts w/ at least $(\delta-\varepsilon)|X|$ neighbors.

Fix $a, b \in B_1$



N(a)
neighbors of a. May not all lie in B_2, but will only lose an ε-percentage of Y outside of B_2.
↪ have $|N(a)| \geq (\delta - \varepsilon)|Y|$, lose an $\varepsilon|Y|$, get $\geq (\delta - 2\varepsilon)|Y|$ for $\cap$.

We want to count $N(N(a)) \cap N(N(b))$.

$|N(a) \cap B_2| \geq (\delta - 2\varepsilon)|Y|$. Sim, $|N(b) \cap B_2| \geq (\delta - 2\varepsilon)|Y|$

For $N(N(a)) \subseteq X$, the pts of X ctd to a by path of length 2, we know $N(N(a))$ & $N(b)$ are large.

$|N(N(a))| \geq (\delta - 2\varepsilon)|X|$.

We want to count the # of neighors of $N(b)$ inside $N(N(a))$

$\square$

11/5 - FA

Def: A tree is a cld graph w/ no cycles (w/ at least 2 vertices & $<\infty$). Alternatively, a tree is a graph T where $V(T)$ can be ordered as $v_1 \ldots v_n$ ($n = \#$ of vertices) s.t. the left degree of v_i is at most 1 for each i, i.e. $N_L(v_i) := \{j \mid j<i, v_j v_i \in E(T)\}$ has $|N_L(v_i)| \le 1$.

$v_1 \quad v_2 \quad v_3 \quad v_4 \quad \cdots \quad v_n$

A simple graph is a pair $G = (V,E)$ where V is the set of vertices & $E \subseteq (V \times V \setminus \Delta)/(xy \sim yx) = \binom{V}{2}$. $H \subseteq G$ is a subgraph if $V(H) \subseteq V(G)$ & $xy \in E(H) \Rightarrow xy \in E(G)$. A graph homomorphism $H \to G$ is a map $\phi: V(H) \to V(G)$ & $xy \in E(G) \Rightarrow \phi(x)\phi(y) \in E(G)$. A hom is an iso if ϕ^{-1} exists & is a hom.

Thm: Let $0 < \delta < \frac{1}{2}$. $\exists$ a constant $c(\delta)$ s.t. every graph on N vertices w/ at least δN^2 edges contains an isomorphic copy of every tree T w/ at most $c(\delta) N$ vertices.
[dense graphs contain all small trees].

Thm: The same holds w/ bipartite graphs of bdd degree d, though you will have $c_d(\delta)$. [trees are bipartite]

By our weak regularity lemma, we need to show that any tree can be embedded in an ε-regular pair on say (X,Y) [b/c any graph will have a bipartite subgraph, & you can pass to an ε-regular pair from that subgraph].

Pf: Let T be a tree on n vertices w/ the listing $v_1, \ldots, v_n$, where $|N_L(v_i)| \le 1$ $\forall i$. Also let $\chi: V(T) \to \{1,2\}$ s.t. the induced subgraph on $\chi^{-1}(\{i\})$ has no edges. [Called a proper coloring; can do this b/c trees are bipartite] We inductively find a map $\phi: V(T) \to V(G)$, where G is an ε-regular graph on (X_1, X_2), w/ edge density δ & $|X_1|, |X_2| \ge M$. (to be picked later), s.t.

(i) $\phi(v_i) \in X_{\chi(v_i)}$ & ϕ is 1-1.

(ii) $v_i v_j \in E(T) \Rightarrow \phi(v_i)\phi(v_j) \in E(G)$

(iii) after $\phi(v_j)$ is chosen, we have a set $S^j_k \subseteq X_{\chi(v_k)}$ where any choice of $\phi(v_k) \in S^j_k$ gives an isomorphic copy of the subgraph on the vertices

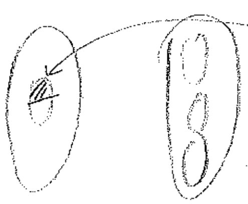
$v_1, \ldots, v_j, v_k$, & $|S_k^j| \geq (\delta-\varepsilon)^{\overbrace{|N_L(v_k) \cap \{v_1,\ldots,v_j\}|}^{=0,1}} \cdot |X_{\chi(v_k)}|$.

To see this is possible, start w/ v_1 & assume $\chi(v_1)=1$, & set $S_k^0 = X_{\chi(v_k)}$. As G is ε-regular, there are at least $(1-\varepsilon)|X_1|$ elts $x \in X_1$ w/ $|N(x)| \geq (\delta-\varepsilon)|X_2|$. (i.e, these x have a lot of neighbors). Pick one of these to be $\phi(v_1)$. Then $S_k^1 = N(\phi(v_1))$ for $v_1 \in N_L(v_k)$, so (iii) holds. For other vertices, we have $S_k^1 = S_k^0$.

Assume that $\phi(v_1), \ldots, \phi(v_{j-1})$ have been chosen to satisfy (i)-(iii). Then we have a set S_j^{j-1} s.t. any choice of $\phi(v_j)$ is good [from (iii)], & $|S_j^{j-1}| \geq (\delta-\varepsilon)|X_{\chi(v_j)}|$. For every right neighbor of v_j say v_k, each has at most 1 left neighbor, i.e. $S_k^{j-1} = X_{\chi(v_k)}$, b/c it's a tree, so (iii) is satisfied. $S_k^j = N(\phi(v_j))$, decreases by same factor as in the first case. Now (ii) & (iii) are clear.

We still need to guarantee this is 1-1: we only need that $(\delta-\varepsilon)|X_i| \geq n$, $i=1,2$, n = size of tree, b/c then we have enough choices for $\phi(v_j)$ s.t. we don't repeat. This gives us the linear bound. M.

For bdd deg graphs, you have not all of your right neighbors in you selection set - need same degree for all sets on rt.



some small percentage - throw this away to get same deg.

11/10 - FA

Thm: Let $\delta > 0$ be given. If $A \subseteq [N]$ w/ δN elts, then $\exists$ a non-trivial 4-term AP in A, provided that $N \geq N_0(\delta)$.

Q: How big is $N_0(\delta)$?

- Szemeredi ('60's) - Not good - tower of length δ^{-1}
- Gowers (~99) - $\delta \gtrsim \frac{1}{(\log(\log N))^c}$ $(c>0)$ [i.e. N a double exp. in δ]

  i.e. $N \gtrsim e^{e^{k\delta^{-c}}}$

- Green-Tao ('08) - $\delta \gtrsim \frac{1}{\log(N)^c}$ $(c>0)$ [i.e. N a single exp.]

Conjecture (Erdös - Turan) If $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$ has $\sum_{n=1}^{\infty} \frac{1}{a_i} = \infty$, then A contains a k-term AP $\forall k$.

- If $c > 1$, conjecture would follow

  [$p_{n+1} - p_n = O(\log(p_n))$ for ∞'ly many n. ← significantly beats average
  density $\delta N \Rightarrow$ ave. gap size expected is $\delta^{-1} = \log(p_n)$]

* What we want is a statement of the form:
  one of the following holds:
  (1) N is small
  (2) A contains approx. the right # of 4-term AP's
  (3) A has increased density on a long AP
 [as in pf of Roth's thm]

Recall: The Gowers U^3 norm of f is $\|f\|^8_{U^3(\mathbb{Z}/N\mathbb{Z})} = \mathbb{E}_{\substack{x,h_1,h_2,h_3 \\ \in \mathbb{Z}/N\mathbb{Z}}} \Delta_{h_1,h_2,h_3} f(x)$, where

$\Delta_{h_1} = \overline{f(x+h)}\, f(x)$

- Generalized von Neumann ineq: $\left| \mathbb{E}_{x,h \in \mathbb{Z}/N\mathbb{Z}} f_1(x) f_2(x+h) f_3(x+2h) f_4(x+3h) \right|$

  $\leq \min_i \|f_i\|_{U^3(\mathbb{Z}/N\mathbb{Z})}$

- If f_i is balanced fcn of A, then we have a way of characterizing (2).

