cts inj. btwn Haus. cpt sp's are embeddings

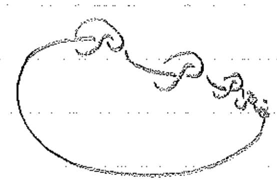
Cor: A closed nonorientable n-mfld N, then N does not embed in $\mathbb{R}^{n+1}$. [of Cor]
$\uparrow$ inject as homeomorphism onto image

Ex: Klein bottle doesn't embed in $\mathbb{R}^3$.

Pf: By 3.28 $H_{n-1}(N;\mathbb{Z}) \supseteq \mathbb{Z}/2\mathbb{Z}$. $\square$

Knots: Let K be a circle in S^3, i.e. $K = \mathrm{Im}(S^1 \hookrightarrow S^3)$

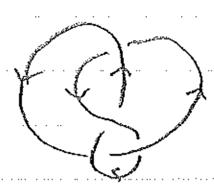
K $\rightarrow$



Untie this: i.e. "homotope" K through embeddings until you get O
$\hookrightarrow$ ~~not precise enough~~: just pull until it disappears.

Using Alex. duality, I can show that
$H_1(S^3-K) \cong H^1(K)$
$\qquad\qquad\qquad\quad \| \mathbb{Z}$

So $H^1(S^3-K) = \mathbb{Z}$. Using this fact, can build a map $S^3-K \to S^1$ realizing this homology class. Then pull back a regular value in S^1, & get an embedded surface (oriented) with "boundary" K.



"Seifert surfaces"

Ex: Lens Space $L(7,5)$. Does it bound a 4-mfld? Yes.

Thm: (Rochlin) Every orientable closed 3-mfld M^3 is homeomorphic to boundary of some 4-mfld.

Thm (Hirsch): Closed orientable 3-mfld's embed in S^5.
Then by Alex duality, $M^3 \subseteq S^5$ has a Seifert 4-mfld.

$$H^3(M^3) \cong H_{5-3-1}(S^5 - M^3) = H_1(S^5 - M^3)$$

get a class in $H^1(S^5 - M^3)$, which gives a map to S^1, then pull back a regular value.

4/2 $\boxed{\text{Homotopy Theory}}$

(can't classify spaces up to h.e., even for 2-cx's. Papakyriakopolous solved the homeomorphism problem for 2-cx's, but the h.e. problem is unsolvable. Key pt: you can't tell if 2 f.pres. gps are isomorphic.

Higher Homotopy Groups X a top. sp, $* \in X$.
We've already seen $\pi_1(X,*)$. This fits into a sequence of functors $\underbrace{\pi_0(X,*)}_{\text{may not be a gp}}, \underbrace{\pi_1(X,*), \pi_2(X,*), \dots, \pi_n(X,*)}_{\text{all gps}}, \dots$

π_1: $\{(I, \partial I) \to (X,*)\} /$ homotopy of pairs $\qquad I = [0,1]$

We'll generalize this:
$I^n = [0,1]^n$, $I^0 = \{*\}$ by convention, $\partial I^0 = \emptyset$

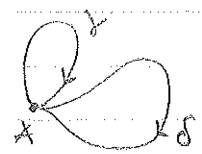
$$(I^n, \partial I^n) \downarrow \{\text{elts where some coord is } 0 \text{ or } 1\}$$

$\emptyset$ not cld & has $\pi_n = 0 \ \forall n$

$\pi_n(X,*) := \{(I^n, \partial I^n) \to (X,*)\} /$ htpy of pairs
$\hookrightarrow$ htpy satisfies $f_t(\partial I^n) = * \ \forall t$

- $n=0$, $\pi_0(X,*) = \{\text{path cpts of } X\}$

- $n=1$: we had a way to compose paths, & $\pi_1(X,*)$ is a gp

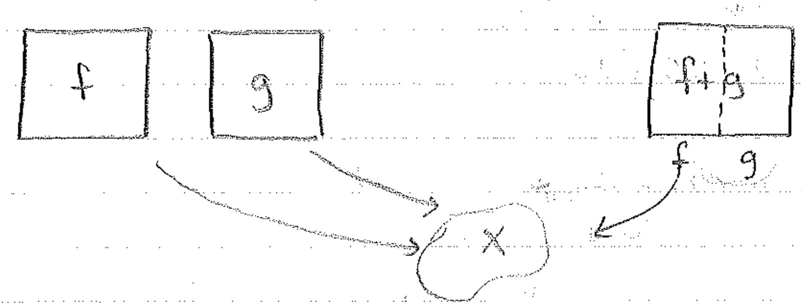


just concatenate after reparamatrizing.

- $n>1$: In the same way, we can glue together 2 maps $f,g:(I^n,\partial I^n)\to(X,*)$

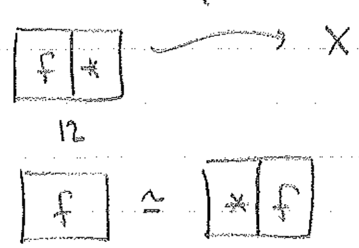
$$f+g:(I^n,\partial I^n)\longrightarrow(X,*)$$
$$(s_1,\dots,s_n)\longmapsto\begin{cases} f(2s_1,s_2,\dots,s_n) & s_1\in[0,1/2]\\ g(2s_1-1,s_2,\dots,s_n) & s_1\in[1/2,1]\end{cases}$$

well-def. product up to htpy of pairs on $\pi_n(X,*)$.

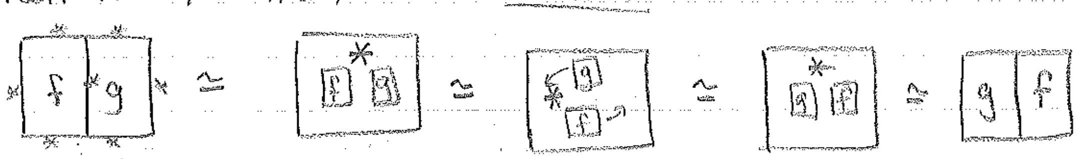


← reparamatrize 1st coord.

- The constant map $*:(I^n,\partial^n)\to(X,*)$ is the identity



- Inverses: $-f(s_1,\dots,s_n)=f(1-s_1,\dots,s_n)$
- When $n>1$, $\pi_n(X,*)$ is abelian:



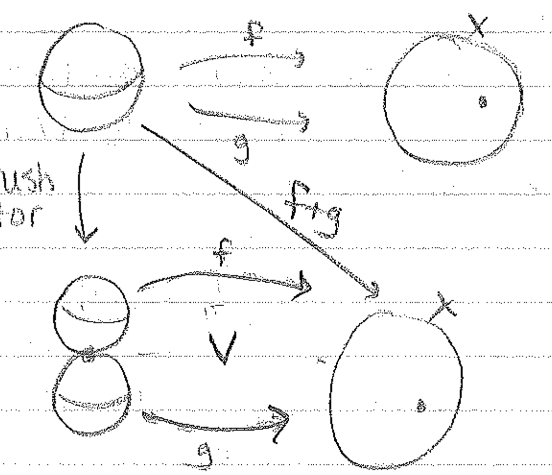
Since we're considering htpy classes of maps of pairs $(I^n, \partial I^n) \to (X, *)$, we can think of $\pi_n(X,*)$ as

$\{ (S^n,*) \longrightarrow (X,*) \} / \text{htpy of pairs}$

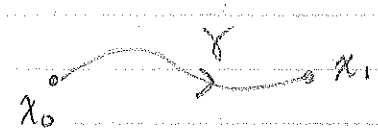
since every map factors through $(S^n,*)$, where $S^n = I^n/\partial I^n$ & $* = \partial I^n / \partial I^n$

$$\begin{tikzcd} (I^n, \partial I^n) \arrow[rr] \arrow[dr] & & (X,*) \\ & (S^n,*) \arrow[ur] & \end{tikzcd}$$

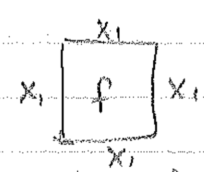
Thought of this way, the product $+$ is



<u>Basepoints</u>: If X is path-cted, there're still change-of-basept isomorphisms. Take a path γ & map

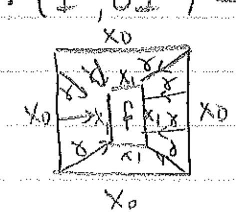


$f : (I^n, \partial I^n) \to (X, x_1)$



. Use γ

to get a map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$



- delete center cube, have map $S^{n-1} \times I \to X$ & patch together γ for each $s \in S^{n-1}$.

This gives a homomorphism $\pi_n(X, x_1) \xrightarrow{\beta_\gamma} \pi_n(X, x_0)$.

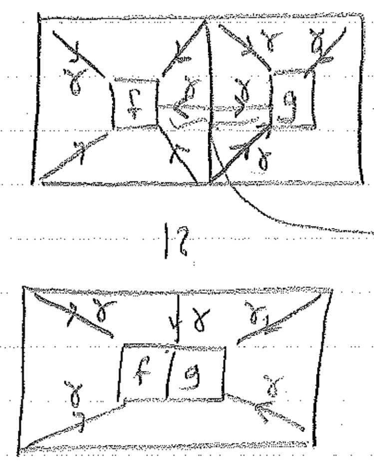
Facts: (routine checks)

(1) $\gamma(f+g) \simeq \gamma f + \gamma g$ concatenation of paths

(2) $(\gamma\eta)f \simeq \gamma(\eta f)$

(3) $1f \simeq f$ (1 = const. path at $*$).

Pf of (1):



doing & then undoing γ: $\gamma * \bar{\gamma} \simeq 1$

so can eat away region btwn.

$\downarrow$

can take care of still having vert γ by adding $\boxed{f|*}$ $\boxed{*|g}$ at start $\square$

β_γ is an isomorphism, with inverse $\beta_{\bar{\gamma}}$, where $\bar{\gamma}(s) = \gamma(1-s)$.

- If $x_0 = x_1$, get action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$, i.e. a map

$$\pi_1(X,*) \longrightarrow \mathrm{Aut}(\pi_n(X,*))$$
$$[\gamma] \longmapsto \beta_\gamma$$

So $\pi_n(X,*)$ is a module over the gp ring of $\pi_1(X,*)$.

