

(d) Let u be a unit. Then $u+a = u(1+u^{-1}a)$. By (b), $u^{-1}a$ is nilpotent, so by (c) $1+u^{-1}a$ is a unit. Since the product of 2 units is a unit, $u+a = u(1+u^{-1}a)$ is a unit.

(20) Let $I = n\mathbb{Z}$, $J = m\mathbb{Z}$. Since $I+J \supseteq I$, k divides n . Similarly $I+J \supseteq J$, so k divides m . Suppose \exists an ideal M s.t. $I, J \subseteq M$. Then since M is a subgp, $M \supseteq I+J$, as well. Thus $I+J$ is the smallest ideal containing $I \cup J$. Therefore, $k = \gcd(n, m)$.

(21) $\text{Ann}(R) = \{a \in R \mid ax = 0 \ \forall x \in R\}$.

We first show $\text{Ann}(R)$ is a subgp:

- $0 \cdot x = 0 \Rightarrow 0 \in \text{Ann}(R)$
- If $a, b \in \text{Ann}(R)$, then $(a+b)x = ax + bx = 0 + 0 = 0 \Rightarrow a+b \in \text{Ann}(R)$.
- If $a \in \text{Ann}(R)$, then $a^{-1}x = a^{-1}(a^{-1}a)x = a^{-2}(ax) = a^{-2}(0) = 0 \Rightarrow a^{-1} \in \text{Ann}(R)$.

Thus $\text{Ann}(R)$ is a subgp of R .

To show $\text{Ann}(R)$ is an ideal, let $r \in R$ & $a \in \text{Ann}(R)$. Then $(ra)x = r(ax) = r(0) = 0 \ \forall x \in X$. Therefore $ra \in \text{Ann}(R)$, so $\text{Ann}(R)$ is an ideal of R .

(22) $I \subseteq R$ an ideal. $\text{rad}(I) = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{Z}\}$

It is clear from the definition that $\text{rad}(I) \supseteq I$.

We first show $\text{rad}(I)$ is a subgp of R :

- $0 \in I \Rightarrow 0 \in \text{rad}(I)$.
- Let $a, b \in \text{rad}(I)$. Then $\exists n, m \in \mathbb{Z}$ s.t. $a^n \in I$ & $b^m \in I$. Let $N \in \mathbb{Z}$ be larger than $n+m$. Then $(a+b)^N = a^N + k_{n-1}a^{n-1}b + \dots + k_i a^i b^{N-i} + \dots + k_1 a b^{N-1} + b^N$, $k_i \in \mathbb{N}$. By the choice of N , for every i , either $i \geq n$ or $N-i \geq m$. Thus either $a^i \in I$ or $b^{N-i} \in I$. Thus each term $k_i a^i b^{N-i} \in I$, since I is an ideal. Therefore $(a+b)^N \in I$, so $a+b \in \text{rad}(I)$.
- Let $a \in \text{rad}(I)$. Then $\exists n \in \mathbb{Z}$ s.t. $a^n \in I$. I is a subgp, so $(a^{-1})^n = (a^n)^{-1} \in I \Rightarrow a^{-1} \in \text{rad}(I)$.

To prove $\text{rad}(I)$ is an ideal, let $r \in R$ & $a \in \text{rad}(I)$. Then $\exists n \in \mathbb{Z}$ s.t. $a^n \in I$. $(ra)^n = r^n a^n \in I$, since I is an ideal. Thus $\text{rad}(I)$ is an ideal of R .

(23) (a) Consider the map $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ defined by $f(a,b) = a+bi$. We will show this is an isomorphism.

• f is a (gp) hom: Let $(a,b), (c,d) \in \mathbb{R} \times \mathbb{R}$. Then:

$$f((a,b)+(c,d)) = f((a+c, b+d)) = (a+c) + (b+d)i = (a+bi) + (c+di) = f(a,b) + f(c,d). \checkmark$$

• f is injective: Let $(a,b), (c,d) \in \mathbb{R} \times \mathbb{R}$ s.t. $f(a,b) = f(c,d)$. Then $a+bi = c+di$, so $a=c$ & $b=d$. Thus $(a,b) = (c,d)$. \checkmark

• f is surj: Let $a+bi \in \mathbb{C}$. Then $(a,b) \in \mathbb{R} \times \mathbb{R}$ & $f(a,b) = a+bi$. \checkmark

Therefore, $\mathbb{R} \times \mathbb{R} \cong \mathbb{C}$.

(b) Suppose $\mathcal{Q}: \mathbb{R}^* \times \mathbb{R}^* \rightarrow \mathbb{C}^*$ is an isomorphism. Then $\exists (a,b) \in \mathbb{R}^* \times \mathbb{R}^*$ s.t. $\mathcal{Q}(a,b) = i$. Since the order of i in \mathbb{C}^* is 4 & \mathcal{Q} is an isomorphism, the order of (a,b) in $\mathbb{R}^* \times \mathbb{R}^*$ is 4. Thus $(a,b)^4 = (1,1)$, so $a^4 = 1$ & $b^4 = 1$.

$$a^4 - 1 = 0$$

$$(a^2-1)(a^2+1) = 0$$

$$(a-1)(a+1)(a^2+1) = 0 \Rightarrow a = \pm 1. \text{ (} a^2+1 \text{ has no zeros in } \mathbb{R} \text{).}$$

$$\text{Similarly, } b = \pm 1.$$

But $(\pm 1, \pm 1)$ have order 2 in $\mathbb{R}^* \times \mathbb{R}^*$, which is a contradiction.

Therefore, \mathcal{Q} is not an isomorphism.

* Note that $\mathbb{R}^*, \mathbb{C}^*$ are gps under multiplication, not addition. This is a particular case of the more general fact that the set of units in a ring form a gp under multiplication.

(24) The elt $(1, \bar{0}) \in \mathbb{Z}_8 \times \mathbb{Z}_2$ has order 8, while every elt in $\mathbb{Z}_4 \times \mathbb{Z}_4$ has order at most 4. (This is because the order of $(\bar{a}, \bar{b}) \in \mathbb{Z}_4 \times \mathbb{Z}_4$ is $\text{lcm}(\underset{\uparrow \text{ in } \mathbb{Z}_4}{|\bar{a}|}, \underset{\uparrow \text{ in } \mathbb{Z}_4}{|\bar{b}|})$, & $|\bar{a}|, |\bar{b}| = 1, 2, \text{ or } 4$ by Lagrange's Thm. Thus

the lcm is 1, 2, or 4 as well - just try all combinations.)

Since a gp isomorphism preserves the order of an elt, there can be no isomorphism between these 2 gps.