If $\|1_A - \delta\|_{U^3(\mathbb{Z}/N\mathbb{Z})} \leq \eta \ll \delta^4$, then have (2).

We need that, given a fcn f w/ $|f| \leq 1$ & $\|f\|_{U^3(\mathbb{Z}/N\mathbb{Z})} \geq \eta > 0$, then $\exists$ a long AP, P, s.t. $\left|\mathbb{E}_{x\in P} f(x)\right| > c(\eta)$.

* What can we say about f w/ $\|f\|_{U^3(\mathbb{Z}_N)} \geq \eta$?

- Recall $\|f\|_{U^3}^8 = \mathbb{E}_{h\in\mathbb{Z}_N} \|\Delta_h f\|_{U^2}^4$, so if $\|f\|_{U^3}^8 \geq \eta^8$, then $\|\Delta_h f\|_{U^2}^4 \geq \eta^8/2$ ← the L^4-hat norm
  for many h. ↳ Let B = set of these h's. In turn, there are many h w/
  $\|\widehat{\Delta_h f}\|_{L^\infty(\hat{\mathbb{Z}}_N)} \geq c(\eta)$, & in particular, there are pairs $(h, \Phi(h)) \in \mathbb{Z}_N^2$ where $|\widehat{\Delta_h f}(\Phi(h))| \geq c(\eta)$ for $h \in B \subseteq \mathbb{Z}_N$.

The observation of Gowers is that the map Φ has some very nice properties:

- $\exists$ many quadruples $h_1+h_2 = h_3+h_4$ w/ $\Phi(h_1)+\Phi(h_2) = \Phi(h_3)+\Phi(h_4)$.
  Let $T := \{(h, \Phi(h)) \in (\mathbb{Z}/N\mathbb{Z})^2 \mid h \in B\}$. i.e. T contains many additive quadruples. Then, by Balog-Szemeredi, $\exists\, S \subseteq T$ w/ $|S-S| \leq C|S|$ & $|S| > c|T|$ (i.e. large set w/ small difference set)
- Fremann's thm then tells us that S is a dense subset of a proper GAP G of dimension at most d. There is one long AP P (at least as big as $|S|^{1/d}$) & the GAP is $\frac{|G|}{|P|}$ disjoint translates of P. S has the same density on one translate as it does on G.
  Recall: A GAP is $\{x + b_1 n_1 + b_2 n_2 + \cdots + b_r n_r \mid 0 \leq b_i \leq M_i\}$.
  Fix all b_i's but largest, & let that n_i range, get an AP & translates.
- $\{(h_1 + m n_1, \Phi(h_1) + m n_2)\} = T|_P$?

So $\Phi(h)$ = linear fcn of h. But this gives us info about $\Delta_h f$, not about f. We use quadratic approx for f.

11/12 — FA

$\mathbb{Z}/N\mathbb{Z}$, N prime.

Aside: $p_{n+1} - p_n = f(p_n)$. How good can f be? $O(x^{1-\varepsilon})$, for some $\varepsilon > 0$.

- If have $[N]$ & N not prime, use $[N']$ w/ N' prime, & from above, we won't change the density of $A \subseteq [N]$ if we embed $A \subseteq [N']$.

- From the Riemann Hypothesis, $f(x) = O_\varepsilon(x^{1/2+\varepsilon})$ $\forall \varepsilon > 0$

  Kramer's Conjecture: $f(x) = O(\log(x)^2)$.

Let f be a real valued fcn on $\mathbb{Z}_N$ w/ $|f| \le 1$ & w/ $\|f\|_{U^3(\mathbb{Z}_N)} \ge \eta > 0$.

Recall, we then have

$$\|f\|_{U^3}^8 = \mathop{\mathbb{E}}_{h \in \mathbb{Z}_N} \|\Delta_h f\|^4_{U^2(\mathbb{Z}_N)} \ge \eta^8$$

Then there exists a set B w/ $|B| \ge (\eta/2) N$ s.t. $\|\Delta_h f\|^4_{U^2(\mathbb{Z}_N)} \ge \eta^8/2$,

and we have a map $\phi : B \to \mathbb{Z}_N \cong \widehat{\mathbb{Z}_N}$ where $|\widehat{\Delta_h(f)}(\phi(h))| \gtrsim \eta^4$, for all $h \in B$.

Prop: With the above setup, we have $\gtrsim c(\eta) N^3$ additive quadruples $(b_1, b_2, b_3, b_4) \in B$ with $\phi(b_1) + \phi(b_2) = \phi(b_3) + \phi(b_4)$. [i.e. $b_1 + b_2 = b_3 + b_4$]

Pf: Note: $\mathop{\mathbb{E}}_{b \in B} |\widehat{\Delta_b f}(\phi(b))|^2 \ge \eta^8$.) def. of Fourier transform

Thus:
$$\mathop{\mathbb{E}}_{b \in B} \mathop{\mathbb{E}}_{x} \mathop{\mathbb{E}}_{y} f(x) f(x+b) f(y) f(y+b)\, e((x-y)\phi(b)/N)$$

$$= \mathop{\mathbb{E}}_{x,y} f(x) f(y) \mathop{\mathbb{E}}_{b} f(x+b) f(y+b)\, e((x-y)\phi(b)/N)$$

$(x - y = -u)$

$$= \mathop{\mathbb{E}}_{x,u} f(x) f(x+u) \mathop{\mathbb{E}}_{b} f(x+b) f(x+u+b)\, e(-u\phi(b)/N)$$) Δ-ineq. & $|f| < 1$.

$$\le \mathop{\mathbb{E}}_{x,u} \Big| \mathop{\mathbb{E}}_{b} \underbrace{f(x+b) f(x+u+b)}_{\Delta_u f(x+b)} \underbrace{e(-u\phi(b)/N)}_{= 1_B(b) e(-u\phi(b)/N) = h_u(b)} \Big|$$

Use C-S:

$$\eta^{16} \le \mathbb{E}_{x,u} \Big| \underbrace{\mathbb{E}_{b} \Delta_u f(x+b) h_u(b)}_{\substack{\shortparallel \\ F_u(x)}} \Big|^2$$

Calculate $\widehat{F_u}(\xi) = \mathbb{E}_{x} \left(\mathbb{E}_{b} \Delta_u f(x+b) h_u(b) \right) e(x\xi/N)$

$$= \mathbb{E}_{x} \mathbb{E}_{b} \Delta_u f(x+b) h_u(b) e((x+b)\xi/N) e(-b\xi/N)$$

$$\overset{(x+b \to x)}{=} \widehat{\Delta_u f}(\xi) \cdot \widehat{h_u}(-\xi)$$

And, by Parseval,

$$\eta^{16} \le \mathbb{E}_{u} \mathbb{E}_{x} |F_u(x)|^2 = \mathbb{E}_{u} \sum_{\xi} |\widehat{\Delta_u f}(\xi)|^2 \cdot |\widehat{h_u}(\xi)|^2$$

and by C-S:

$$\eta^{32} \le \mathbb{E}_{u} \Big(\underbrace{\sum_{\xi} |\widehat{\Delta_u f}(\xi)|^4}_{\lesssim 1} \Big)^{1/2} \Big(\sum_{\xi} |\widehat{h_u}(\xi)|^4 \Big)^{1/2}$$

$$\le \mathbb{E}_{u} \Big(\sum_{\xi} |\widehat{h_u}(\xi)|^4 \Big)^{1/2}$$

Use C-S again:

$$\eta^{64} \lesssim \mathbb{E}_{u} \Big(\sum_{\xi} |\widehat{h_u}(\xi)|^4 \Big) \overset{\text{def of } h_u}{=} \mathbb{E}_{u} \sum_{\xi} \mathbb{E}_{x_1,x_2,x_3,x_4} \mathbb{1}_B(x_1) \cdots \mathbb{1}_B(x_4)\, e(-\phi(x_1)u/N - \cdots + \phi(x_4)u/N) \cdot e((x_1 + \cdots - x_4)\xi/N).$$

$$= \mathbb{E}_{x_1,\dots,x_4} \underbrace{\mathbb{1}_{\{\phi(x_1)+\phi(x_2) = \phi(x_3)+\phi(x_4)\}}}_{\text{sum over } u} \cdot \underbrace{N \cdot \mathbb{1}_{\{x_1+x_2 = x_3+x_4\}}}_{\text{sum over } \xi}$$

$$= N^{-3} \cdot \#\{\underbrace{(x_1,x_2,x_3,x_4)}_{\text{add. quad.}} \mid \phi(x_1)+\phi(x_2) = \phi(x_3)+\phi(x_4)\}$$

So $cN^3\eta^{64}$ many add. quadruples w/ the desired property. □

Now we set $\Gamma = \{(b, \phi(b)) \mid b \in B\}$, which lives in $\mathbb{Z}_N^2$. Γ has many additive quadruples. Thus, by Balog-Szemeredi & Freiman, this gives us a subset Γ' of Γ which is contained in a GAP, w/ dimension not too large (bdd in terms of η). We can find a translate of the longest AP P where $\Gamma' \cap P$ has the same density. On $\Gamma' \cap P$ we have $\{(b_1 + nv, \phi(b_1) + n\mu)\}_{n=1}^{M}$, an AP. $\Rightarrow$ ϕ linear on $\Gamma' \cap P$.