4/2 Problem Session

— 4 on exam: • $(m|_{\text{each factor}})^*$ an $\cong$ on cohom. pull-back to each factor gives $\gamma \mapsto \alpha+\beta$

$m^*: H^*(S^4) \longrightarrow H^*(S^4 \times S^4)$

$$\mathbb{Z}[\gamma]/(\gamma^2) \qquad \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2),\ |\alpha|=|\beta|=4$$
$|\gamma|=4$

• Given a gp G, the action of $G\setminus\{e\}$ has no fixed pts, & each action is a homeo: $g: S^4 \to S^4$

$\tau(g) = 1+0+0+0+(-1)^4 = 2 \neq 0$
Lefschetz # $\Rightarrow$ fix pt ↯

[using inverses to show fixed-pt-free]

$H_0 \to H_0$ deg $=1$
$H_1, H_2, H_3 = 0$
$H_4 = \pm 1$

→ problem true for any even-dim'l sphere

H-sp in dim 0, 1, 3, 7
[w/o ID & assoc.

b/c $\mathbb{R}', \mathbb{C}, \mathbb{H}, \mathbb{O} \cong_{\text{homeo}} \mathbb{R}', \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$
algebras

S^1 = rotations of $\mathbb{R}^2$ about O = $\{z = x+iy \mid |z|=1\}$
$S^3 = \{\overbrace{x+iy}^{\alpha=} + jz + kw \mid \|\alpha\| = 1\}$
↪ quaternions

— #1 on HW 3: $2\alpha+\beta = 195 \Rightarrow \beta$ odd $\Rightarrow y \smile y = (-1)^{\beta^2} y \smile y$
$\Rightarrow y^2 = -y^2$ ↯

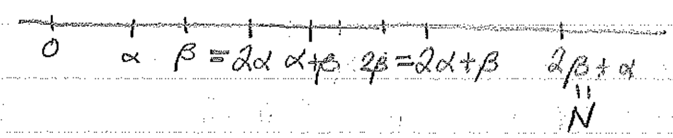
$R = \mathbb{Q}[x,y]/(x^3, y^3, x^2y^2)$
If $R = H^*(M^N)$, then N must be even
Claim: N must be a multiple of 7.
Get $\alpha = 3\beta$, then $2\alpha+\beta = 195$
$\Rightarrow 6\beta + \beta = 195$
$7\beta = 195 \Rightarrow$ 195 divisible by 7.

Pf: Claim: $7 \mid N$

If $\alpha = \beta$, then $N = 3\alpha$, doesn't make sense b/c top dim'l cohom. cannot have 2 lin. ind. things (xy^2 & x^2y).
$H^N(M;\mathbb{Q}) \ncong \mathbb{Q}^2$.

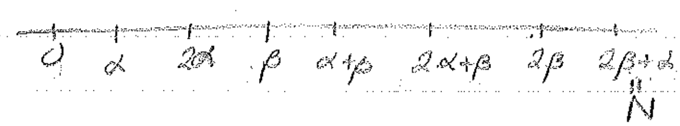
If $\alpha < \beta$, $N = 2\beta + \alpha$.

I. $2\alpha \geq \beta$, then $2\alpha + \beta \geq 2\beta$



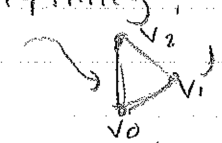
By Poincare duality, $2\beta + \alpha - (2\alpha + \beta) = \alpha \Rightarrow \beta = 2\alpha$
↯ by rank of vector spaces.

II. $2\alpha < \beta$, then $2\alpha + \beta < 2\beta$
So duality: $2\beta + \alpha - (2\alpha) = 2\alpha + \beta$



So $\beta = 3\alpha \Rightarrow N = 7\alpha$

\#1 on exam: $\alpha \in H^1(X;\mathbb{Z}) \Rightarrow \alpha^2 = 0$

- Skew-commutivity $\Rightarrow 2\alpha^2 = 0$. Only holds at cohom. level, so not immediately clean at chain level. (i.e, could be all 2-torsion)
  So, write down a representative $2\alpha^2 = \delta\psi$. If $\delta\psi = \delta(2\phi) = 2\delta\phi$, done. Take ψ & make it 2ϕ: $\phi = 0$ on all 1-cells that aren't a face of anything.
  If ψ odd  change ψ appropriately, i.e. increase value on 1 side by 1 & subtract it from somewhere else (not changing $\delta\psi$).
  Know $\delta\psi$ = even

• Different Idea: $\alpha \in H^1(X;\mathbb{Z}) \to \langle \pi_1, \mathbb{Z} \rangle$
α realized by a map: $X \to S^1$ b/c
so $\alpha = f^*([S^1])$
↳ fund. cohom. class
But $[S^1]^2 = 0 \Rightarrow \alpha^2 = f^*([S^1]^2) = 0$
$= f^*([S^1])^2 =$
takes some formality...

4/4 $\mathbb{Q}[x,y]/(x^3y^3, x^2y^2)$ is not the rat'l cohom. ring of any closed orientable mfld. Hint: consider that the cup product pairing is nonsingular.

Homotopy Groups $\pi_n(X,*)$
Given a path γ based at $*$, $x_0 \to x_1$, have β_γ an iso.
$\beta_\gamma : \pi_n(X,x_1) \to \pi_n(X,x_0)$. If γ is a loop based at $*$, get a map
$$\pi_1(X,*) \to \mathrm{Aut}(\pi_n(X,*))$$
$$\gamma \longmapsto \beta_\gamma$$
So $\pi_1(X,*)$ acts on $\pi_n(X,*)$. Given a G-action of a gp G on an abel. gp A, you naturally get a $\mathbb{Z}G$-module structure on A by extending linearly.
Recall: $\mathbb{Z}G$ is the integral gp ring. Ring of formal $\mathbb{Z}$-linear comb's of elts of G.
ex: F_2 = free gp on a, b.
$\mathbb{Z}F_2$ contains things like $72ab^{-1}ab^{-3} + ab + 2a^{-1} + \cdots$
Kaplansky's Conjecture: If G is torsion-free, then $\mathbb{Z}G$ has no zero divisors. (open)
π_n is a functor: Given a map $(X,*) \xrightarrow{f} (Y,*)$, there is an induced map $f_* : \pi_n(X,*) \to \pi_n(Y,*)$, & $(f \circ g)_* = f_* \circ g_*$, $\mathbb{1}_* = \mathbb{1}$. If $f \simeq g$, then $f_* = g_*$.

Goodwilly Calculus – π_n like Taylor Series of analytic fcn.

This tell us a lot about the space.

Whitehead's Thm: If X & Y are h.e. to cell cx's, & $f:(X,*)\to(Y,*)$ is continuous map s.t. $f_*:\pi_n(X,*)\to\pi_n(Y,*)$ is an isomorphism $\forall n$, then f is a homotopy equivalence.

Ex: (1) Let $\mathcal{L}$ be the long line.

$[0,\infty)$ $\quad$ 0 1 2 ... $\quad$ is a cell cx:

ω = natural #s (as an ordinal) = 0 skeleton
attach one cells.

Now, take the 1st uncountable ordinal & do the "same" thing (add 1 cells) $\Rightarrow$ long ray

Attach 2 of them $\Rightarrow$ long line. Not contractible

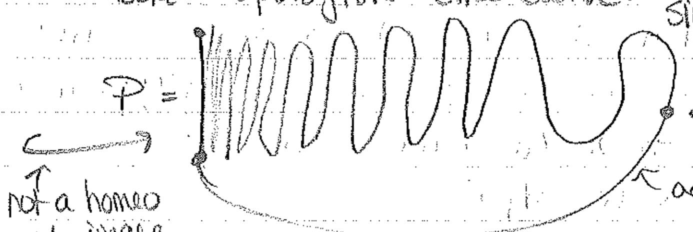
But: $\pi_n(\mathcal{L},*)=0\ \forall n$

Have $* \xrightarrow{f} \mathcal{L}$, but not a h.e., even though $f_* \cong \forall n$,
b/c $\mathcal{L}$ not CW cx (zero skeleton doesn't have the discrete top.)
↳ has the order top

(2) Pseudocircle

Take topologist's sine curve: $\sin(1/x)$

$P =$



ctd but not path-ctd

not a h.e.

$*$

$[0,\infty)$

not a homeo onto image

but embedding of ray into plane

add this to get "circle"

P is not contractible. (HW)

any map $S^n\to P$ is nullhomotopic (a path-ctd subset of P lifts to $[0,\infty)$)

[By Whitehead's Thm, P cannot be h.e. to cell cx]

Cor: A cell cx w/ all $\pi_n=0$ is contractible.

Pf: Any map $*\to X$ will be a h.e. by Whitehead's Thm.

Generally, you need the map f, since you can construct examples:

Ex: $\mathbb{RP}^2$ $\qquad$ $S^2 \times \mathbb{RP}^\infty$

These have same htpy gps (we'll see in a minute)

But not h.e., b/c $H^n(\mathbb{RP}^2; \mathbb{F}_2)$ $\forall n > 2$, but $S^2 \times \mathbb{RP}^\infty$ has cohom. in every dimension.

Here, there's no single map realizing all the $\cong$'s of π_n's.

- Products

$\prod_{\alpha \in \Lambda} (X_\alpha, *_\alpha)$. Then $\pi_n\left(\prod_{\alpha \in \Lambda} (X_\alpha, *_\alpha), (*_\alpha)_{\alpha \in \Lambda}\right) = \prod_{\alpha \in \Lambda} \pi_n(X_\alpha, *_\alpha)$

Pf: map into a product is product of map into factors.

- Covering Spaces

$p: \tilde{X} \to X$ a cov. sp. Pick $\tilde{*} \in \tilde{X}$ in preimage of $*$.

Then $p_*: \pi_n(\tilde{X}, \tilde{*}) \to \pi_n(X, *)$ is an isomorphism $\forall n \geq 2$.

Pf surj: $(S^n, *) \to (X, *)$ has a ! lift to $(\tilde{X}, \tilde{*})$.

$\therefore$ This defines an elt of $\pi_n(\tilde{X})$.

inj: p_* has no kernel: $(S^n, *) \to (X, *)$ is nullhtpc if it extends to a map of the ball D^{n+1}. We can lift this map to $\tilde{X}$ (uniquely), b/c D^{n+1} is s.ctd, so this defines a nullhtpy upstairs. $\square$

$\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, $\pi_1(S^2 \times \mathbb{RP}^\infty) = \mathbb{Z}/2\mathbb{Z}$

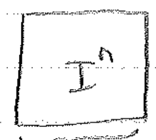
univ. cover of $S^2 \times \mathbb{RP}^\infty$ is $S^2 \times \underset{\uparrow \text{ contractible}}{S^\infty} \simeq_{he} S^2$

& univ. cover of $\mathbb{RP}^2$ is S^2. So univ. covers are isomorphic $\Rightarrow$ $\pi_n(\mathbb{RP}^2, *) \cong \pi_n(S^2 \times \mathbb{RP}^\infty, *)$ $\forall n$.

ex: $f, g: X \to Y$, cell cx's, that induce same maps on π_n's but $f \not\simeq_{he} g$.

