

## Final Practice Problems

1.  $\int \frac{x^3}{\sqrt{x^2+9}} dx$

Method I:  $u$ -sub

Let  $u = x^2 + 9$ . Then  $du = 2x dx$  and  $x^2 = u - 9$ . Substituting into our integral yields:

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+9}} dx &= \int \frac{(u-9) du}{\sqrt{u}} \\ &= \frac{1}{2} \int (\sqrt{u} - 9u^{-1/2}) du \\ &= \frac{1}{3} u^{3/2} - 9u^{1/2} + C \\ &= \frac{1}{3} (x^2 + 9)^{3/2} - 9(x^2 + 9)^{1/2} + C\end{aligned}$$

Method II: Trig sub

Let  $x = 3 \tan \theta$ . Then  $dx = 3 \sec^2 \theta d\theta$ . Substituting into our integral yields:

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+9}} dx &= \int \frac{(3 \tan \theta)^3 3 \sec^2 \theta d\theta}{\sqrt{9 \sec^2 \theta}} \\ &= 27 \int \tan^3 \theta \sec \theta d\theta \\ &= 27 \int \tan \theta (\sec^2 \theta - 1) \sec \theta d\theta \quad \text{Let } u = \sec \theta \quad du = \sec \theta \tan \theta d\theta \\ &= 27 \int (u^2 - 1) du \\ &= 27(1/3 u^3 - u) + C \\ &= 9 \sec^3 \theta - 27 \sec \theta + C \\ &= 9 \left( \frac{\sqrt{x^2+9}}{3} \right)^3 - 27 \cdot \frac{\sqrt{x^2+9}}{3} + C\end{aligned}$$

2.  $\int (\sin x + x)e^x dx$

We will break this into two integrals:

$$\int \sin xe^x dx = \sin xe^x - \int \cos xe^x dx \quad f = \sin x, f' = \cos x, g = e^x, g' = e^x$$

$$\int \sin xe^x dx = \sin xe^x - \left( \cos xe^x + \int \sin xe^x dx \right) \quad f = \cos x, f' = -\sin x, g = e^x, g' = e^x$$

$$\int \sin xe^x dx = \sin xe^x - \cos xe^x - \int \sin xe^x dx$$

$$2 \int \sin xe^x dx = \sin xe^x - \cos xe^x$$

$$\int \sin xe^x dx = \frac{1}{2} \sin xe^x - \frac{1}{2} \cos xe^x + C$$

$$\int xe^x dx = xe^x - \int e^x dx \quad f = x, f' = 1, g = e^x, g' = e^x$$

$$= xe^x - e^x + C$$

Adding these two integrals together, we get

$$\int (\sin x + x)e^x dx = \int \sin xe^x dx + \int xe^x dx = \frac{1}{2} \sin xe^x - \frac{1}{2} \cos xe^x + xe^x - e^x + C$$

3.  $\int_{-\infty}^{\infty} \frac{dx}{25 + 4x^2}$

First, I will compute the indefinite integral  $\int \frac{dx}{25 + 4x^2}$ :

$$\begin{aligned} \int \frac{dx}{25 + 4x^2} &= \frac{1}{25} \int \frac{dx}{1 + \frac{4x^2}{25}} \\ &= \frac{1}{25} \int \frac{dx}{1 + \left(\frac{2x}{5}\right)^2} && \text{Let } u = \frac{2x}{5}, du = \frac{2}{5} \\ &= \frac{1}{25} \cdot \frac{5}{2} \int \frac{du}{1 + u^2} \\ &= \frac{1}{10} \arctan u + C \\ &= \frac{1}{10} \arctan \left( \frac{2x}{5} \right) \end{aligned}$$

This is a doubly improper integral, so I will split the integral at 0 (you could choose any

number to split the integral).

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{dx}{25+4x^2} &= \int_{-\infty}^0 \frac{dx}{25+4x^2} + \int_0^{\infty} \frac{dx}{25+4x^2} \\
 &= \lim_{a \rightarrow -\infty} \int_a^0 \frac{dx}{25+4x^2} + \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{25+4x^2} \\
 &= \lim_{a \rightarrow -\infty} \left[ \frac{1}{10} \arctan \left( \frac{2x}{5} \right) \right]_a^0 + \lim_{b \rightarrow \infty} \left[ \frac{1}{10} \arctan \left( \frac{2x}{5} \right) \right]_0^b \\
 &= \lim_{a \rightarrow -\infty} \left[ 0 - \frac{1}{10} \arctan \left( \frac{2a}{5} \right) \right] + \lim_{b \rightarrow \infty} \left[ \frac{1}{10} \arctan \left( \frac{2b}{5} \right) - 0 \right] \\
 &= \frac{1}{10} \cdot \frac{\pi}{2} + \frac{1}{10} \cdot \frac{\pi}{2} \\
 &= \frac{\pi}{10}
 \end{aligned}$$

4. Solve  $\frac{dy}{dx} = xe^{x^2 - \ln(y^2)}$  if  $y(0) = 1$ .

This is a separable differential equation:

$$\begin{aligned}
 \frac{dy}{dx} &= xe^{x^2} \cdot e^{\ln(y^{-2})} \\
 \frac{dy}{dx} &= xe^{x^2} \cdot y^{-2} \\
 y^2 dy &= xe^{x^2} dx \\
 \int y^2 dy &= \int xe^{x^2} dx \\
 \frac{1}{3}y^3 &= \frac{1}{2}e^{x^2} + C
 \end{aligned}$$

Plugging in  $y(0) = 1$  we get  $\frac{1}{3} = \frac{1}{2} + C$ , so  $C = \frac{1}{6}$ . Plugging this back into our equation and solving for  $y$ , we have:

$$\begin{aligned}
 y^3 &= \frac{3}{2}e^{x^2} + \frac{1}{2} \\
 y &= \sqrt[3]{\frac{3}{2}e^{x^2} + \frac{1}{2}}
 \end{aligned}$$

5. *Set up a differential equation for the following problem.* A 1500 gallon tank contains 600 gallons of water with 5 pounds of salt dissolved in it. Water enters the tank at a rate of 9 gallons per hour, and the water entering the tank contains  $\frac{1}{2}$  lb of salt per gallon. A well-mixed solution leaves the tank at a rate of 6 gallons per hour. How much salt is in the tank when it overflows?