11/14 - FA $\boxed{\text{Motivation}}$

$\langle f,g\rangle = \mathbb{E}_{x\in\mathbb{Z}_N} f(x)g(x)$, f,g real-valued. We have $|\langle f,g\rangle|^2 = \mathbb{E}_{x,y} f(x)g(x)f(y)g(y)$

← our goal is to get an exp. like this.

$\overset{y=x+h}{=} \mathbb{E}_h \mathbb{E}_x \Delta_h f(x)\Delta_h g(x)$

$= \mathbb{E}_h \langle \Delta_h f, \Delta_h g\rangle$

* f correlates w/ g if $\langle f,g\rangle$ is large.

Recall: $\|f\|_{U^3} \geq \eta \Rightarrow \mathbb{E}_h \langle \Delta_h f, e(\cdot\xi_h)\rangle \geq \eta$

↑ linear in h from our discussion of Φ (on a small region)

$\Downarrow$

$\mathbb{E}_h \mathbb{E}_x \Delta_h f(x) e(x\xi_h) \geq \eta$

b/c: We know smth about $\Delta_h f$ & we want to know about f. If ξ_h linear, we'll get g quadratic.

If ξ_h is of the form $2h$, then $x\xi_h = 2xh$

Note: $\Delta_h(e^{x^2}) = e((x+h)^2 - x^2) = e(2xh+h^2) = \Delta_h(g)$

Szemeredi's Thm: Given any set of integers, finite, say F, & $A \subseteq \mathbb{Z}$ w/ $\bar{\delta}(A) > 0$, then $\exists\ \lambda, x \in \mathbb{Z}$, $\lambda \neq 0$, w/ $x + \lambda F \subseteq A$.

Multi-dim'l version: If $F \subseteq \mathbb{Z}^n$ & $|F| < \infty$, then every set $A \subseteq \mathbb{Z}^n$ w/ $\bar{\delta}(A) > 0$ contains ∞-ly many homothetic copies of F. (transl & dilate)

→ This follows from the special case of F an n-dim'l corner, smth of the form

$\{(x_1,\ldots,x_n), (x_1+h, x_2, \ldots, x_n), \ldots, (x_1,\ldots,x_{n-1}, x_n+h)\}$

Q: Are there 4-term AP's in the set of squares of ~~rationals~~? integers (by clearing denoms)

a^2, b^2, c^2, d^2 No.

$b^2 - a^2 = c^2 - b^2 = d^2 - c^2 \neq 0$

Prop: If $f: \mathbb{Z}_N \to [-1,1]$ and $\mathbb{E}_h |\widehat{\Delta_h f}(2\lambda h)|^2 \geq \eta$, then $\exists\ \xi$ w/

$|\mathbb{E}_x f(x) e\left(\frac{\lambda x^2 + x\xi}{N}\right)| \geq \eta^{1/2}$. [optimal case: Φ linear everywhere]

Pf: $\mathbb{E}_h \mathbb{E}_{x,y} f(x)f(y)f(x+h)f(y+h)e((x-y)2\lambda h/N)$

$(u = x-y) \quad = \mathbb{E}_{x,u,h} f(x)f(x+u)f(x+h)f(x+u+h)e(2\lambda hu/N)$

We use the identity $|x|^2 - |x+u|^2 - |x+h|^2 + |x+u+h|^2 = 2uh$, & set $g(x) = f(x)e(\lambda x^2/N)$ to get

$\mathbb{E}_{u,h}\, g(x)\,g(x+u)\,g(x+h)\,g(x+h+u)$, the U^2-norm, still bdd below by η, so somewhere the Fourier transform of g is large:

$$|\hat{g}(\xi)| \geq \eta^{1/2} \text{ for some } \xi, \text{ or } \left|\mathbb{E}_x f(x)e((\lambda x^2 + x\xi)/N)\right| \geq \eta^{1/2}$$

$\square$.

11/17 - FA

We have ϕ, which gives the large Fourier coeff. for $\Delta_h f$, and there is an arithmetic progression P were ϕ is linear i.e. $\phi(h) = 2\lambda h + \mu$.

Prop: With $\phi(h)$ & h as above, we have that for each $x \in \mathbb{Z}_N$ $\exists\, r_x \in \mathbb{Z}_N$ s.t. $\mathbb{E}_{x\in\mathbb{Z}_N} \left| \mathbb{E}_{h\in P+x} f(h) e((\lambda h^2 + r_x h)/N) \right| \geq \eta^2$ (where we had $\|f\|_{U^3} \geq \eta$).

($|f| \leq 1$)

$\uparrow$ family of quadratics, one for each translate of P

Pf: Expand $\mathbb{E}_{h\in P} |\widehat{\Delta_h f}(2\lambda h + \mu)|^2 \geq \eta^2$ to get

$$\mathbb{E}_{h\in P} \mathbb{E}_{x,y} f(x) f(y) f(x+h) f(y+h) e((x-y)(2\lambda h + \mu)/N)$$

$x-y=u$

$$= \mathbb{E}_{h\in P} \mathbb{E}_{x,u} f(x) f(x+u) f(x+h) f(x+u+h) e(u(2\lambda h + \mu)/N).$$

Split the u-sum: $\mathbb{E}_{u\in\mathbb{Z}_N} g(u) = \mathbb{E}_{l\in P} \mathbb{E}_{y\in\mathbb{Z}_N} g(l+y)$, to get

$$= \mathbb{E}_{x,y\in\mathbb{Z}_N} \mathbb{E}_{h,l\in P} f(x) f(x+h) f(x+l+y) f(x+h+l+y) e((l+y)(2\lambda h + \mu)/N)$$

And for some y_0, we have $\swarrow$ from Δ-ineq.

$$\mathbb{E}_{x\in\mathbb{Z}_N} \left| \mathbb{E}_{x,l\in P} f(x+h) f(x+l+y_0) f(x+h+l+y_0) e\left(\frac{2\lambda h l}{N} + \frac{\mu l}{N} + \frac{2\lambda h y_0}{N}\right) \right| \geq \eta^2$$

Use $2\lambda h l = \lambda((h+l)^2 - h^2 - l^2)$, & collect 3 fcns,

$$\left.\begin{aligned} g_1(h,x) &= f(x+h) e(-\lambda h^2/N + 2\lambda h y_0/N), && \text{supp} = P \\ g_2(l,x) &= f(x+l+y_0) e(-\lambda l^2/N + \mu l/N), && \text{supp} = P \\ g_3(h+l,x) &= f(x+h+l+y_0) e(\lambda(h+l)^2/N), && \text{supp} = P+P \end{aligned}\right\} \begin{array}{l}\text{in terms of } h - \\ \text{think of } x \text{ as} \\ \text{fixed}\end{array}$$

The inner sum becomes

$$\frac{N^2}{|P|^2} \mathbb{E}_{h,l\in\mathbb{Z}_N} g_1(h) g_2(l) g_3(h+l)$$

$\uparrow$ (renormalize so $h,l \in \mathbb{Z}_N$ but they're supp on P, so ok)