4/7 <u>Homotopy Groups</u>

<u>Relative groups</u>: Given $(X, A, *)$, $* \in A \subseteq X$. Want: rel. hom. gps $\pi_n(X, A, *)$.



Let $J^{n-1} = \overline{(\partial I^n - I^{n-1})}$

$I^{n-1} = \{(s_1, \dots, s_n, 0)\} \subseteq I^n$

Define the set $\pi_n(X, A, *) := \{(I^n, I^{n-1}, J^{n-1}) \to (X, A, *)\} /$ htpy of triples

i.e., $J^{n-1} \to *$ & $I^{n-1} \to A$ $\forall$ times t.

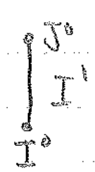
Could also think of this in terms of balls:

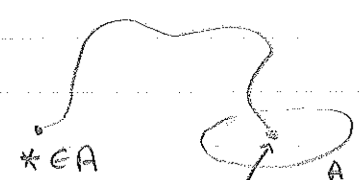
$\pi_n(X, A, *) = \{(D^n, \partial D^n, *) \to (X, A, *)\} /$ htpy of triples (to get this from above, crush J^{n-1} to a pt.)

Addition in $\pi_n(X, A, *)$ defined as before $(n \geq 2)$, but note that for $n=1$, last coord. is not available for gp operation

$\pi_n(X, A, *)$ abel. for $n \geq 3$.

When $n = 1$.

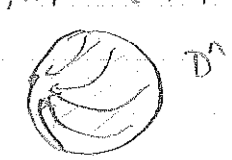




In $\pi_n(X, *)$, an equiv. class $[(S^n, *) \to (X, *)]$ is trivial if it's nullhtpc, i.e. if it extends to a map of D^{n+1}. For relative gps, we have:

<u>Compression Criterion</u>: $(D^n, \partial D^n, *) \to (X, A, *)$ is trivial in $\pi_n(X, A, *)$ $\Leftrightarrow$ $D^n \to X$ is htpc rel ∂D^n to a map into A.

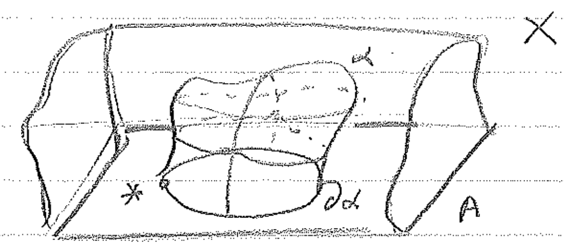
By def, $[(D^n, \partial D^n, *) \to (X, A, *)]$ is trivial when we can homotope it to a map $(D^n, \partial D^n, *) \to (A, A, *)$, just by retracting D^n to a pt



]

Given $(X, A, *)$, $\exists$ LES of htpy gps:

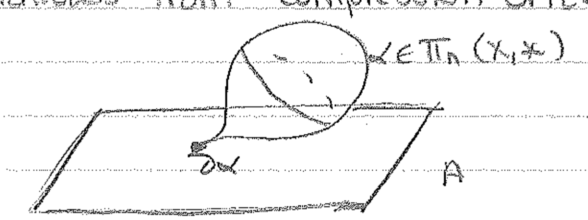
$\pi_{n+1}(X,A,*) \to \pi_n(A,*) \xrightarrow{i_*} \pi_n(X,*) \xrightarrow{j_*} \pi_n(X,A,*) \xrightarrow{\partial} \pi_{n-1}(A,*)$



$\partial\alpha = [\alpha|_{\partial D^n}]$
$= [\alpha|_{I^{n-1}}]$

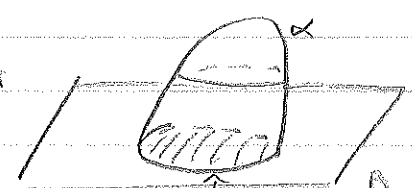
Exactness at $\pi_n(X,*)$ follows from compression criterion.

At $\pi_n(X,A,*)$,



So $i_A \circ \partial \circ j_* = 0$

If $\partial\alpha = 0$, then after filling in, we get a pure class in $\pi_n(X,*)$

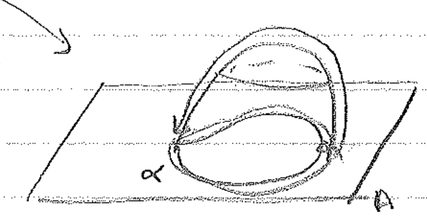


Can fill in since $\partial\alpha = 0$

At $\pi_{n-1}(A,*)$, if $i_*\alpha = 0$,

If $\alpha \in \mathrm{Im}\,\partial$, $\exists \beta$ s.t. $\alpha = \partial\beta$ & $\beta : D^n \to X$ that gives a null htpy of α in X

$\Rightarrow i_*\alpha = 0$



Htpy of α to a pt gives a map $D^n \to X \Rightarrow \alpha \in \mathrm{Im}\,\partial$

4/9 Connectivity Properties (Higher)

Def: A space is simply ctd if it is ctd (path) & $\pi_1(X) = 1$.

↳ ok to drop basept b/c the isomorphism class of π_1 is ind of the basept if X is ctd.

Def: A space X is n-connected if $\pi_i(X) = 0$ for all $i \le n$.

- $i=0 \Rightarrow$ ctd, so n-ctd $\Rightarrow$ ctd.

This is equivalent to:

(1) Every map $S^i \to X$ is null-homotopic (doesn't need to be pted map b/c sp is ctd)

(2) Every map $S^i \to X$ extends to a map $D^{i+1} \to X$.

Def: An n-ctd pair, (X,A), is a pair s.t. $\pi_i(X,A) = 0$ $\forall i \le n$. If $n=0$, a 0-ctd pair (X,A) means every path component of X intersects A. (pick a pt in X, & can always homotope into A)

Whitehead's Thm: If X, Y (h.e. to) cell cx's & $f: X \to Y$ is a map s.t. $f_*: \pi_n(X,*) \to \pi_n(Y,*)$ is an isomorphism $\forall n$, then f is a homotopy equivalence. Furthermore, if f is inclusion of a subcx, then Y deformation retracts onto X.

Def: A map $f: X \to Y$ btwn cell cx's is cellular if $f(X^{(k)}) \subseteq Y^{(k)}$ [k-skeleta] ["nice" b/c not dimension-increasing, but can still be bad]

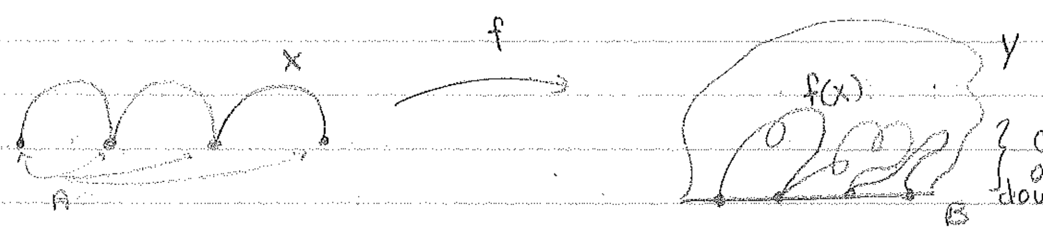
Cellular Approximation Thm: Every map $f: X \to Y$ btwn cell cx's is homotopic to a cellular map. Moreover, if f is cellular on a subcx $A \subseteq X$, then the homotopy may be taken to be stationary on A.

Cor: $\pi_k(S^n) = 0$ when $k < n$.

Pf: $f: S^k \to S^n$ (rep htpy class in π_k) By cell. approx. thm, can homotope f to g s.t. $g \equiv *$
↑ the 0-cell of $S^n = e^0 \cup e^n$

□

Compression Lemma: Let (X, A) be a cellular pair, (Y, B) be any pair w/ $B \neq \emptyset$. $\forall n$ s.t. $X - A$ has n-cells, assume that $\pi_n(Y, B, *) = 0$ $\forall$ choices of $*$. Then every map $f: (X, A) \to (Y, B)$ is htpc rel A to a map $X \to B$. (When $n = 0$, $\pi_0(Y, B, *) = 0$ means (Y, B) a 0-ctd pair)



can push all this down into B!
↓
b/c (for 1-cells) $\pi_1(Y, B$ ... so can push them down.

Pf: Assume inductively that $X^{(k-1)}$ has already been homotoped into B. Consider an open k-cell e^k in $X - A$, w/ characteristic map $\Phi: D^k \to X$. Apply f to get $f\Phi: (D^k, \partial D^k) \to (Y, B)$. Since $\pi_k(Y, B, *) = 0$, our compression criterion allows us to homotope $f\Phi$ rel ∂D^k into B. Doing this on all of the k-cells in $X - A$ at once & doing the constant htpy on A, we may homotope $f|_{X^{(k)} \cup A}$ into B. By the htpy extension property (0.16), this htpy extends to X.

Note: Outside of $X^{(k)} \cup A$, this could still be bad. If there's a bound on the dim of cells of $X - A$, just do this induction step fin. many times. If not, perform faster & faster htpy's. This produces the htpy we want, & cts b/c eventually stationary on each skeleton. □

Pf (Whitehead's Thm):
Case I: Assume $f: X \to Y$ is inclusion of a subcx.
The LES of pair (Y,X) & our hypoth of $\simeq$ on π_i
$\Rightarrow \pi_n(Y,X,*) = 0 \ \forall n \Rightarrow$ (by lemma) Y def retr. onto X.
Case II: Replace Y w/ the htpy equiv M_f, $M_f = X \times I \sqcup Y / (x,1) \sim f(x)$.
If f is cellular, (M_f, X) is a cellular pair, so done by case I.
(X here is $X \times \{0\}$)
Else, use cell. approx. thm to homotope f to be cellular, which doesn't change the htpy type of M_f, so done □

4/11 Revisit pf. of Whitehead's Thm:
Case I: $f: X \to Y$ inclusion of a subcx. Consider (Y,X):

$$\to \pi_n(X) \underset{\simeq}{\xrightarrow{f_*}} \pi_n(Y) \to \underbrace{\pi_n(Y,X)}_{\Rightarrow = 0} \to \pi_{n-1}(X) \underset{\simeq}{\xrightarrow{f_*}} \pi_{n-1}(Y) \to \underbrace{\pi_{n-1}(Y,X)}_{\Rightarrow = 0} \to$$

Then by compression lemma, $Y \searrow X$.
Case II: Homotope f to be cellular. Replace Y w/
$M_f = X \times I \sqcup Y / (x,1) \sim f(x)$. $M_f \simeq_{he} Y$ $(M_f \searrow Y)$. Since f is cellular, $(M_f, X \times \{0\})$ is a cellular pair, and the inclusion $X \times \{0\} \xrightarrow{i} M_f$ is s.t. $i_*: \pi_n(X) \to \pi_n(M_f)$ is an iso, still. Done by case 1.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y \arrow[d, "h.e."] \\ & M_f \end{tikzcd}$$
commutes up to htpy.

