$y_p(t) = te^{-t}\sin 2t$

$y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + te^{-t}\sin 2t$

initial conditions: $y'(t) = -c_1 e^{-t}\cos 2t - 2c_1 e^{-t}\sin 2t - c_2 e^{-t}\sin 2t + 2c_2 e^{-t}\cos 2t + e^{-t}\sin 2t - te^{-t}\sin 2t + 2te^{-t}\cos 2t$

$c_1 = 1$
$-c_1 + 2c_2 = 0 \Rightarrow 2c_2 = 1 \Rightarrow c_2 = 1/2$

Solution: $y(t) = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t$

<u>Note</u>: The computation in this problem is (much) longer than what will be on the exam.

(d) $y'' - 2y' + y = \frac{e^t}{1+t^2}$

gen. sol'n to homogeneous eqn: $r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$
$r = 1$ (mult. 2)

$y(t) = c_1 e^t + c_2 te^t \rightarrow y_1(t) = e^t,\ y_2(t) = te^t$

particular sol'n: (variation of parameters) $f(t) = \frac{e^t}{1+t^2}$

$y_p(t) = e^t v_1(t) + te^t v_2(t)$, $\quad W[y_1, y_2] = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$

$v_1(t) = \int \frac{-f(t)y_2(t)}{a\,W(y_1,y_2)}\,dt$, $\quad v_2(t) = \int \frac{f(t)y_1(t)}{a\,W(y_1,y_2)}\,dt$

$v_1(t) = \int -\frac{e^t \cdot te^t}{(1+t^2)e^{2t}}\,dt = -\int \frac{t}{1+t^2}\,dt = -\frac{1}{2}\ln(1+t^2)$

$v_2(t) = \int \frac{e^t \cdot e^t}{(1+t^2)e^{2t}}\,dt = \int \frac{1}{1+t^2}\,dt = \arctan t$

$y_p(t) = -\frac{1}{2}e^t \ln(1+t^2) + te^t \arctan t$

Solution: $y(t) = c_1 e^t + c_2 te^t - \frac{1}{2}e^t \ln(1+t^2) + te^t \arctan t$.

(e) $y^{(4)} + 2y'' + y = 0 \rightarrow r^4 + 2r^2 + 1 = 0 \rightarrow (r^2+1) = 0 \rightarrow r = \pm i$, mult. 2

$y(t) = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t$

(f) $\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

gen. sol'n to homog. eqn: $r^2 + 4r + 3 = 0 \Rightarrow r = -1, r = -3$, w/ assoc. e-vects $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $r = -3$ & $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $r = -1$.

$$X(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$$

particular sol'n: (variation of parameters)

$$X(t)^{-1} = \begin{pmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{pmatrix}$$

$$\int X(t)^{-1} f(t)\, dt = \int \begin{pmatrix} \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \\ \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} dt = \int \begin{pmatrix} e^{2t} - \frac{3}{2}te^{3t} \\ 1 + \frac{3}{2}te^{t} \end{pmatrix} dt$$

$$= \begin{pmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} \end{pmatrix}$$

$$\vec{x}_p(t) = X(t)\cdot \begin{pmatrix} \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t} \\ t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t} \end{pmatrix} = \begin{pmatrix} te^{-t} + \frac{1}{2}e^{-t} + t - 4/3 \\ te^{-t} - \frac{1}{2}e^{-t} + 2t - 5/3 \end{pmatrix}$$

solution: $\vec{x}(t) = X(t)\cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \vec{x}_p(t)$

$$\vec{x}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} te^{-t} + \frac{1}{2}e^{-t} + t - 4/3 \\ te^{-t} - \frac{1}{2}e^{-t} + 2t - 5/3 \end{pmatrix}$$

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

[Either form of the solution is acceptable.]

(15) $100u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0$

$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$

$u(x,0) = \sin 2\pi x - \sin 5\pi x, \quad 0 \le x \le 1.$

$\alpha^2 = 100, \quad f(x) = \sin 2\pi x - \sin 5\pi x, \quad L = 1.$

$$c_n = 2\int_0^1 (\sin 2\pi x - \sin 5\pi x)\sin n\pi x\, dx$$

$$= 2\int_0^1 \sin 2\pi x \sin n\pi x\, dx - 2\int_0^1 \sin 5\pi x \sin n\pi x\, dx$$

$$= \underbrace{2\int_0^1 \tfrac{1}{2}\left(\cos(2\pi x - n\pi x) - \cos(n\pi x + 2\pi x)\right)dx}_{= \begin{cases} 1 & \text{if } n=2 \\ 0 & \text{else} \end{cases}} - \underbrace{2\int_0^1 \tfrac{1}{2}\left(\cos((5-n)\pi x) - \cos((5+n)\pi x)\right)dx}_{= \begin{cases} 1 & \text{if } n=5 \\ 0 & \text{else} \end{cases}}$$

$\Rightarrow c_2 = 1, \quad c_5 = -1, \quad \& \quad c_n = 0 \ \forall\, n \ne 2, 5$

So, $u(x,t) = e^{-400\pi^2 t}\sin 2\pi x - e^{-2500\pi^2 t}\sin 5\pi x.$

(16) $xu_{xx} + u_t = 0$. Suppose $u(x,t) = X(x)T(t)$.

Then $u_{xx} = X''(x)T(t)$ & $u_t = X(x)T'(t)$

so the PDE becomes:

$$x X''(x)\, T(t) + X(x)\, T'(t) = 0$$
$$x X''(x)\, T(t) = -X(x)\, T'(t)$$
$$\frac{x X''(x)}{X(x)} = \lambda = -\frac{T'(t)}{T(t)}$$

Thus we get: $x X''(x) - \lambda X(x) = 0$
$T'(t) + \lambda T(t) = 0$

(17) $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$, period 4, Fourier cosine series.
$\hookrightarrow L = 2$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0,1,2,\dots$$
$$= \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 \cos\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^1 \quad \text{if } n \geq 1$$
$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} \frac{2}{n\pi} & n = 1,5,9,\dots \\ -\frac{2}{n\pi} & n = 3,7,11,\dots \\ 0 & n \text{ even} \end{cases}$$
$$a_0 = \int_0^1 1\,dx = 1$$
$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)}\cos\frac{(2m-1)\pi x}{2}$$

(18) $a^2 u_{xx} = u_{tt}$

$$u(x,0) = \begin{cases} \frac{2x}{L}, & 0 \leq x \leq L/2 \\ \frac{2(L-x)}{L}, & L/2 < x \leq L \end{cases}$$
$$u_t(x,0) = 0$$

$$u_n(x,t) = c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}, \quad n = 1,2,\dots, \quad c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$
$$c_n = \frac{2}{L}\int_0^{L/2} \frac{2x}{L}\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^L \frac{2(L-x)}{L}\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{4}{L^2}\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx + \frac{4}{L^2}\int_{L/2}^L (L-x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{4}{L^2}\cdot\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{4}{L^2}\cdot\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$
$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} 8/n^2\pi^2 & \text{if } n = 1,5,9,\dots \\ -8/n^2\pi^2 & \text{if } n = 3,7,11,\dots \\ 0 & \text{if } n \text{ even} \end{cases}$$

So, our solution is
$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$
$$u(x,t) = \frac{8}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{L}\cos\frac{(2m-1)\pi a t}{L}$$