$\rightarrow$ L^∞, then C-S, then Plancherel/Parseval

$$= \frac{N^2}{|P|^2} \sum_{\xi\in\hat{\mathbb{Z}}_N} \hat{g}_1(\xi) \hat{g}_2(\xi) \hat{g}_3(-\xi) \leq \frac{N^2}{|P|^2} \max_\xi |\hat{g}_1(\xi)| \cdot \underbrace{\|g_2\|_{L^2(\mathbb{Z}_N)}}_{\leq \frac{|P|^{1/2}}{N^{1/2}}} \cdot \underbrace{\|g_3\|_{L^2(\mathbb{Z}_N)}}_{\leq \frac{(|P|+|P|)^{1/2}}{N^{1/2}} \leq \frac{(2|P|)^{1/2}}{N^{1/2}}}$$

Now we have

$$\frac{N}{|P|} \max_{\xi} |\hat{g}_1(\xi)| \gtrsim \eta^2$$

This means

$$\Big| \mathop{\mathbb{E}}_{h\in P} f(x+h) e(-\lambda h^2/N + 2\lambda h y_0/N)\, e(h\xi/N) \Big| \gtrsim \eta^2$$

Shift $h+x \to h$, & we get smth of the form $e(-\lambda h^2/N + \underset{\swarrow \text{ dep on } x, \lambda, \mu, \text{ etc}}{r_x} h)$

[Note: If it doesn't depend on h, it disappears in the abs. val.] $\square$

<u>Recall</u>: If $\psi_1(x) = \alpha x$ is linear & P is a progression of length R in $\mathbb{Z}_N$, then we can partition P into $\{P_i\}_{i=1}^m$, where each $|P_i| \approx R^{1/4}$, P_i an AP, and $|e(\psi_1(x)/N) - e(\psi_1(y)/N)| < R^{-1/4}$ $\forall x,y \in P_i$ $\forall i$. (i.e. AP's on which the characters are almost constant). This follows from: Dirichlet's approx thm: $\exists q \le R^{1/2}$ w/ $\|q\alpha/N\| \le R^{-1/2}$. We need an analogue of this, which led to the notion of Bohr sets, which we will also find an analogue of.

For $\psi_2(x) = \alpha x^2 + \beta x$, we have a consequence of Weyl's thm:
$\exists q \le R^{1/2}$ s.t. $\|q^2\alpha/N\| \le R^{-1/8}$. So we then split P into (mod q) progressions and split these into progressions of size $R^{1/32}$. On some P_i, we have

$$\psi_2(a+jq) = \underbrace{\alpha a^2 + \beta a}_{\substack{\text{disappear} \\ \text{in difference}}} + \underset{\substack{\uparrow \\ \approx \\ \text{const}}}{j^2 q^2 \alpha} + j\,\underbrace{(2\alpha a q + \beta q)}_{\substack{\text{repartition these for where it's constant.} \\ \text{(linear again)}}}$$

In the end, they're approx. same on progression of size $R^{1/128}$

<u>Prop</u>: P is our progression (from ϕ) & let $|P| = R$. For each $x \in \mathbb{Z}_N$, we have a partition of $P+x$ into $\{P_{x,i}\}_{i=1}^M$ s.t. $\mathop{\mathbb{E}}_{x\in\mathbb{Z}_N} \mathop{\mathbb{E}}_{j=1}^{M} \Big|\mathop{\mathbb{E}}_{h\in P_{x,j}} f(h)\Big| \ge c(\eta) > 0$

11/19 - FA

$\underbrace{\mathbb{E}_{x\in\mathbb{Z}_N}\mathbb{E}_{i=1}^{m}}_{\text{runs through a uniform covering of } \mathbb{Z}_N \text{ of size } m.} \left|\mathbb{E}_{h\in P_{i,x}} f(x)\right| \geq c(\eta)$, where $P+x$ has been partitioned into $\{P_{i,x}\}_{i=1}^{m}$, on which the phase is approx constant

So $\mathbb{E}_x \mathbb{E}_i \mathbb{E}_h f(x) = 0$

Thus we can add the 2 & so, w/ $f = 1\!\!1_A - \delta$,

$\exists$ some i,x w/ $|A\cap P_{i,x}| \geq \left(c(\eta)/2+\delta\right)\cdot|P_{i,x}|$

We start with $A\subseteq[N]$ w/ δN elts. We want to pass to $\mathbb{Z}_N$ s.t. AP's in $\mathbb{Z}_N$ lift to actual AP's in A. Let's adjust N s.t. $5|N$.
Break up [N] into sets:

$$[1,\tfrac{N}{5}],\ (\tfrac{N}{5},..,\tfrac{2N}{5}],\ \dots,\ (\tfrac{4N}{5},..,N]$$

Let

$$\left[\tfrac{2N}{5},\tfrac{3N}{5}\right]\cap A = A_1$$

To avoid overlapping we can count

$$C = \mathbb{E}_{x,h\in\mathbb{Z}_N} 1\!\!1_{A_1}(x)\,1\!\!1_{A_1}(x+h)\,1\!\!1_A(x+2h)\,1\!\!1_A(x+3h),$$ b/c then $h\leq N/5$

& so $0\leq x+2h,\ x+3h\leq N$.

If A_1 has only a few pts, say $|A_1|\leq \delta/10\cdot N$. Then $|A\setminus A_1|\geq \frac{9}{10}\delta N$, & so, wlog, $[1,2N/5]$ has $\geq\frac{9}{20}\delta N$ pts, & thus the density of $A\cap[1,2N/5] = \frac{(9/20)\delta N}{2/5\, N} = \frac{9}{8}\delta$. So we have a density increment, & repeat the argument, which we can only do fin. many times.

Else, we assume $\|f\|_{U^3}\leq\eta$. (for $\gamma\geq\eta$, get a density increment, as above).

Then, $C = \mathbb{E}_{x,h} 1_{A_1}(x) 1_{A_1}(x+h) 1_A(x+2h) \overset{= 1_A - \delta}{f_A}(x+3h) + \delta \mathbb{E}_{x,h} 1_{A_1}(x) 1_{A_1}(x+h) 1_A(x+2h)$

$$= \mathbb{E}_{x,h} 1_{A_1}(x) 1_{A_1}(x+h) 1_A(x+2h) f_A(x+3h)$$

$$+ \delta \mathbb{E}_{x,h} \cdots f_A(x+2h) + \delta^2 \mathbb{E}\, 1_{A_1}(x) 1_{A_1}(x+2h)$$

$= \frac{\delta^4}{100} + O(\delta \|f\|_{U^3})$, so we get the right count of AP's in A_1, which will lift to AP's in A. $\square$

*We actually need $N \gg e^{e^{\delta^{-2^{20}}}}$.

This argument generalizes to k-term progressions:

We need: $\mathbb{E}_x \left| \mathbb{E}_{h \in P} f(h) e(\phi_x(h)) \right| \geq c(\eta)$ if $\|f\|_{U^k} \geq \eta$. Then, with this statement, the proof goes through. $\phi_x(h)$ will be a polynomial of degree $< k-1$. This is harder to prove.

Recall: For 3-AP's, ϕ was linear

4-AP's ϕ was quadratic

It was conjectured that there was an inverse statement, that is $\left|\mathbb{E}_x f(x) e(\phi(n))\right| \geq c(\eta)$. But this is false. However, Green-Tao-Ziegler ('11) give a substitute, which leads to a proof of Szemeredi. The inverse statement is true in a finite field.

Question: Let $A \subseteq \mathbb{N}$, $|A| = n$. What is the largest subset $B \subseteq A$ s.t. B is sum-free, i.e. $\nexists\, x,y,z \in B$ w/ $x+y=z$?

Map $A \to [0,1)$. Note that the interval $[1/3, 2/3)$ is sum-free on $[0,1)$. Pick $\theta \in (0,1)$ s.t. $|\theta A \cap [1/3, 2/3)| \geq n/3$. Then our map is $a \mapsto \theta a$. We want the θA part to be about $1/3$.

