⑬ $\mathbb{Z}\times\mathbb{Z}/\langle(1,1)\rangle$. The cosets of $\langle(1,1)\rangle$ are of the form $(n,0)+\langle(1,1)\rangle$ for $n\in\mathbb{Z}$, since every elt $(a,b)\in\mathbb{Z}\times\mathbb{Z}$ can be written as $(a,b)=(a-b,0)+(b,b)$, & $(b,b)\in\langle(1,1)\rangle$, while $(a-b,0)$ is of the form $(n,0)$. Thus $\mathbb{Z}\times\mathbb{Z}/\langle(1,1)\rangle$ is an infinite gp, & so by the FTFGAG, $\mathbb{Z}\times\mathbb{Z}/\langle(1,1)\rangle$ is isomorphic to $\underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{r}$ for some r.

It is clear that the map $\varphi:\mathbb{Z}\times\mathbb{Z}/\langle(1,1)\rangle\to\mathbb{Z}$ defined by $\varphi((n,0)+\langle(1,1)\rangle)=n$ is an isomorphism. (it's a good exercise to check the details). Thus $\mathbb{Z}\times\mathbb{Z}/\langle(1,1)\rangle\cong\mathbb{Z}$.

⑭ Since $|G|>1$, let $g\in G$ be a non-zero element. Then $\langle g\rangle$ is a nontrivial subgp of G. By assumption, G has no nontrivial proper subgps, & thus $G=\langle g\rangle$, so G is cyclic. However, $G\neq\mathbb{Z}$, as $\mathbb{Z}$ has many proper nontrivial subgps (eg. $2\mathbb{Z}$), and therefore G is a finite cyclic gp. However, a finite cyclic gp has a proper subgp of order k for each $k\in\mathbb{N}$ which divides $n=|G|$. Since all proper subgps of G are trivial, the only divisors of n are 1 & n; thus n is prime.

⑮ A_n is the subgp of S_n consisting of all even permutations, so for $n=4$ A_4 contains e, all 3-cycles, & all products of 2 disjoint 2-cycles. Recall that the Klein-4 gp V has 4 elements, all of which have order 2. Thus is a subgroup $H\leq A_n$ is isomorphic to V, it cannot contain any 3-cycle (recall that a 3-cycle has order 3). We will show that $H=\{e,(12)(34),(13)(24),(14)(23)\}$ (all elts in A_n except 3-cycles) is isomorphic to V.

Let $a=(12)(34)$, $b=(13)(24)$, & $c=(14)(23)$. Then:
$ab=ba=c$, $ac=ca=b$, & $bc=cb=a$
Thus H is abelian, & since all elts are order 2, $H\cong\mathbb{Z}_2\times\mathbb{Z}_2$ by the FTFGAG. As $V\cong\mathbb{Z}_2\times\mathbb{Z}_2$, it follows that $H\cong V$.

(you could also draw the gp table for H & show it's the same as that of V).

⑯ Suppose $G/Z(G)$ is cyclic, & let $aZ(G)$ be the generator of $G/Z(G)$. Then cosets of $Z(G)$ in G are of the form $a^nZ(G)$, for some $n\in\mathbb{Z}$. It follows that every elt $g\in G$ can be written as $a^nz=g$ for some $n\in\mathbb{Z}$ & some $z\in Z(G)$.
Let $g=a^{n_1}z_1$, $h=a^{n_2}z_2\in G$. Then

$$\begin{aligned} gh &= (a^{n_1}z_1)(a^{n_2}z_2)\\ &= a^{n_1}a^{n_2}z_1z_2 \quad\leftarrow \text{using that } z_1,z_2\in Z(G), \text{ so}\\ &= a^{n_2}a^{n_1}z_2z_1 \quad\leftarrow \text{commute w/ all elts of } G.\\ &= a^{n_2}z_2a^{n_1}z_1\\ &= hg \end{aligned}$$

Therefore G is abelian.

⑰ Suppose G is a gp $\ni \forall g \in G$, $g^2 = e$.

(a) Then $g = g^{-1}$ $\forall g \in G$. Let $x, y \in G$.

Then
$$(xy)(xy) = e$$
$$x(xyxy)y = xey$$
$$(xx)yx(yy) = xy$$
$$(e)yx(e) = xy$$
$$yx = xy.$$

Therefore G is abelian.

(b) We prove this by induction on the order of G.

Base case: If $|G| = 1$, then $|G| = 2^0$, so this holds.

Assume: If $|G| \leq m$ $\ni \forall g \in G$ $g^2 = e$, then $|G|$ is a power of 2.

Induction Step: Suppose $1 < |G| \leq m+1$ $\ni \forall g \in G$, $g^2 = e$. Let $g \in G \setminus \{e\}$. Since G is abelian by (a), $\langle g \rangle \trianglelefteq G$. Consider $G' = G/\langle g \rangle$. Then as $|\langle g \rangle| = 2$, $|G'| = |G|/2$, so $|G'| \leq m$.

Let $a\langle g \rangle \in G'$. Then $(a\langle g \rangle)^2 = a^2\langle g \rangle = e\langle g \rangle = e_{G'}$, so every element in G' squares to the identity. By the induction hypothesis, $|G'|$ is a power of 2. Since $|G| = 2 \cdot |G'|$, the order of G is also a power of 2.