Variables: Let  $t$  be time in minutes and  $A(t)$  be the amount of salt in the tank at time  $t$  in pounds.

Volume:  $600 + 3t$

Salt In:  $9 \cdot \frac{1}{2} = 4.5$  pounds of salt in one minute.

Salt Out:  $6 \cdot \frac{A(t)}{600+3t}$  pounds of salt out in one minute

Differential Equation:  $\frac{dA}{dt} = 4.5 - 6 \cdot \frac{A(t)}{600 + 3t}$ ,  $A(0) = 5$ .

6. Find the Taylor series centered at 0 of  $f(x) = \ln(1 + x^2)$ . For what values of  $x$  does the series converge?

We know  $T_\infty \ln(1 + u) = \sum_{n=0}^{\infty} \frac{u^n}{n}$ , so  $T_\infty \ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n}$ . To find the values of  $x$  for which the series converges, we will use the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{x^{2n+2}}{n+1} \cdot \frac{n}{x^{2n}} = \lim_{n \rightarrow \infty} \frac{nx^2}{n+1} x^2 \lim_{n \rightarrow \infty} \frac{n}{n+1} = x^2 = L.$$

By the Ratio Test, the series converges when  $L < 1$ , which is when  $x^2 < 1$ , so it converges for  $-1 < x < 1$ . Similarly, it diverges when  $L > 1$ , so it diverges for  $x < -1$  and  $x > 1$ . When  $L = 1$ , the Ratio Test is inconclusive, so we will plug  $x = \pm 1$  into our sum and first consider the series  $\sum_{n=0}^{\infty} \frac{1}{n}$ . We can use the integral test with  $f(x) = \frac{1}{x}$  to see that this diverges: the hypotheses are satisfied, since  $f$  is positive and decreasing, and  $f(n) = \frac{1}{n}$ . By the  $p$ -chart, we know that  $\int_1^{\infty} \frac{1}{x} dx$  diverges, so therefore the series diverges, as well. Similarly for  $x = -1$ .

Therefore, the Taylor series converges exactly when  $-1 < x < 1$ .

7. Find all values of  $x$  for which  $\sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{3^n}$  converges.

We will use the Root Test.

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(-1)^n(x-2)^n}{3^n} \right|} = \lim_{n \rightarrow \infty} \frac{|x-2|}{3} = \frac{|x-2|}{3} = L$$

By the Root Test, the series converges when  $L < 1$ , which is when  $|x-2| < 3$ , so when  $-1 \leq x \leq 5$ . It diverges when  $L > 1$ , so when  $x < -1$  or when  $x > 5$ . We now only need to check what happens when  $L = 1$ , which is when  $x = -1$  or  $x = 5$ . When  $x = -1$ , the series becomes  $\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n} = \sum_{n=0}^{\infty} 1$  which diverges by the term test, as  $\lim_{n \rightarrow \infty} 1 = 1 \neq 0$ . When  $x = 5$ , the series becomes  $\sum_{n=0}^{\infty} \frac{(-1)^n(3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ , which also diverges by the term test, as  $\lim_{n \rightarrow \infty} (-1)^n$  diverges.

Therefore, the series converges exactly when  $-1 < x < 5$ .

8. Consider the three points  $A(0, 2, 5)$ ,  $B(1, -3, 0)$  and  $C(0, -2, 1)$ .

(a) Write an equation for the plane that passes through  $A, B$ , and  $C$ .

$\vec{AB} = \begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}$ ,  $\vec{AC} = \begin{pmatrix} 0 \\ -4 \\ -4 \end{pmatrix}$ , so  $\vec{n} = \vec{AB} \times \vec{AC} = \begin{pmatrix} 40 \\ 4 \\ -4 \end{pmatrix}$ . Then the equation of

the plane is  $\vec{n} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c$ , which becomes  $40x + 4y - 4z = c$ . To find the constant  $c$ , we plug in any point. I will plug in  $A$ :  $80 - 20 = c$ , so  $c = 60$ . Thus the equation is  $40x + 4y - 4z = 60$ .

(b) Find the distance from the plane to the origin.

First we will write the parametric equation of the line through the origin in the direction of  $\vec{n}$ :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 40 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 40t \\ 4t \\ -4t \end{pmatrix}.$$

Next, I will find the intersection point of this line with the plane from part (a), by plugging the components of the line into the equation for the plane:

$$40(40t) + 4(4t) - 4(-4t) = 60$$

$$160t + 16t + 16t = 60$$

$$192t = 60$$

$$t = \frac{5}{16}.$$

Plugging the back into the equation of the line, we see that the line intersects the plane at the point  $(50/4, 5/4, -5/4)$ .

Finally, we find the distance from the origin to this point:

$$\sqrt{(50/4)^2 + (5/4)^2 + (-5/4)^2}$$