[due to Erdös]

Alon & Kleitman: $n/3 + 1$

Bourgain: $n/3 + 2$

Erdös's Conjecture: If $n/3 + g(n)$, for some g, then $g(n) = o(n)$.

Proved by Eberhard-Green-Manners

11/24 - FA

Weighted Generalized von Neumann Inequality. We need to define a quantity:

Set Ψ_k to be the collection of systems of linear forms, $\mathcal{L}:(l_1,\dots,l_j): \mathbb{Z}_N^n \to \mathbb{Z}_N^j$, s.t. (l_i, l_m) are linearly independent for all $i,m \le j$, where $j \le k$ & $n \le 2k$. For a function ν on $\mathbb{Z}_N$, w/ $\mathbb{E}_{x\in\mathbb{Z}_N} \nu(x) = 1$, define $\tau_k(\nu) = \sup_{\mathcal{L}\in\Psi_k} \left| \mathbb{E}_{x\in\mathbb{Z}_N^n} \nu(l_1(x))\cdots\nu(l_j(x)) - 1 \right|$.

Ex: We care about

$$\mathbb{E}_{x,h_1,h_2} \nu(x)\nu(x+h_1)\nu(x+h_2)\nu(x+h_1+h_2) = \|\nu\|^4_{U^2(\mathbb{Z}_N)}$$

This should be close to 1 if $\tau_k(\nu)$ is small.
$\hookrightarrow \approx 1 + O(\tau_k)$

Generalized von Neumann: Let f be a fcn on $\mathbb{Z}_N$ & $0 \le |f| \le \nu$, where $\mathbb{E}_x \nu = 1$. Then $\mathbb{E}_{x,h} f_1(x) f_2(x+h)\cdots f_k(x+(k-1)h) = (1+O_k(\tau_k))\|f_i\|_{U^k(\mathbb{Z}_N)} + O_k(\tau_k)$
$[\tau_k = \tau_k(\nu)]$

Pf ($k=3$): (C-S argument) Set $T \equiv T(f_1,f_2,f_3) = \mathbb{E}_x f_1(x)f_2(x+h)f_3(x+2h)$, where $0 \le |f_1|, |f_2|, |f_3| \le \nu$. We have

$$|T| \le \mathbb{E}_x |f_1(x)| \left|\mathbb{E}_h (f_2(x+h)f_3(x+2h))\right| \le \mathbb{E}_x \nu(x) \left|\mathbb{E}_h f_2(x+h)f_3(x+2h)\right|$$

Write $\nu(x) = \nu^{1/2}(x)\cdot\nu^{1/2}(x)$, so C-S will give an L^1-norm of ν, not an L^2-norm of ν.

Then, by C-S,

$$T^2 \le \Big(\underbrace{\mathbb{E}_x \nu(x)}_{=1}\Big)\Big(\mathbb{E}_x \nu(x) \mathbb{E}_{h_1,h_2} f_2(x+h_1)f_2(x+h_2)f_3(x+2h_1)f_3(x+2h_2)\Big)$$

$$\overset{(h_2 = h_1+h_2)}{=} \mathbb{E}_x \mathbb{E}_{h_1,h_2} \nu(x) f_2(x+h_1) f_2(x+h_1+h_2) f_3(x+2h_1) f_3(x+2h_1+2h_2)$$

$$\overset{(x = x-h_1)}{=} \mathbb{E}_x \mathbb{E}_{h_1,h_2} \nu(x-h_1) f_2(x) f_2(x+h_2) f_3(x+h_1) f_3(x+h_1+2h_2)$$

$$\overset{(\Delta\text{-ineq})}{\le} \mathbb{E}_{x,h_2} \nu(x)\nu(x+h_2) \left|\mathbb{E}_{h_1} \nu(x-h_1) f_3(x+h_1) f_3(x+h_1+2h_2)\right|$$

$h_1 = x + h_1$

$$= \mathop{\mathbb{E}}_{x,h_2} \nu(x)\nu(x+h_2)\left|\mathop{\mathbb{E}}_{h_1} \nu(2x-h_1) f_3(h_1) f_3(h_1+2h_2)\right|$$

<u>Note</u>: we have 5 linear forms, which are pairwise lin. ind.

Again, write $\nu(x) = \nu^{1/2}(x)\nu^{1/2}(x)$ & same for $\nu(x+h_2)$

Then by C-S,

$$T^4 \le \Big(\underbrace{\mathop{\mathbb{E}}_{x,h_2} \nu(x)\nu(x+h_2)}_{\le(\tau_2)}\Big)\Big(\mathop{\mathbb{E}}_{x,h_1,h_2,h_3} \nu(x+h_2)\nu(x)\nu(2x-h_1)\nu(2x-h_3) f_3(h_1) f_3(h_3) f_3(h_1+2h_2) f_3(h_3+2h_2)\Big)$$

$\left.\begin{array}{l} h_3 = h_1 + h_3' \\ 2h_2 \to h_2 \end{array}\right\} \Rightarrow$ $f_3(h_1)\cdots f_3(h_3+2h_2)$ is the expression in the Gower's Norm. Doesn't depend on x.

$$\le (1 + \overset{\text{in terms of } \tau_2}{\dots}) \mathop{\mathbb{E}}_{h_1,h_2,h_3} W(h_1,h_2,h_3) f_3(h_1)\cdots f_3(h_3+2h_2), \text{ where}$$

$$W(h_1,h_2,h_3) = \mathop{\mathbb{E}}_{x} \nu(x)\cdots\nu(2x-h_3).$$

- If $W(h_1,h_2,h_3) = 1$, then 2^{nd} part is just $\|f_3\|_{U^2}^4$. So let's compare W to 1, & try to bound

$\mathop{\mathbb{E}}_{h_1,h_2,h_3}(W(h_1,h_2,h_3) - 1) f_3(h_1)\cdots f_3(h_1+2h_3)$. Note that a bound on

(*) $\mathop{\mathbb{E}}_{h_1,h_2,h_3} |W(h_1,h_2,h_3) - 1| \nu(h_1)\cdots\nu(h_1+2h_3)$ is sufficient, by Δ-ineq.

By C-S, writing $\nu = \nu^{1/2}\cdot\nu^{1/2}$ for each, we have

$$(*) \le \Big(\underbrace{\mathbb{E}\, \nu(h_1)\cdots\nu(h_1+2h_3)}_{(\le(\tau_4))}\Big)\Big(\mathop{\mathbb{E}}_{h_1,h_2,h_3} \nu(h_1)\cdots\nu(h_1+2h_3)|W(h_1,h_2,h_3) - 1|^2\Big)$$

and the latter sum we expand:

$$\mathop{\mathbb{E}}_{h_1,h_2,h_3} \mathop{\mathbb{E}}_{x,x'} \nu(x)\nu(x')\cdots\nu(2x-h_3)\nu(2x'-h_3)\nu(h_1)\cdots\nu(h_1+2h_3)$$

$$-2 \mathop{\mathbb{E}}_{h_1,h_2,h_3} \nu(x)\cdots\nu(2x-h_3)\nu(h_1)\cdots\nu(h_1+2h_3)$$

$$+ \mathop{\mathbb{E}}_{h_1,h_2,h_3} \nu(h_1)\cdots\nu(h_1+2h_3) = (1-2+1) + O(\tau_{16})$$

b/c all pairwise lin ind. forms above.

$\square$

<u>Note</u>: Same for $k>3$; just C-S, & repeat the linear forms argument

Problem: Before, we had $\|\mathbb{1}_A - \alpha\|_{U^k}$ was ok to work with, since $|\mathbb{1}_A - \alpha| \leq \underbrace{1+\alpha}_{\leq 2}$, & we can use this (gen. von N) for $f = \frac{\mathbb{1}_A - \alpha}{2}$, so $|f| \leq 1$. For suitable fcns ν, μ, we have that $\frac{\nu+\mu}{2}$ is also going to be suitable (i.e. has small τ_k). We'll show this, & that small $\tau_k \Rightarrow$ small Gower's norm. Finally, we need to know what we should replace α with — since we'll be looking at sparse sets (i.e. the primes).