□

• Can we compute $\pi_n(X)$? Usually no — very hard.
Harder than homology b/c excision fails.
$\pi_k(S^n) = ?$ (open problem)

Ex: $\pi_0(S^1) = 0$
$\pi_1(S^1) \simeq \mathbb{Z}$
$\pi_k(S^1) = 0 \ \forall k > 1$, since $\pi_k(S^1) \simeq \pi_k(\mathbb{R}) = 0$ b/c $\mathbb{R}$ contractible.
univ. cover

S^1 is nice in that its univ. cover is contractible.
Def: X is aspherical if its univ. cover exists & is contractible.

ex Ex: A surface F that is not homeomorphic to S^2 or $\mathbb{RP}^2$ is aspherical. B/c for all such F, the univ. cover is homeo. to the plane, $\mathbb{R}^2$. (try a direct argument when F is closed)
$\pi_k(F) = 0 \ \forall k > 1$ for such F.

Def: Given a gp, G, a $K(G,1)$ is an aspherical cell cx w/ $\pi_1(K(G,1)) \cong G$.
More generally, a $K(G,n)$ is a space s.t. $\pi_n(K(G,n)) = G$ & $\pi_k(K(G,n)) = 0 \ \forall k \neq n$. These are called Eilenberg-Mac Lane spaces.

- If X, Y are $K(G,1)$'s, then $X \simeq_{he} Y$: Build a map:
Let $X^{(0)} \to Y^{(0)}$ be a fcn (easier to imagine if they're singletons - crush a max'l tree in each).
$X^{(1)} \to Y^{(1)}$ s.t. this map realizes the isomorphism on $\pi_1(X) \to \pi_1(Y)$
Since that's an $\cong$ (on π_1's), we can extend over 2-cells. B/c if can extend over 2-cell in X, that loop is $0 \in \pi_1(X) \Rightarrow 0 \in \pi_1(Y) \Rightarrow$ can extend over 2-cell in Y.
$X^{(2)} \to Y^{(2)}$
Then can extend map over 3-cells b/c map $S^2 \to Y$, & $\pi_2(Y) = 0 \Rightarrow$ can extend to $X^{(3)} \to Y^{(3)}$.
Etc.
Then by Whitehead, $X \simeq_{he} Y$.

Back to spheres:

$\pi_k(S^2) = ?$

Observations:

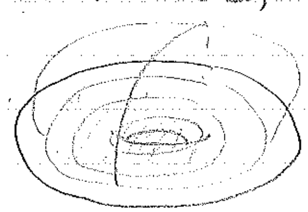
- $\pi_2(S^2) \neq 0$ i.e. $\mathbb{1}$ not nullhom. In fact, the degree completely determines a map $S^2 \to S^2$, & so $\pi_2(S^2) \cong \mathbb{Z}$.
- $\pi_3(S^2) \cong \mathbb{Z}$, & the generator is called the Hopf fibration — fibration of $\mathbb{R}^2$ by round tori of revolution, cut by planes, & each torus filled w/ circles $\Rightarrow \mathbb{R}^2$ filled w/ circles. If crush each circle to pt, get S^2:

$$\begin{array}{ccc} S^1 & \to & S^3 \\ & & \downarrow \pi \\ & & S^2 \end{array}$$

use les of fibration to get $\pi_2 \to$ gen by π_1

4/12 Hopf Fibration: there is a homotopically essential map $S^3 \to S^2$. (In fact, $\pi_3(S^2) \cong \mathbb{Z}$ & is gen by this map)

S^3:



$S^1 \times S^1 \subset S^3$. $S^3 \setminus (S^1 \times S^1)$ is 2 solid tori: $\mathring{D}^2 \times S^1 \sqcup \mathring{D}^2 \times S^1$

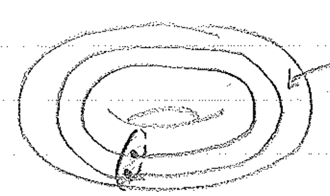
Think of $S^3 = \mathbb{R}^3 \cup \{\infty\}$

OR: Torus can have center filled w/ disks $\times I$, & becomes sphere — so complement in S^3 a sphere. Add the disk $\times I$, get:  = solid torus.

[Called a Heegaard splitting, a decomp of a 3-mfld as (Ball w/ handles) $\sqcup$ (ball w/ handles)]

So S^3 = union two solid tori (glued along their boundries = some torus). We want to use these to fill up S^3 with circles.

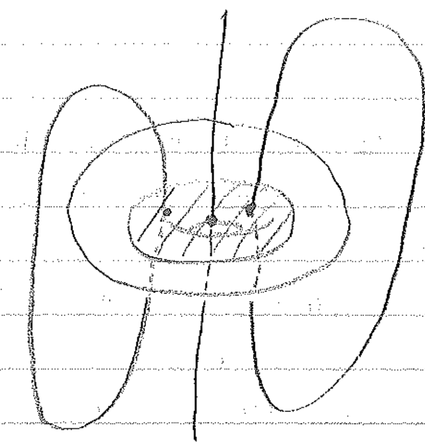
Inner torus:



$D^2 \times S^1 \leftarrow$ fill it w/ all pts $\times S^1$

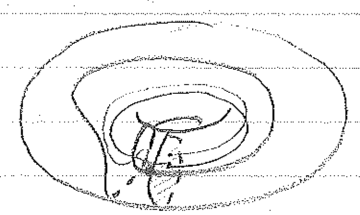
This fills up the solid torus w/ circles.

Same thing for the outside solid torus:

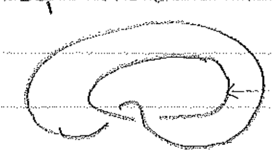


these circles include the boundaries of the disks of the inner torus, & vice versa.

These 2 families of circles don't "match up" on T^2.
Fix this by putting in some "twist":
Cut solid torus along disk & rotate one side 360° & reglue:



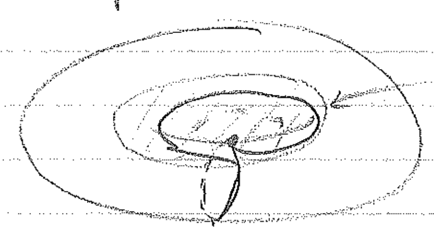
- one central curve unchanged & all others wrap around it once:



Note: This is homeo. to original solid torus, & top. still a product $D^2 \times S^1$.

← family of S^1's

Do same thing to outside solid torus so that the circles match up on T^2:



cut here & twist s.t. circle becomes one of our family, rotate top 360° counterclockwise looking down

So now we have $S^3 = \cup S^1$'s in a nice way. In fact, it's very nice:

Consider the quotient space $S^3/\sim$ where $\sim$ is an equiv. rel. that has classes equal to S^1's (ie, crush each circle to a pt). On each solid torus, it's just D^2, but $\sim$ agrees on boundary. This quotient sp. is homeomorphic to $D^2 \sqcup D^2$ glued along their boundaries (by a homeo). Thus $S^3/\sim \cong S^2$. So we have a quotient map

$\pi : S^3 \to S^2 \cong S^3/\sim$. This is called the <u>Hopf fibration</u>. Every pt $x \in S^2$ has a nbhd $U \ni x$ s.t. $\pi^{-1}(U) \cong U \times S^1$ & $\pi|_{\pi^{-1}(U)}$ is product projection onto U.

What we have is an example of a <u>fiber bundle</u>. (map where all fibers homeo, & locally the map looks like projection in a product).

From a general discussion of fiber bundles, we get a LES of htpy gps:

$$\cdots \to \pi_{n+1}(S^2) \to \underbrace{\pi_n(S^1)}_{\text{fiber}} \xrightarrow{i_*} \underbrace{\pi_n(S^3)}_{\text{total sp}} \xrightarrow[\text{projection}]{\pi_*} \underbrace{\pi_n(S^2)}_{\text{base}} \xrightarrow{\partial} \pi_{n-1}(S^1) \to \pi_{n-1}(S^3) \to \cdots$$

<u>Ex</u>: Ever covering sp is a fiber bundle where the fiber is discrete.

(bundle has homotopy lifting property for disks – allows us to define $\partial \to$ next time)

$$\to \underset{0}{\underset{\|}{\pi_3(S^1)}} \to \pi_3(S^3) \xrightarrow{\pi_*} \pi_3(S^2) \to \underset{0}{\underset{\|}{\pi_2(S^1)}} \to 0$$

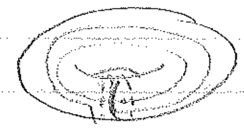
$\Rightarrow \pi_3(S^3) \cong \pi_3(S^2)$

$\underset{\mathbb{Z}}{\underset{\|}{}}$ ← we'll see – every map $S^3 \to S^3$ determined by its degree.

Barman: mapping cyl of covering map → just need lower set to be fibration

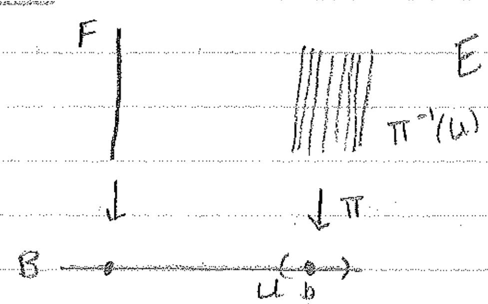
4/16 Hopf Fibration

$S^3 = \cup S^1$'s
$= D^2 \times S^1 \cup_\partial S^1 \times D^2$ ← glue along boundary

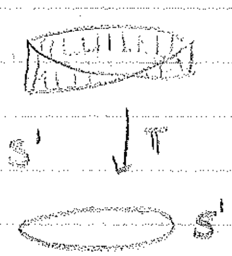


Crushing S^1's to a pt gives map $\pi: S^3 \to S^2$. Every $x \in S^2$ has a nbhd U s.t. $\pi^{-1}(U) \cong U \times S^1$ s.t. $\pi|_{\pi^{-1}(U)}$ = projection to 1st factor.
[In fact, delete any pt y from S^2 & let $U = S^2 \setminus \{y\}$. Then the above holds.]