(c) By the FTFGAG, G is isomorphic to one of:
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$$
$$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$$
$$\mathbb{Z}_8 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$$
$$\vdots$$
$$\mathbb{Z}_{2^n} \qquad (\text{where } |G| = 2^n)$$

In all but the first case, the gp has an elt of order at least 4. However, by assumption, G has no such elt. Thus, $G \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

⑱ (a) Let $g \in G$, consider the inner automorphism $i_g \in \text{Aut}(G)$ defined by $i_g(x) = gxg^{-1}$. Since H is characteristic, $i_g(H) = H$ for all $g \in G$. Thus $gHg^{-1} = H$ $\forall g \in G$, $\ni$ so $H \trianglelefteq G$.

(b) $K \leq H \trianglelefteq G$, $\ni$ K is characteristic in H. Let $g \in G$, $\ni$ let i_g be as in (a). Since $H \trianglelefteq G$, $i_g(H) = H$. Thus $i_g \in \text{Aut}(H)$. Since K is characteristic in H, $i_g(K) = K$. Since g was arbitrary, $gKg^{-1} = K$ for all $g \in G$, $\ni$ so $K \trianglelefteq G$.

⑲ Let $a \in R$, a comm. ring w/ unity, $\ni$ suppose $\exists n$ s.t. $a^n = 0$. Moreover, assume n is minimal with respect to this property.

(a) Suppose $a \neq 0$. Then $a^{n-1} \cdot a = a^n = 0$. Since $a \neq 0$ $\ni$ $a^{n-1} \neq 0$ by assumption, a is therefore a zero divisor.

(b) Let $x \in R$. Then since R is commutative, $(ax)^n = a^n x^n = 0 \cdot x^n = 0$. Thus ax is nilpotent.

(c) $(1+a)(1 - a + a^2 - \cdots + (-1)^{n-1}a^{n-1}) = 1 \pm a^n = 1 \pm 0 = 1$, so $1+a$ is a unit.

(d) Let u be a unit. Then $u+a = u(1+u^{-1}a)$. By (b), $u^{-1}a$ is nilpotent, so by (c) $1+u^{-1}a$ is a unit. Since the product of 2 units is a unit, $u+a = u(1+u^{-1}a)$ is a unit.

(20) Let $I = n\mathbb{Z}$, $J = m\mathbb{Z}$. Since $I+J \supseteq I$, k divides n. Similarly $I+J \supseteq J$, so k divides m. Suppose $\exists$ an ideal M s.t. $I, J \subseteq M$. Then since M is a subgp, $M \supseteq I+J$, as well. Thus $I+J$ is the smallest ideal containing I & J. Therefore, $k = \gcd(n,m)$.

(21) $\mathrm{Ann}(R) = \{a \in R \mid ax = 0 \ \forall x \in R\}$.
We first show $\mathrm{Ann}(R)$ is a subgp:
- $0 \cdot x = 0 \Rightarrow 0 \in \mathrm{Ann}(R)$
- If $a, b \in \mathrm{Ann}(R)$, then $(a+b)x = ax + bx = 0 + 0 = 0 \Rightarrow a+b \in \mathrm{Ann}(R)$.
- If $a \in \mathrm{Ann}(R)$, then $(-a)(x) = -(ax) = -0 = 0$, so $-a \in \mathrm{Ann}(R)$.

Thus $\mathrm{Ann}(R)$ is a subgp of R.

To show $\mathrm{Ann}(R)$ is an ideal, let $r \in R$ & $a \in \mathrm{Ann}(R)$. Then $(ra)x = r(ax) = r(0) = 0 \ \forall x \in X$. Therefore $ra \in \mathrm{Ann}(R)$, so $\mathrm{Ann}(R)$ is an ideal of R.

(22) $I \subseteq R$ an ideal. $\mathrm{rad}(I) = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{Z}\}$
It is clear from the definition that $\mathrm{rad}(I) \supseteq I$.

We first show $\mathrm{rad}(I)$ is a subgp of R:
- $0 \in I \Rightarrow 0 \in \mathrm{rad}(I)$.
- Let $a, b \in \mathrm{rad}(I)$. Then $\exists n, m \in \mathbb{Z}$ s.t. $a^n \in I$ & $b^m \in I$. Let $N \in \mathbb{Z}$ be larger than $n+m$. Then
$$(a+b)^N = a^N + k_{N-1}a^{N-1}b + \cdots + k_i a^i b^{N-i} + \cdots + k_1 a b^{N-1} + b^N, \quad k_i \in \mathbb{N}$$
By the choice of N, for every i, either $i \geq n$ or $N-i \geq m$. Thus either $a^i \in I$ or $b^{N-i} \in I$. Thus each term $k_i a^i b^{N-i} \in I$, since I is an ideal. Therefore $(a+b)^N \in I$, so $a+b \in \mathrm{rad}(I)$.
- Let $a \in \mathrm{rad}(I)$. Then $\exists n \in \mathbb{Z}$ s.t. $a^n \in I$. I is a subgp, so $-a^n \in I$. Since $(-a)^n = a^n \in I$ if n is even & $(-a)^n = -a^n \in I$ if n odd, it follows that $-a \in \mathrm{rad}(I)$. (Or, you could note that $(-a)^{2n} = a^n \in I$.)

To prove $\mathrm{rad}(I)$ is an ideal, let $r \in R$ & $a \in \mathrm{rad}(I)$. Then $\exists n \in \mathbb{Z}$ s.t. $a^n \in I$. $(ra)^n = r^n a^n \in I$, since I is an ideal. Thus $\mathrm{rad}(I)$ is an ideal of R.