12/3 - FA

Recall the Gowers norms are:

$$\|f\|_{U^k}^{2^k} = \mathbb{E}_{x,h_1,\dots,h_k} \Delta_{h_1,\dots,h_k} f(x)$$

$$= \mathbb{E}_{x,h_1,\dots,h_k} \prod_{\omega\in\{0,1\}^k} f(x+\omega\cdot h)$$

We can write $\|f\|_{U^k}^{2^k} = \langle f, Df\rangle$, where $Df(x) = \mathbb{E}_{h_1,\dots,h_k} \prod_{\substack{\omega\in\{0,1\}^k \\ \omega\neq 0}} f(x+\omega\cdot h)$

Prop: Let ν be "pseudo-random" & $F_1,\dots,F_K \le \nu$. Then

$\|DF_1\cdots DF_K\|_{U^k}^{*} = O_K(1)$

Pf: We need to show that $\langle f, DF_1\cdots DF_K\rangle = O_K(1)\cdot\|f\|_{U^k}$

$$\langle f, DF_1\dots DF_K\rangle = \mathbb{E}_{x\in\mathbb{Z}_N} f(x)\prod_{i=1}^{K}\Big(\mathbb{E}_{h^{(i)}\in\mathbb{Z}_N^k}\prod_{\omega\neq 0} F_i(x+\omega\cdot h^{(i)})\Big)$$

Add a redundant sum over $h\in\mathbb{Z}_N^k$ and write $h^{(j)} = h+H^{(j)}$ for each $j\ge k$.

The average over h gives

$$\mathbb{E}_{\substack{x\in\mathbb{Z}_N \\ h\in\mathbb{Z}_N^k}} \mathbb{E}_{H^{(1)}}\cdots\mathbb{E}_{H^{(K)}} f(x)\prod_{\omega\neq 0} F_1(x+\omega\cdot H^{(1)}+\omega\cdot h)\cdots F_K(x+\omega\cdot H^{(K)}+\omega\cdot h)$$

Let $f_{0,H}(x) = f(x)$, & $\underset{\omega\neq 0}{f_{\omega,H}(x)} = F_1(x+\omega\cdot H^{(1)}+\omega\cdot h)\cdots F_K(x+\omega\cdot H^{(K)}+\omega\cdot h)$

This is an expectation of a Gowers inner product

$$\mathbb{E}_{H\in\mathbb{Z}_N^{kK}} \langle (f_{\omega,H})_{\omega\in\{0,1\}^k}\rangle$$

By Gowers-Cauchy-Schwartz,

$\le \|f\|_{U^k}\,\mathbb{E}_H \prod_{\omega\neq 0}\|f_{\omega,H}\|_{U^k}$ Use Holder on the 2^k-1 $f_{\omega,H}$'s (& since we're in a prob. sp, norms are incr)

& it suffices to show $\mathbb{E}_H \|f_{\omega,H}\|_{U^k}^{2^k} = O_K(1)$ for $\omega\neq 0$.

With $\omega\neq 0$, $\omega\cdot H$ is a uniform cover of $\mathbb{Z}_N^K$, & then we need to estimate $\mathbb{E}_{x\in\mathbb{Z}_N^k}\mathbb{E}_{x,U^{(1)},\dots,U^{(K)}\in\mathbb{Z}_N}\prod_{j=1}^{K}\prod_{\omega\in\{0,1\}^k} F_j(x+U^{(j)}+h\cdot\omega)$.

$\mathbb{E}_{h\in\mathbb{Z}_N^k} \mathbb{E}_x \prod_{j=1}^{k} \mathbb{E}_{u^{(j)}} \prod_{\omega} F_j(x + u^{(j)} + h\cdot\omega)$. The x-sum is redundant (subst.)

$$\leq \mathbb{E}_h \Big(\underbrace{\mathbb{E}_u \prod_{\omega} \nu(u + h\cdot\omega)}_{} \Big)^k$$

summing over shifts of u, so by the Δ-ineq in L^k, & the condition that $\mathbb{E}_{u\in\mathbb{Z}_N} \nu(u+h\cdot\omega) \leq \sum_{\substack{\omega,\omega'\in \\ \{0,1\}^k,\ \omega\neq\omega'}} \tau(h\cdot(\omega-\omega'))$

from the corelation condition

So, $\mathbb{E}_h \big(\mathbb{E}_{u\in\mathbb{Z}_N} \nu(u+h\cdot\omega)\big) \leq O(k)\, \mathbb{E}_{x\in\mathbb{Z}_N} \tau^k(x)$ because $h\cdot(\omega-\omega')$ is a uniform cover of $\mathbb{Z}_N$.

The correlation condition then completes the proof. $\square$

<u>QAP Norm</u> (Quasi-algebraic-predual) (Gowers) We have that $\|\cdot\|$ on $\mathbb{R}^n$ is a QAP w/ respect to X, a bounded set, if $\exists$ an operator D, an increasing fcn C, and a decr. fcn c s.t.

(1) $\langle f, Df\rangle \leq 1 \ \forall f\in X$.

(2) $\langle f, Dg\rangle \geq c(\varepsilon)$ for some $g\in X$ whenever $\|f\|>\varepsilon$.

(3) $\|DF_1 \cdots DF_k\|^* \leq C(k)$

For us, X = all fcns bdd by given pseudorandom fcn ν, D is as above (let $g=f$ in (2), (3) discussed above)

<u>Prop</u>: The Gowers norms are QAP norms for appropriate X.

To finish the relative Szemeredi, we only need the proof of the transference principle: Then we have $f = g + h$, $0\leq f\leq \nu$, where $|g|\leq 2$ & $\|h\|_{U^k} \leq \eta$ (small given parameter). From generalized von Neumann,

12/5 - FA

<u>Relative Szemeredi Thm</u>: Let ν be a pseudo-random measure. If $0 \le f \le \nu$ & $\mathbb{E}_{x \in \mathbb{Z}_N} f \ge \delta$ for some fixed δ, then

$$\mathbb{E}_{x \in \mathbb{Z}_N} \underbrace{f(x) \cdots f(x+(k-1)h)}_{k\text{-term AP}} \ge C_k(\delta).$$

<u>Ex</u>: Take a random subset, A, of $\mathbb{Z}_N$, w/ prob. $p = \frac{1}{\log N}$. $\mathbb{1}_A$ not a pseudo-random measure b/c expectation not $= 1$. But we "expect" $\nu = \log(N)\, \mathbb{1}_A$ is (for some fixed k, w/ N suff. large)

* we'd like to get AP's in sparse sets, which is why we chose $1/\log N$

From the p-r condition, A should have long AP's, & in fact, if $B \subseteq A$ w/ $\mathbb{E}_{x \in A} \mathbb{1}_B \ge \delta$, then B contains "long" AP's (morally).

We're extending our results from dense subset to subsets that sit inside subsets that are pretty much random.

We'll have $B = \{\text{primes}\}$ & we'll need to construct an A w/ our desired properties.

<u>von Mangoldt Function</u> ($\approx$; we'll ignore prime powers)

$$\Lambda(x) = \begin{cases} \log x & \text{if } x \text{ prime} \\ 0 & \text{else} \end{cases}$$

<u>Prime Number Thm</u>: $\mathbb{E}_{x \in [N]} \Lambda(x) = 1 + o(1)$.

<u>PNT in AP's</u>: $\mathbb{E}_{x \in [N]} \Lambda(x) \cdot \mathbb{1}_{x \equiv a \pmod b} = \frac{1}{\phi(b)} + o(1)$

where $(a,b) = 1$, & $b \le \log(N)$

Fix $W = \prod_{p \le \omega(N)} p$ (p prime, $\omega(N) = o(\log(\log N))$; $\frac{1}{\omega(N)} \to 0$ as $N \to \infty$)

- we'll remove contribution of all primes smaller than $\omega(N)$.