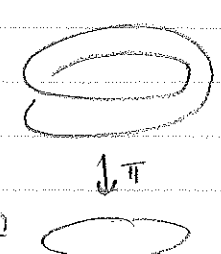
Fiber Bundle: A fiber bundle is (F, E, B, π), where $\pi: E \to B$ is a surj. map btwn spaces s.t. for each $b \in B$, $\exists$ open $U \ni b$ s.t. $\pi^{-1}(U) \cong U \times F$ and $\pi|_{\pi^{-1}(U)}$ = projection of $U \times F$ onto 1st factor. F is the fiber, B is the base, & E is the total space.



These are not products in general.
Ex: (1) Hopf Fibration $S^3 \to S^2$.
$S^3 \ncong S^2 \times S^1$
(2) Mobius Band
$F = I$
$B = S^1$
⇒ fiber bundle over S^1



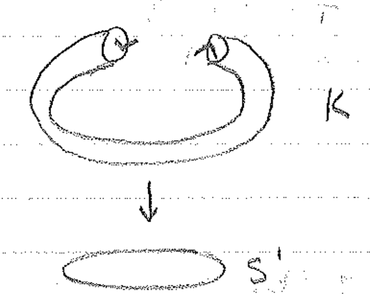
(3) $\pi: S^1 \to S^1$ a deg n covering map
a nontrivial bundle
F = discrete set of pts
Note: $B \nsubseteq E$, & there may not be a section of π, i.e. $s: B \to E$ s.t. $\pi(s) = \mathrm{id}$



(4) In general, a cov. sp is a fiber bundle where F is discrete.

Def: A map $s: B \to E$ s.t. $\pi \circ s = \mathbb{1}_B$ is a section.

(5) Klein Bottle fibers over S^1



(6) Mapping torus of homeomorphisms, $h: X \to X$

$T_h = X \times [0,1] / (x,0) \sim (h(x), 1)$

Projection onto $[0,1]$ induces a map $T_h \to S^1$, which is a bundle.

Let $E \xrightarrow{p} B$ be any map. We say p has the homotopy lifting property wrt a space X if given a htpy $g_t: X \to B$ and a lift $\tilde{g}_0: X \to E$ of g_0, then there exists $\tilde{g}_t: X \to E$ of g_t.

A fibration is a map $p: E \to B$ that has the htpy lifting property for every X.

ex: products w/ projections are fibrations:

$B \times F \xrightarrow{\pi} B$ projection, given $g_t: X \to B$, can lift $\tilde{g}_t = (g_t(x), h(x))$, if $\tilde{g}_0(x) = (g_0(x), h(x))$

Thm: If $p: E \to B$ has HLP wrt disks D^k $\forall k \geq 0$. (a Serre fibration), pick a basept $b_0 \in B$ & a basept $x \in F = p^{-1}(b_0)$, then $p_*: \pi_n(E,F) \to \pi_n(B)$ is an isomorphism $\forall n \geq 1$.

So, if B is path-ctd, then the LES for p_* is:

$$\cdots \to \pi_n(F) \to \pi_n(E) \xrightarrow{p_*} \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E) \to 0$$

<u>Prop</u>: A fiber bundle has HLP $\forall$ cellular X (in partic. wrt disks). [∴ thus there is a LES]

Consider the Hopf fibration $\pi : S^3 \to S^2$.

$$\underset{\overset{\shortparallel}{0}}{\pi_3(S^1)} \to \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2) \to \underset{\overset{\shortparallel}{0}}{\pi_2(S^1)} \to \underset{\overset{\shortparallel}{0}}{\pi_2(S^3)} \to \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \to \underset{\overset{\shortparallel}{0}}{\pi_1(S^3)} \to \cdots$$

univ. cover contr.

In HW: $\pi_k(S^k) \cong \mathbb{Z}$, b/c det by degree — homotope map so it's smooth most places.

4/18 Hopf Fibration is part of an infinite family: bundle: $S^1 \to S^{2n+1} \xrightarrow{p} \mathbb{C}P^n$ (note: common to write a bundle like a ses) defined by:

Think of S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1} - \{0\}$. S^{2n+1} intersects each cx line in a circle. This gives you a bundle $S^1 \to S^{2n+1} \xrightarrow{p} \mathbb{C}P^n$.

- Locally a product (ie. local trivialization):

  $U_i \subseteq \mathbb{C}P^n$, $U_i = \{[z_0 : \cdots : z_n] \mid z_i \neq 0\}$. Define

  $h_i : p^{-1}(U_i) \to U_i \times S^1$

  $h_i(z_0, \ldots, z_n) := ([z_0 : \cdots : z_n], z_i/|z_i|)$

  check this is a homeo w/ inverse

  $g_i([z_0 : \cdots : z_n], \lambda) = \lambda |z_i| z_i^{-1}(z_0, \ldots, z_n)$

- When $n=1$, $\mathbb{C}P^1 = S^2$ (Riemann sphere), so $S^1 \to S^3 \to S^2$, the Hopf fibration.

Computing π_n: Why is this hard? B/c excision fails.

$\pi_n(X,A) \not\cong \pi_n(X/A)$

Ex: FACT: $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. (difficult computation)

Consider the inclusion $S^2 \hookrightarrow S^3$ as the "equator"

(1) $\pi_4(S^3, S^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, from LES of (S^3, S^2):

$$\cdots \to \pi_4(S^2) \xrightarrow{i_* = 0} \pi_4(S^3) \hookrightarrow \pi_4(S^3, S^2) \twoheadrightarrow \pi_3(S^2) \xrightarrow{i_* = 0} \pi_3(S^3) \to \cdots$$

$i_* = 0$ b/c S^2 is nullhomotopic in S^3; $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$.

$\pi_3(S^2) \cong \mathbb{Z}$ by Hopf fib. Proj, so $\exists$ section $\Rightarrow$ SES splits

$\Rightarrow \pi_4(S^3, S^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

(2) $\pi_4(S^3/S^2) = \pi_4(S^3 \vee S^3) \cong \pi_4(S^3 \times S^3) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$S^3 \vee S^3$ is 3-skeleton of $S^3 \times S^3$, & then add a 6-cell $(e^3 \times e^3)$

i.e. $S^3 \times S^3 = S^3 \vee S^3 \cup e^6$

By simplicial approximation: $\pi_4(X)$ only depends on the 5-skeleton. $S^4 \to X$ null hom. can extend to 5-ball & homotope into 5-skel.

More gen, $\pi_k(X)$ depends only on $X^{(k+1)}$ [from $X^{(k+1)} \hookrightarrow X$ & LES $(X, X^{(k+1)})$]

Here the 5-skel. is the 3-skel, so get iso. π_4's.

Ex: $S^1 \hookrightarrow S^2$ as the equator also a counterex

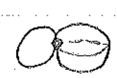
(1) $\pi_3(S^2, S^1)$

(2) $\pi_3(S^2 \vee S^2) \not\cong \pi_3(S^2 \times S^2)$

has 4-cell, so previous trick doesn't work

Actually, $\pi_3(S^2 \vee S^2) \cong \mathbb{Z}^3$ (need a spectral seq.)

Ex:

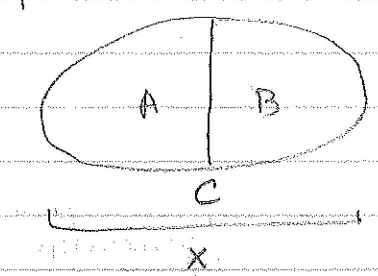
$X =$  $\pi_2(X)$ not f.g. b/c $\tilde{X} =$ 

so ∞ly many lin. ind maps of spheres into $\tilde{X}$.

(in this case, f.g. as a $\pi_1(X)$-module — b/c only one sphere after mod out by deck trans.)

Excision for Htpy Gps

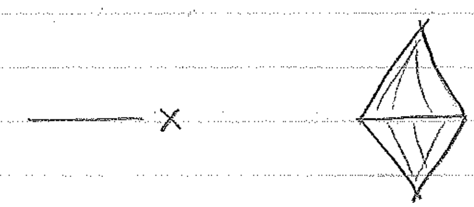
Not all bad: Assume X is cellular, A, B subcx's of X with $X = A \cup B$ & $C = A \cap B \neq \emptyset$ & ctd. Assume (A, C) is m-ctd and (B, C) is n-ctd. Then, $\pi_i(A, C) \xrightarrow{\iota_*} \pi_i(X, B)$ is an isomorphism when $i < m+n$ and ι_* is surj when $i = m+n$.



[i.e., can throw away B if in small enough dim & pairs are ctd enough]

Pf: Technical ✓

Cor: (Freudenthal Suspension Thm): The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ & surjective for $i = 2n-1$. In fact, if X is $(n-1)$-ctd, the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an iso for $i < 2n-1$ & surj for $i = 2n-1$.



$SX = C_+X \cup_X C_-X$

$S^i \xrightarrow{f} X$

$S^{i+1} \xrightarrow{g} SX$ by $g(S^{i+1}) = f(S^i) \times I /$ crush endpts

$S^i \times I /$ crush top & bottom $\cong S^{i+1}$

✓ Check this is the susp. map

Pf: $\underset{\text{LES of } (C_+X, X)}{\pi_i(X) \cong} \pi_{i+1}(C_+X, X) \xrightarrow{\iota_*} \pi_{i+1}(SX, C_-X) \underset{\text{LES of } (SX, C_-X)}{\cong} \pi_{i+1}(SX)$

For either pair, (C_+X, X) is n-ctd by LES of pair if X is $(n-1)$-ctd
$\Rightarrow$ (by thm) ι_* has desired properties. □

Cor: $\pi_n(S^n) \cong \mathbb{Z}$, gen by $\mathbb{1}$, $n \geq 1$. In particular, $\deg: \pi_n(S^n) \to \mathbb{Z}$ is an $\cong$.

Pf: Consider suspension maps $\underbrace{\mathbb{Z} \cong \pi_1(S^1) \twoheadrightarrow}_{\text{by cov. sp.}} \pi_2(S^2) \xrightarrow{\cong} \pi_3(S^3) \xrightarrow{\cong} \cdots$
by the thm

$\Rightarrow$ $\pi_n(S^n)$ are all cyclic, gen. by $\mathbb{1}$ (b/c susp. of $\mathbb{1}$ is $\mathbb{1}$). Note: $\pi_n(S^n)$ is infinite: there are deg k maps $S^k \to S^n$ for all k, n. (use homology to show these maps are not homotopic & so are diff elts of $\pi_n(S^n)$) $\square$

4/21 Wed. 4pm - problem session

Last time: Freudenthal suspension thm $\Rightarrow$ $\pi_n(S^n) \simeq \mathbb{Z}$ when $n \geq 1$.

More generally, $\pi_n(\bigvee_{\alpha \in \Lambda} S^n_\alpha) \simeq \bigoplus_{\alpha \in \Lambda} \pi_n(S^n_\alpha)$, $n \geq 2$

Pf: • $|\Lambda| < \infty$:

$$\bigvee_\alpha S^n_\alpha \simeq \left(\prod S^n_\alpha\right)^{(n)}$$

. With the usual cell structure, $\prod_\alpha S^n_\alpha$ only has cells in dimensions divisible by n, so the pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$ is $(2n-1)$-ctd. Then for the LES of the pair,

$\bigvee_\alpha S^n_\alpha \hookrightarrow \prod_\alpha S^n_\alpha$ induces isomorphisms on π_n if $n \geq 2$. By previous observation, $\pi_n(\prod_\alpha S^n_\alpha) \simeq \bigoplus_\alpha \pi_n(S^n_\alpha) \simeq \bigoplus_\alpha \mathbb{Z}_\alpha$

• $|\Lambda| = \infty$: There's a map induced by all inclusions $S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$:

$\bigoplus_\alpha \pi_n(S^n_\alpha) \to \pi_n(\bigvee_\alpha S^n_\alpha)$

- surj b/c S^n cpt:

$$S^n \to \bigvee S^n_\alpha$$

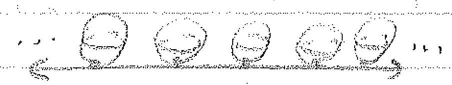
only hits fin. many factors (since $\bigvee S^n_\alpha$ has weak top - b/c cpt sets only contain fin many cells in weak top)

- inj for same reason:

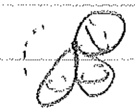
if $S^n \to \bigvee S^n_\alpha$ extends over ball $D^{n+1} \to \bigvee S^n_\alpha$, ball also hits only fin many factors, & done by 1st case. $\square$

Htpy gps $\pi_k(\bigvee S^n_\alpha)$ intractible. (hard even when k close to n: $\pi_3(S^2 \vee S^2) \simeq \mathbb{Z}^3$)

$\underline{Ex}$: $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$
$\pi_2(S^1 \vee S^2) \cong \bigoplus_{-\infty}^{\infty} \mathbb{Z}$ by covering sp's. & preceding ex:
univ. cover of $S^1 \vee S^2$ (gen by inclusion maps of each S^2)



$\mathbb{R}$



$\simeq$ wedge of spheres by crushing line to a pt.

We have cpt X with infinitely generated $\pi_2(X)$.
But $\pi_2(S^1 \vee S^2)$ is fin.-gen. as a $\pi_1(S^1 \vee S^2)$-module (i.e. up to action of π_1, there's only one S^2).
Even this is not always the case.
exercise in Hatcher: $\pi_3(S^1 \vee S^2)$ is $\underline{\text{not}}$ fin. gen, as a $\pi_1(S^1 \vee S^2)$ module.
Even worse, there's an example due to Stallings of a cpt 2-cx X s.t. $\pi_2(X)$ is not f.g. as a $\pi_1(X)$-mod.

How do you build a space X w/ $\pi_1(X)$ = your favorite gp G?
Any gp G has a presentation $G = \langle \Lambda \mid R \rangle$.

$X = \bigvee_{\alpha \in \Lambda} S^1_\alpha \cup \left(\bigcup_{r \in R} D^2_r \right)$ ← according to relations in presentation

Then V.K. $\Rightarrow$ $\pi_1(X) = G$. Now, we can do this & get a sp. with no higher htpy gps by adding high dim'l cells as follows:

(1) Suppose $\pi_2(X) \neq 0$. Attach a 3-cell for each elt of $\pi_2(X)$ to kill that elt. Get new space, X'.
(2) Attach 4-cells to X', ... continue forever.
(3) Let $K = X \cup X' \cup X'' \cup \cdots$. Then $\pi_1(K) = G$ & $\pi_n(K) = 0$ for $n \geq 2$. K is a $K(G,1)$ ($\pi_1 = G$ & univ. cover contr, by Whitehead's Thm, b/c all htpy gps of $\tilde{K}$ vanish)

How to build a $K(G,n)$ for G abel, $n \geq 2$:
Let $X = (\bigvee_{\alpha \in A} S^n_\alpha) \cup (\bigcup e^{n+1}_\beta)$, w/ $\phi_\beta : S^n \to \bigvee_\alpha S^n_\alpha$ $n \geq 2$.
Claim: $\pi_n(X) = \bigoplus_{\alpha \in A} \mathbb{Z} / \{[\phi_\beta]\}$
[Aside: for $i<n$, $\pi_i(X)=0$ by cell. approx. $\Rightarrow X$ $(n-1)$-ctd]
Pf: Consider the LES of $(X, \bigvee_\alpha S^n_\alpha)$:
$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \xrightarrow{\partial} \pi_n(\bigvee_\alpha S^n) \to \pi_n(X) \to \pi_n(X, \bigvee S^n_\alpha)$

$\pi_{n+1}(X, \bigvee_\alpha S^n_\alpha) \cong \pi_{n+1}(\bigvee_\alpha S^{n+1}_\alpha) \simeq \bigoplus_\beta \mathbb{Z}_\beta$ one for each ϕ_β; $\pi_n(X, \bigvee S^n_\alpha) = 0$

(using slight variant of excision for htpy gps:
Prop: If (X,A) r-ctd & A s-ctd, $r,s \geq 0$. Then $\pi_i(X,A) \to \pi_i(X/A)$ is an $\cong$ for $i \leq r+s$ & surj for $i = r+s+1$.

This X has $\pi_n(X) \simeq \bigoplus_{\alpha \in A} \mathbb{Z}_\alpha / \{[\phi_\beta]\}$. (generic abel. gp, G)
Now kill higher htpy gps as before to get a $K(G,n)$.

4/23 B115 Last time we built a $K(G,n)$ for any G when $n=1$ & any abel. G when $n>1$ by starting with $(\bigvee_\alpha S^n_\alpha) \cup (\bigcup_\beta e^{n+1}_\beta) = X$ with $\pi_n(X) \simeq G$ & then added cells to kill higher htpy gps.

Thm: The htpy type of $K(G,n)$ is completely determined by G & n.
Pf: We just need to show that given K, Y cell-cx's w/ $\pi_n(K) \simeq \pi_n(Y) \simeq G$ & $\pi_k(K) \simeq \pi_k(Y) = 0$, $k \neq n$, we have a map $K \to Y$ realizing an $\cong$ $\pi_n(K) \to \pi_n(Y)$.

Lemma: Let X be a ctd space of the form $(\bigvee_{\alpha \in A} S^n_\alpha) \cup (\bigcup_\beta e^{n+1}_\beta)$. If Y is a space & $\pi_n(X) \xrightarrow{\phi} \pi_n(Y)$ is a homomorphism, then there is a map $f: X \to Y$ s.t. $f_* = \phi$

Lemma ⇒ Thm: Given the $K(G,n)$ we built and a hom. $\Phi: G \to \pi_n(Y)$, then lemma gives a map $f: K^{(n+1)} \to Y$ s.t. $f_* = \Phi$. Then, given an $(n+2)$-cell e^{n+2} w/ attaching map $\psi: S^{n+1} \to K^{(n+1)}$, can extend f over e^{n+2} since $f \circ \psi \simeq c_{x_0}$ b/c $\pi_{n+1}(Y) = 0$ (b/c $K(G,n)$ space).
Continue to extend f to a map $K \to Y$.
[Since only extended f over skeleta, doesn't change induced map b/c of fact in margin] □

* π_n only depends on $(n+1)$-skeleton by cellular approx. *

Pf of Lemma: (Morally same as above)
$\pi_n(X) \cong \bigoplus_{\alpha \in A} \mathbb{Z}_\alpha / \langle [\phi_\beta] \rangle$. Given $\Phi: \pi_n(X) \to \pi_n(Y)$, Know where each generator of $\pi_n(X)$ is sent.
$X^{(0)} = \{*\}$. Define $f(\{*\}) = \{*\}$ (basept of Y)
Define $f|_{S^n_\alpha} \overset{(any)}{=}$ a representative of $\Phi \circ i: S^n_\alpha \to Y$
So we've defined f on $X^{(n)}$. Given an $(n+1)$-cell e^{n+1}_β w/ attaching map $\psi_\beta: \partial e^{n+1}_\beta \to X^{(n)}$. $\psi_\beta: \partial e^{n+1}_\beta \to X^{(n)}$ is nullhtpc in X (by def, b/c extends over $\boxed{\text{"}S^{n}\text{"}}$ that cell). Since Φ is a homomorphism, $f \circ \psi_\beta$ is nullhtpc in Y, so we can extend f to e^{n+1}_β. Continue. □

More generally: Given a map $X^{(n-1)} \to Y$, the only obstruction to extending to $X^{(n)}$ is $\pi_{n-1}(Y)$.
[i.e. is the map of the $\partial e^n = S^{n-1}$ nullhtpc in Y?]

Cor: (of $n=1$ case) Let G be a gp. Then G has well-defined cohomology gps, $H^k(G;R)$ & homology gps, $H_k(G;R)$, defined by $H^k(K(G,1);R)$, & $H_k(K(G,1);R)$, resp.

Given $(X,*)$ & $(Y,*)$, let $\langle X,Y\rangle = \{(X,*)\to(Y,*)\}/_{\text{htpy}}$
basept-pres. htpy.

<u>Thm</u>: There is a natural bijection (if X a cell cx)
$H^n(X;G) = \langle X, K(G,n)\rangle$

<u>Idea</u>: $n=1$: Think $G=\mathbb{Z}$, $K(G,1)=S^1$
$H^1(X;\mathbb{Z}) = \text{Hom}(H_1(X),\mathbb{Z}) = \text{Hom}(\pi_1(X),\mathbb{Z})$
b/c $\pi_1 = (H_1)^{ab}$

then "more generaley" statement lets us extend to
map $X^{(2)}\to S^1$ then $\pi_i(S^1)=0\ \forall i\geq 2 \Rightarrow$ extend to
map $X\to S^1$.