<u>Def</u>: Let the modified von Mangoldt fcn be

$$\tilde{\Lambda}(x) = \begin{cases} \frac{\phi(W)}{W} \log(Wx+1) & Wx+1 \text{ prime} \\ 0 & \text{else} \end{cases}$$

Note that $\underset{x \in [N/W]}{\mathbb{E}} \tilde{\Lambda}(x) = 1 + o(1)$ b/c # primes $\equiv 1 \pmod{w} \le N$

$\rightarrow \frac{N}{\phi(w)} \cdot \frac{w}{N} \cdot \frac{\phi(w)}{w} \quad \ldots$

$\uparrow$ normalization

Goldston - Yildirim :

We want to find an approximation to Λ. We use :

$$\Lambda(x) = \sum_{d \mid x} \mu(d) \log(x/d) \quad , \quad \mu = \text{Mobius fcn},$$

$$\mu(d) = \begin{cases} +1 & \text{if even \# of distinct prime factors} \\ -1 & \text{if odd \# ''} \\ 0 & \text{if } d \text{ div. by square prime } p^2 \mid d \end{cases}$$

'' coeff's of $\prod_p (1-p)$

$$\prod_p (1-p^{-s}) = \frac{1}{\zeta(s)}$$

$$\left[\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \right]$$

- Truncated divisor sum :

$$\Lambda_R(x) = \sum_{\substack{d \mid x \\ d \le R}} \mu(d) \log(R/d)$$

take positive part of log.

$$= \sum_{d \mid x} \mu(d) \log(R/d)_+ \quad \left(\text{if } d \ge R, \ \log(R/d) \text{ goes away}_{=0}\right)$$

[We use $R = N^{\gamma}$, small power depending on k, later on]

- Set

$$\nu(x) = \begin{cases} \frac{\phi(w)}{w} \cdot \frac{\Lambda_R^2(wx+1)}{\log R}, & x \in [\varepsilon_k N, 2\varepsilon_k N] \\ 1, & \text{else} \end{cases}$$

In $[\varepsilon_k N, 2\varepsilon_k N]$, we WTS $\nu(x) \gtrsim \tilde{\Lambda}(x)$. We have, if $wx+1$ prime, (b/c else $\tilde{\Lambda}(x) = 0$)

b/c our primes are $\ge R$, & only 2 divisors

$$\nu(x) = \frac{\phi(w)}{w} \cdot \frac{\log(R)^2}{\log R} = \frac{\phi(w)}{w} \cdot \log R \approx \gamma_k \log(N) ; \text{ so}$$

$\uparrow R = N^{\gamma}$

$$\tilde{\Lambda}(x) \le \frac{1}{\gamma_k} \nu(x)$$

<u>Note</u> $\tilde{\Lambda}$ will be our f, & we need $f \le \nu$ for Rel. Sz. Thm

12/8 · FA

Recall: Truncated Divisor Sum

$$\Lambda_R(x) = \sum_{d \mid x} \mu(x) \log(R/d)_+$$

μ: Möbius fcn; $(\cdot)_+$: if $d>R$, take 0

We want correlation estimates:

Let $\psi_1, \dots, \psi_m$ be linear functions, $\psi_i = \sum_{j=1}^{t} L_{ij} x_j + b_i$, with:

- $\lambda_1 \psi_i + \lambda_2 \psi_j \neq 0$ for any $i \neq j$ & $(\lambda_1, \lambda_2) \neq (0,0) \in \mathbb{Q}^2$ (pairwise ind.)
- the L_{ij}'s are rational & $|L_{ij}| \leq \frac{\sqrt{w(n)}}{2}$ (where $w(n) \to \infty$ slowly as before)

Let $W = \underbrace{\prod_{p \leq w(n)} p}_{\text{prod. of small primes}}$ & $\theta_i = W\psi_i + 1$. We want the following estimate:

$$\mathop{\mathbb{E}}_{x \in B} \Lambda_R^2(\theta_1(x)) \cdots \Lambda_R^2(\theta_m(x)) = (1 + \underbrace{o(1)}_{= o_{m,t}(1)}) \left(\frac{W \log R}{\varphi(W)} \right)^m$$

φ: Euler totient fcn

B: box of suff. large size in $\mathbb{Z}^t$ $= \prod_j I_j$ where $|I_j| \geq R^{10m}$

By the def of Λ_R,

$$\text{LHS} = \mathop{\mathbb{E}}_{x \in B} \prod_{i=1}^{m} \sum_{\substack{d_i, d_i' \leq R \\ d_i, d_i' \mid \theta(x)}} \mu(d_i)\mu(d_i') \log(R/d_i)_+ \log(R/d_i')_+$$

$$= \sum_{d_1, \dots, d_m' \leq R} \prod_{i=1}^{m} \mu(d_i)\mu(d_i') \log(R/d_i) \log(R/d_i') \underbrace{\mathop{\mathbb{E}}_{x \in B} \prod_{i=1}^{m} \mathbb{1}_{d_i, d_i' \mid \theta_i(x)}}_{\text{periodic in } D = \operatorname{lcm}(d_1, \dots, d_m') \leq R^{2m} \text{ b/c each } d_i, d_i' \leq R} \quad (*)$$

Rather summing over the box B, we tile it w/ copies of $(\mathbb{Z}/D)^t$.

So,

$$\mathop{\mathbb{E}}_{x \in B} \mathbb{1}_{d_i, d_i' \mid \theta_i(x)} = \mathop{\mathbb{E}}_{x \in \mathbb{Z}_D^t} \mathbb{1}_{d_i, d_i' \mid \theta_i(x)} + \underbrace{O(R^{-8m})}_{\text{error from tiling}}$$

The error of using ↗ in $(*)$ is $O(R^{-6m} \log(R)^{2m})$. So we need to estimate:

$$\sum_{d_1,\dots,d_m' \le R} \left(\prod_{i=1}^{m} \mu(d_i)\mu(d_i') \cdots \log(R/d_i')_+ \right) \mathbb{E}_{x \in \mathbb{Z}_D^t} \mathbb{1}_{d_i,d_i' | \theta_i(x)}$$

($\mu(d_i) \neq 0$ when d_i square-free.)

The Chinese Remainder Thm gives us

$$\mathbb{E}_{x \in \mathbb{Z}_D^t} \prod_{i=1}^{m} \mathbb{1}_{d_i,d_i' | \theta_i(x)} = \underbrace{\prod_{p | D}} \mathbb{E}_{x \in \mathbb{Z}_D^t} \prod_{i:\, p | d_i, d_i'} \mathbb{1}_{\theta_i(x) \equiv 0 \pmod p} \quad (\dagger)$$

can just be $\prod_p$, b/c if $p \nmid D$, this prod. is empty so $=1$.

Now we define

$$\omega_X(p) = \mathbb{E}_{x \in \mathbb{Z}_D^t} \prod_{i \in X} \mathbb{1}_{\theta_i(x) \equiv 0 \pmod p} \quad (\le 1), \text{ where}$$

$$X_{d_1,\dots,d_m}(p) = \{ 1 \le i \le m \mid p | d_i \}$$

Then we have

$$(\dagger) = \prod_p \omega_{X_{d_1,\dots,d_m}(p) \cap X_{d_1',\dots,d_m'}(p)}(p) \qquad [\mu \text{ is multiplicative}]$$

So, we have

$$(\dagger\dagger) \sum_{d_1,\dots,d_m' \in \mathbb{Z}} \prod_{i=1}^{m} \mu(d_i)\mu(d_i') \log(R/d_i)_+ \log(R/d_i')_+ \prod_p \omega_{X_{d_1,\dots,d_m}(p) \cap X_{d_1',\dots,d_m'}(p)}(p)$$

(not $\le R$ b/c if $> R$, $\log(R/d_i) = 0$)

We'll use an integral representation (a Mellin transform) for $\log(x)_+$:

$$\log(x)_+ = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{x^z}{z^2}\, dz \quad \text{where } \Gamma_1 = \frac{1}{\log R} + it, \quad -\infty < t < \infty,$$ a vertical contour in the right half plane.