$n>1$: same principle holds.

4/25 Another reason to like $K(G,1)$'s:

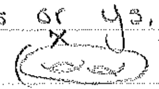
<u>Ex</u>: $K(\mathbb{Z},1)=S^1$
$K(\mathbb{Z}/2\mathbb{Z},1)=\mathbb{RP}^\infty$

- $\mathbb{RP}^2 \neq K(\mathbb{Z}/2\mathbb{Z},1)$ b/c $\pi_2(\mathbb{RP}^2)=\mathbb{Z}$; so $\mathbb{RP}^2$ not aspherical. Fix by adding cells – get $\mathbb{RP}^\infty$, & univ. cover is S^∞ (contractible) $\Rightarrow \mathbb{RP}^\infty$ is aspherical.

$\to$ Every $K(\mathbb{Z}/2\mathbb{Z},1)$ has infinite dimension

$\to$ If G has torsion, $K(G,1)$ is ∞ dim'l.
(non-obvious) [By thm, $\mathbb{Z}$ is torsion-free]

Actually, this is how you prove gps are torsion-free, unless you have a very explicit normal form for the gp, for example

- $\mathbb{F}_2$ on $\{x,y\}$. $w\in\mathbb{F}_2$ has a normal form $w = x^{\varepsilon_1}y^{\varepsilon_2}\cdots x^{\varepsilon_{n-1}}y^{\varepsilon_n}$ w/ no obviously canceling pairs of xs or ys. So $w\neq 1 \Rightarrow w^k\neq 1\ \forall k\neq 0$.
- Let $G=\pi_1($  $) = \langle a,b,c,d \mid [a,b][c,d]\rangle$
You can prove directly that G is torsion-free, but it's not too easy. But by thm, since $\tilde{X}=\mathbb{R}^2$, so $X=K(G,1) \Rightarrow G$ torsion-free.

$\boxed{\text{Hurewicz Theorem}}$ Let (a cell cx) X be $(n-1)$-ctd for $n \geq 2$. Then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$.

i.e., the first nontrivial homotopy gp coincides w/ the first nontrivial (reduced) homology gp.

Relative version: If (X,A) is $(n-1)$-ctd for $n \geq 2$, w/ $A \neq \emptyset$ & 1-ctd, then $H_i(X,A) = 0$ $i < n$ and $\pi_n(X,A) \cong H_n(X,A)$.

Pf: By excision for htpy gps (version 2: namely Prop. 4.28), we have $\pi_i(X,A) \cong \pi_i(X/A)$ for $i \leq n$, & also $H_i(X,A) \cong \tilde{H}_i(X/A)$, so the relative case follows from the absolute case.

In the pure case, X is $(n-1)$-ctd (i.e. $\pi_i(X) = 0$ $\forall i \leq n-1$)

Claim: Can crush $X^{(n-1)}$ to a pt w/o changing the htpy type. → quotient map & use Whitehead. (4.16)

Assume $X^{(n-1)} = \{*\} \Rightarrow \tilde{H}_i(X) = 0$ for $i < n$.

We're only interested in $\pi_n(X)$. Since π_n depends only on $X^{(n+1)}$, replace X with $X^{(n+1)}$ (i.e. assume X is $(n+1)$-dim'l). Since $X^{(n-1)} = \{*\}$, $X = (\bigvee_{\alpha \in \Lambda} S^n_\alpha) \cup (\bigcup_{\beta \in \Omega} e^{n+1}_\beta)$

Let $\phi_\beta : D^{n+1}_\beta \to X$ be the characteristic maps (ptd maps)

$\pi_n(X) \cong \oplus \mathbb{Z}_\alpha / \langle [\phi_\beta|_{\partial D^{n+1}_\beta}] \rangle$, but how do we relate it to H_n?

By les of pair,

$$\pi_n(X) = \operatorname{coker}(\pi_{n+1}(X, X^{(n)}) \xrightarrow{\partial} \pi_n(X^{(n)}))$$

$$[\pi_{n+1}(X, X^{(n)}) \to \pi_n(X^{(n)}) \to \pi_n(X) \to \underset{0?}{\pi_n(X, X^{(n)})} \to \cdots]$$

$$= \operatorname{coker}(\bigoplus_{\beta \in \Omega} \mathbb{Z}_\beta \longrightarrow \bigoplus_{\alpha \in \Lambda} \mathbb{Z}_\alpha)$$

$$= \operatorname{coker}(H_{n+1}(\overset{X^{(n+1)}}{\overset{\shortparallel}{X}}, X^{(n)}) \xrightarrow{\partial} H_n(X^{(n)}, \underset{\{*\}}{\underset{\shortparallel}{X^{(n-1)}}}))$$

$$= H_n(X)$$

$\square$

Cor (Strong Whitehead Thm): If X, Y are s. ctd, & $f: X \to Y$ is a map with $f_*: H_n(X) \to H_n(Y)$ an isomorphism $\forall n$, then f is a htpy equivalence.

Pf: By using the mapping cylinder, assume f is an inclusion of a subcx.

$\pi_1(Y, X) = 0$ b/c 1-ctd

relative Hurewicz thm says that 1st non-zero $\pi_n(Y, X)$ is 1st non-zero $H_n(Y, X)$. But $H_n(Y, X) = 0$ $\forall n$ by hypothesis $\Rightarrow$ $\pi_n(Y, X) = 0$ $\forall n$. Then the les of $(Y, X) \Rightarrow$ $\pi_n(X) \xrightarrow{f} \pi_n(Y)$ are $\cong$ $\forall n$, so done by Whitehead's thm. $\square$

4/28 Homotopy & Cohomology

Thm (informal): Cohomology gps of X are spaces of maps from X.

(Precisely): Given pted spaces $(X, *)$, $(Y, *)$, let $\langle X, Y \rangle = \{(X, *) \to (Y, *)\} / \text{htpy of pairs}$. There is a (natural) bijection $H^n(X; G) = \langle X, K(G, n) \rangle$. Here, X a cell cx, G an abel. gp.

We've seen this in $H^1(X; \mathbb{Z}) = \langle X, S^1 \rangle$.

$$\begin{aligned} &\parallel \\ &\text{Hom}(H_1(X), \mathbb{Z}) \\ &\parallel \\ &\text{Hom}(\pi_1(X), \mathbb{Z}) \end{aligned}$$

(every map $\pi_1 \to$ Abel. gp factors thru $(\pi_1)^{Ab} = H_1$)

everything here determined by a map $f: X \to S^1$ (b/c π_1 gives info on what to do on 2-skeleton, & since $S^1 = K(\mathbb{Z}, 1)$, can extend over rest of X, b/c $\pi_i(S^1) = 0$ $\forall i \geq 2$.

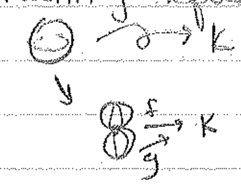
Q: Why is $\langle X, K(G,n)\rangle$ a group?

Let K = cell cx. Is $\langle X, K\rangle$ a gp in any way?

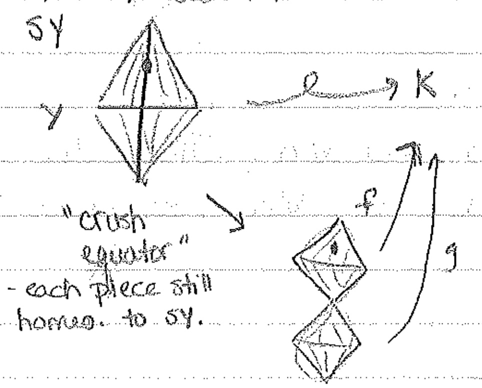
If $X = S^n$, yes b/c $\langle S^n, K\rangle = \pi_n(K)$. ← addition of maps from

More generally, have a gp operation for $X = SY$:

Crushing equator:



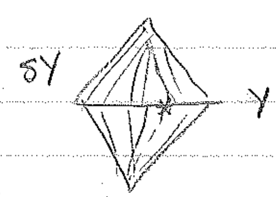
In $\langle SY, K\rangle$, addition is:



→ basept issue: the basept chosen is only in one have of space 8.

Solution: Take a "larger" basept, i.e. use reduced suspension instead: Given a pted $(Y,*)$

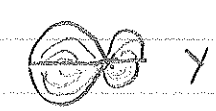
$\Sigma Y = SX/_{\{*\}\times I}$



ex: do this for a disk w/ $*$ in the interior → get donut w/ "hole" of diameter 0.

So $\langle \Sigma Y, K\rangle$ is always a gp.

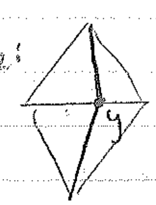
$\Sigma Y =$



• We really want $\langle X, K\rangle$ to be an abelian gp.

Lovely fact: $\langle \Sigma Y, K\rangle = \langle Y, \Omega K\rangle$ ← space of all loops on K.

because:



$Y \longrightarrow K$

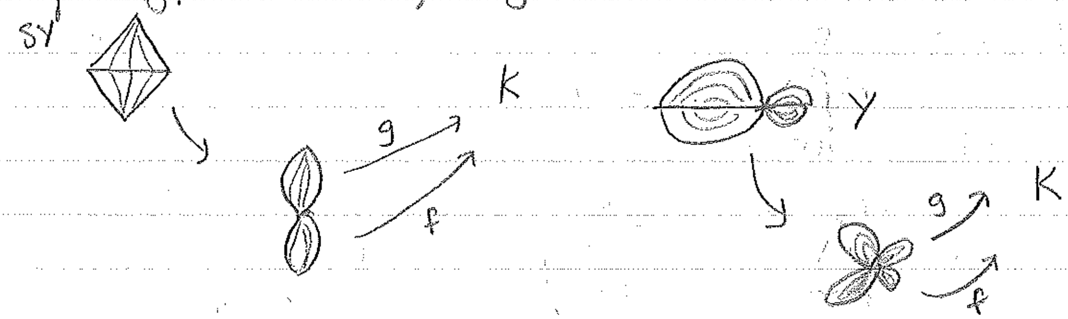
$y \longmapsto k$

$y \times I$ in ΣY ⟼ loop in K
← reduced

Fact: $\Omega K(G,n) = K(G, n-1)$

4/30 $H^n(X;G) \simeq \langle X, K(G,n)\rangle$.