Then,

$$(\dagger\dagger) = \left(\frac{1}{2\pi i}\right)^{2m} \int_{\Gamma_1} \cdots \int_{\Gamma_1} F(z,z') \prod_{i=1}^{m} \frac{R^{z_i + z_i'}}{z_i^2 z_i'^2}\, dz_i\, dz_i' \quad \text{for } z = (z_1,\dots,z_m),\ z' = (z_1',\dots,z_m'), \text{ \&}$$

$$F(z,z') = \sum_{d_1,\dots,d_m'} \left(\prod_{i=1}^{m} \frac{\mu(d_i)\mu(d_i')}{d_i^{z_i} d_i'^{z_i'}} \right) \prod_p \omega_{X_{d_1,\dots,d_m}(p) \cap X_{d_1',\dots,d_m'}(p)}(p)$$

$$= \prod_p E_p(z,z'), \text{ where } E_p(z,z') = \sum_{X,X' \subseteq \{1,\dots,m\}} \frac{(-1)^{|X|+|X'|} \omega_{X \cap X'}(p)}{p^{\sum_{i \in X} z_i + \sum_{i \in X'} z_i'}}$$

Recall: $\displaystyle\sum_{d=1}^{\infty} \frac{\mu(d)}{d^z} = \frac{1}{\zeta(z)}$

So $F_p(z,z')$ is well-def. on $\{\mathrm{Re}(z_1),\dots,\mathrm{Re}(z_m') > 1\} \subseteq \mathbb{C}^{2m}$

So F conv. absolutely in that region.

12/12 - FA

$D_\sigma^m = \{z_j, z_j' \mid -\sigma < Re(z_j), Re(z_j') \leq 100\}$

$$\|f\|_{C^k(D_m^\sigma)} = \sup_{\alpha_1, \ldots, \alpha_m'} \|\partial z_1^{\alpha_1} \cdots \partial z_m'^{\alpha_m'} f\|_{L^\infty(D_\sigma^m)}$$

over all $\alpha_i \geq 0$
$\sum_{i=1}^m (\alpha_i + \alpha_i') \leq k$

<u>Lemma</u>: $R>0$ is real, $G(z,z')$ is analytic on D_σ^m for some $\sigma<0$, & $\|G\|_{C^m(D_\sigma^m)} = \exp(O_{m,\sigma}((\log R)^{1/3}))$. Then

$$(2\pi i)^{-2m} \int_{\Gamma_1} \cdots \int_{\Gamma_1} G(z,z') \prod_{i=1}^m \frac{\zeta(1+z_i+z_i')}{(1+z_i)\zeta(1+z_i')} \cdot \frac{R^{z_i+z_i'}}{z_i^2 \cdot z_i'^2} dz_i dz_i'$$

$$= \left(1 + o_{R\to\infty}(1)\right) G(0,0,\ldots,0) \log^m R$$

<u>Pf</u>: Exercise. □

Then we have our estimate:

$$(1+o(1))\left(\frac{W}{\varphi(W)} \log R\right)^m = \mathbb{E}_{x\in B_1} \Lambda_R^2(\theta_1(x)) \cdots \Lambda_R^2(\theta_m(x))$$

← renormalize to get 1, our linear forms condition.

Set $v(x) = \begin{cases} \frac{\varphi(W)}{W} \cdot \frac{\Lambda_R^2(Wx+1)}{\log R}, & x \in [\varepsilon_k N, 2\varepsilon_k N] \\ 1 & \text{else} \end{cases}$ on $\mathbb{Z}/N\mathbb{Z}$. This

will be our pseudorandom function - we've shown the linear forms condition, & we need to show the correlation condition:

$G_0(0,0,\ldots,0) = \prod_{p\mid\Delta}(1+O_m(p^{-1/2}))$, where $\Delta = \prod_{1\leq i<j\leq m} |h_i - h_j|$ & this is for the forms $x_1 + h_i$.

<u>Prop</u>: Pick $\varepsilon_k = \frac{1}{2^k(k+4)!}$ & $R = N^{k^{-1}2^{-k-5}}$, N a large prime. Then $v(x)$ is k-pseudorandom: v satisfies
$(k2^{k-1}, 3k-4, k)$-linear forms condition & the 2^k-correlation condition

Pf of Green-Tao: (Fix k) On $\mathbb{Z}_N$ we define

$$k2^{k+5} f(x) = \begin{cases} \frac{\varphi(W)}{W} \Lambda(Wx+1) & x \in [\varepsilon_k N, 2\varepsilon_k N] \;\&\; Wx+1 \text{ prime} \\ 0 & \text{else} \end{cases}$$

(supp on shift of primes, i.e. when $Wx+1$ is prime)

We have

$$\mathbb{E}_{x \in \mathbb{Z}_N} f(x) = k^{-1} 2^{-k-5} \varepsilon_k (1 + o(1))$$

- $f \le \nu$, which we showed previously.

We also have $\|\nu - 1\|_{U^k(\mathbb{Z}_N)} = o(1)$, from the linear forms condition.
By Gowers' version of the transference & fact this is a QAP norm, we can write $f \le \nu \Rightarrow f = g + h$, where $\|g\|_\infty \le 2$ & $\|h\|_{U^k(\mathbb{Z}_N)} = o(1)$ $(N \to \infty)$

If we have $T = \mathbb{E}_{x,h \in \mathbb{Z}_N} f(x) \cdots f(x+(k-1)h) > 0$, we're done, b/c we've found a translate-dilate of a k-term prog. in primes.
Substitute $g+h$ for f, as we did w/ the balance fcn argument:

$$T = \mathbb{E}_{x,h}\, g(x) \cdots g(x+(k-1)h) + \underbrace{\text{errors}}_{\text{terms w/ at least on } h}$$

$h \le \frac{\nu+1}{2}$, which is also a pseudorandom fcn; so by weighted von Neum,

$$= \underbrace{\mathbb{E}_{x,h}\, g(x) \cdots g(x+(k-1)h)}\; + \underbrace{O(\|h\|_{U^k})}_{\to 0 \text{ if } N \text{ suff. large}}$$

(As $\mathbb{E}_x g \geq k^{-1} 2^{-k-5} \varepsilon_k (1 + o(1)) - o(1)$, Szemeredi's Thm says

$$\geq C_k(k^{-1} 2^{-k-5} \varepsilon_k)$$

So, $T \geq C_k + o(1)$ $\square$

To guarantee a k-term AP, we need have $N \geq 2^{2^{2^{2^{2^{2^{2^{100k}}}}}}}$

Roth $\Leftrightarrow$ Vinogradov
(tr-inv. eqns in dense sets) (linear eqns in primes) w/ asymptotics

Szemeredi $\Rightarrow$ same w/ primes
$\hookrightarrow$ getting the asymptotics for AP's in primes
Green-Tao did this

Linear eqns in primes
Let $\mathcal{L} = (l_1, \ldots, l_r)$ be r linear eqns w/ $\mathbb{Z}$-coeff's.
Def: The rank $h(l_1)$ to be the # of non-zero coeff's (# of vars)
For $\mathcal{L}$, this is $\min_{\substack{\lambda_1, \ldots, \lambda_r \in \mathbb{Q} \\ \text{not all zero}}} h(\lambda_1 l_1 + \cdots + \lambda_r l_r)$. (Ext'n of how linearly ind these are)

If $h(\mathcal{L}) \geq 3$, then $\sum_{\substack{\mathcal{L}(x)=0 \\ 1 \leq x_i \leq N}} \Lambda(x_1) \cdots \Lambda(x_n) \sim C\left(\prod_p \mu_p\right) N^{n-1}$, where

$\mu_p = \mathbb{E}_{\substack{x \in U_p^r \\ \mathcal{L}(x) = 0}} 1$, $U_N = \mathbb{Z}_N^*$

Classically, we had this for $h(\mathcal{L}) \geq 2R+1$, so it grows w/ R.
Thus, this is a huge improvement.

Polynomial version: $p_1, \ldots, p_k$ polynomials of 1 var h &
$p_i(0) = 0 \ \forall i$, then

$\mathbb{E}\, f(x+p_1(h)) \cdots f(x+p_k(h))$ can be bdd below as in Szemeredi's Thm. ($p_i = ih$, this is Sz's Thm)
$\to$ this is also done in primes by Tao & Ziegler

$\to$ multidim'l version of lin eqns extended to primes
Not for poly in primes