General setting: $\langle X,K\rangle$ for X & K cell cxs. We noticed last time that $\langle SY,K\rangle$ is almost a gp, but there are basept issues, fixed by using $\langle \Sigma Y, K\rangle$. [For a cellular pair $(Y,*)$, $SY \simeq_{he} \Sigma Y$] $\langle \Sigma Y, K\rangle$ is always a gp. Addition, $f+g$:



Two Things:

(1) I want X on LHS of $\langle \cdot,\cdot\rangle$, not ΣY, & X may not be ΣY for any Y.

(2) I want $\langle\cdot,\cdot\rangle$ to be abelian.

Use the following adjoint relation:

$\langle \Sigma Y, K\rangle = \langle Y, \Omega K\rangle$, where ΩK is the (ptd) loop space of K.

What is the loop space of K?

As a set, $\Omega K = \{ (S^1,*) \to (K,*) \}$
$= \{ (I,\partial I) \to (K,*) \}$
$\subseteq K^I = \{ I \to K \}$

Give this the compact-open topology:
Given X,Y top. sp's, the cpt-open topology on Y^X is the top. defined as follows:

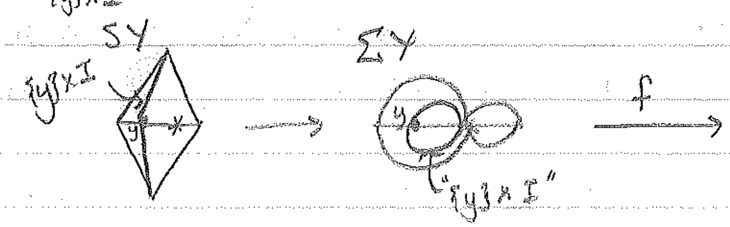
Note: $Y^X = \{X \to Y\}$. Notation chosen to respect cardinal arithmetic, i.e. in analogy w/,

Given cpt $C \subseteq X$ & open $U \subseteq Y$, let $W(C,U) = \{ f: X \to Y \mid f(C) \subseteq U \}$. This is a subbasis for the cpt-open top $\mathcal{T}$.

If Y is a metric space, this is the top. of uniform convergence on cpt sets.

Since ΩK is now a top. sp, $\langle Y, \Omega K\rangle$ is defined.
Take basept of ΩK to be the (constant) trivial map $S^1 \to K$.

Adjoint relationship $\langle \Sigma Y, K\rangle = \langle Y, \Omega K\rangle$ is the observation:
Given $f: \Sigma Y \to K$ & a pt $y \in Y$, get a loop $f|_{\text{"}\{y\}\times I\text{"}}$ in K.



Consider a special case, $Y = S^n$:
$\langle S^n, K\rangle = \pi_n(K)$
$\langle SS^n, K\rangle = \langle \Sigma S^n, K\rangle = \langle S^n, \Omega K\rangle = \pi_n(\Omega K)$
$\langle S^{n+1}, K\rangle = \pi_{n+1}(K)$

* $\Omega K(G,n)$ is a $K(G,n-1)$. This is nice.
* Also nice: <u>Thm</u> (Milnor): If K is a cell cx, then ΩK is h.e. to a cell cx.

Also notice that Ω is a functor: $X \xrightarrow{f} Y$ yields
$\Omega X \xrightarrow{\Omega f} \Omega Y$.
loop $\longmapsto f(\text{loop})$

Think of gp structure on $\langle Y, \Omega K\rangle (= \langle \Sigma Y, K\rangle)$ as composition of paths.

$\langle Y, \Omega K\rangle$ is <u>abelian</u>:
We have an Eilenberg-MacLane sp. K, so ΩK still an E-M sp, so $\Omega(\Omega K) = \Omega^2 K$ also an E-M sp, called the <u>double loop sp</u>
$\langle Y, \Omega^2 K\rangle$ still a gp, & it's abelian, by same kind of argument used for $\pi_4(K) = \langle S^2, \Omega^2 K\rangle$.

4/2 Thm: $H^n(X;G) \cong \langle X, K(G,n)\rangle$

In general: Given X, K, when is $\langle X,K\rangle$ a gp?
$X = \Sigma Y$: $\langle \Sigma Y, K\rangle = \langle Y, \Omega K\rangle$
a gp by either:
(1) crushing equator on left & "stacking"
(2) concatenating loops on right.

How can we get this to be abelian?
If we have $X = \Sigma^2 Y$, $\langle \Sigma^2 Y, K\rangle = \langle Y, \Omega^2 K\rangle$ is abel,
by similar reasoning as in case of $\pi_2(K)$.
- Some technicalities when you're visualizing this:
  - rather than thinking about loops into the loop sp, think about squares into K. This is ok, since $Y^{(X\times Z)} \overset{bij}{\longleftrightarrow} (Y^X)^Z$ is a homeomorphism if X & Z are nice (loc. cpt & Hausdorff)
  For us: $(K^I)^I \cong K^{I^2}$, so $\Omega^2 K = \{(I^2, \partial I^2) \to (K, *)\}$
  $\underset{\Omega^2 K}{\cup\!|}$

Now I want to specify Y to be my X, i.e. want the situation $\langle X, K(G,n)\rangle$.
We've seen that $\Omega K(G,n) = K(G,n-1)$.
So given any X, G, n
$\langle X, \Omega^2 K(G,n+2)\rangle \overset{bij}{\leftrightarrow} \langle X, K(G,n)\rangle$ b/c RHS htpy equiv.
Notice that we need G abel: If $n \geq 0$, then $K(G,n+2)$ only exists for abel. G. On the other hand, $\langle X, K(G,n)\rangle$ is defined for any G, so take $H^1(X;G) := \langle X, K(G,1)\rangle$ (not a gp, but useful — the cohom. of X w/ non-abel. coeff's)

What about the actual theorem? It turns out that if you give me some K, then if you have an Ω-spectrum (see next page) ...

If K_0 $\overset{=K}{}$ itself is a double loop space, then the $\langle X, K_n\rangle$'s are all abel. gps. And then

$h^n(X) = \langle X, K_n\rangle$ is a reduced cohom. theory on ptd sp's.

This seq of (htpy types of) ptd sp's: $K_0, K_1, K_2, \ldots, K_n, \ldots$ is called an <u>Ω-spectrum</u>, if it is equipped w/ (weak) htpy equivalences $K_n \underset{he}{\simeq} \Omega K_{n+1}$. (This is a counterintuitive thing; there's some sort of stability going on...)

- If 2 ptd cohom theories agree at a pt, they agree everywhere, which yields the thm.

<u>An amazing thing</u>: Converse is true (<u>Brown Representability</u>)
Every reduced cohom. theory looks like this, i.e. is of the form $\langle X, K_n\rangle$, for K_n some Ω-spectrum.

5/5 <u>Spectra</u>: Ω-spectrum was a seq. $K_0, K_1, \ldots, K_n, \ldots$ w/ $K_n \underset{he}{\simeq} \Omega K_{n+1}$

<u>Thm</u>: (Brown Representablity thm) Every reduced cohom. theory on ptd spaces is given by $h^n(X) = \langle X, K_n\rangle$.

Ordinary cohom: take $K_n = K(G, n)$

$$\begin{array}{cccc} K(\mathbb{Z},0) & K(\mathbb{Z},1) & K(\mathbb{Z},2) & K(\mathbb{Z},3) \\ \shortparallel & \shortparallel & \shortparallel & \\ \mathbb{Z} & S^1 & \mathbb{C}P^\infty & \end{array}$$

<u>Recall</u>:
there is a bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ (generalizes the Hopf fibration)
$S^{2n+1} \subset \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$

Let $n = \infty$, & get $S^1 \to S^\infty \to \mathbb{C}P^\infty$ [ie. direct limit]

LES of fibration $\Rightarrow$ $\underset{\underset{n>1}{0}}{\pi_n(S^1)} \to \underset{0 \text{ b/c contr.}}{\pi_n(S^\infty)} \to \pi_n(\mathbb{C}P^\infty) \to \pi_{n-1}(S^1)$

$\Rightarrow$ only nonzero htpy gp is $\pi_2(\mathbb{C}P^\infty) \cong \pi_1(S^1) \cong \mathbb{Z}$

$\Omega -$ is right adjoint to red. susp. $\Sigma -$, i.e. $\langle \Sigma Y, K\rangle \cong \langle Y, \Omega K\rangle$
Is there a 'de-looping functor? I.e., is there a functor $B(-)$ s.t. $\Omega BX \cong X$? I.e., if $X \simeq \Omega Y$, can I get Y in some nice (functorial) way?
YES! (but!) when X is $\simeq_{h.e.}$ to a top. gp. Good news:
If $X \simeq \Omega Y$, then X is h.e. to a top. gp, G. (Milnor - by a direct construction)
Even better (in our situation) If G is abelian (like $\mathbb{Z}$ or S^1), then BG is h.e. to a top. gp, which is also abel.
So we can iterate to get Ω-spectrum.

What group is $\mathbb{C}P^\infty$? Let $\mathcal{H}$ be a Hilbert space (square summable sequences w/ $\|\cdot\|_2$), let $\mathcal{U}$ be the gp of isometries of $\mathcal{H}$ (the unitary gp). Then

$$\mathbb{C}P^\infty \simeq_{h} \mathcal{U}/Z(\mathcal{U})$$

center of $\mathcal{U}$: $Z(\mathcal{U}) \cong S^1$

BG is super important: Called the classifying space of G (or assoc. to G). It classifies all bundles of a certain type: "Pre-Milnor" def. of classifying sp. BG:
Def: There is a (weakly) contractible [all htpy gps vanish] space EG & a bundle $G \to EG \xrightarrow{p} BG$ together w/ a cts right G-action $EG \times G \to EG$ fixing fibers, on which G acts freely & transitively.
A principal G-bundle is a bdle $\underset{\cong}{\overset{G}{F}} \to E \to X$ as above, but w/ E not necessarily contractible.
ex: Think of $GL_n(\mathbb{R})$ acting on the bundle of frames (choice of a basis) on a smooth mfld.

BG is called the classifying sp b/c: every principal G-bdle is obtained by pulling back $EG \to BG$ under a map $X \to BG$:

$$\begin{array}{ccccc} F & \to & E & \to & X \\ & & & & \uparrow f \\ F & \to & f^*E & \to & Y \end{array}$$

G abel $\Rightarrow$ get our spectrum $K(G,n)$ by $\overset{\overset{K(G,0)}{\cong}}{G}, \dots, B^n G, \dots$
What about the general case? • Let $\mathcal{U}$ be the infinite unitary gp (isometries of Hilbert sp). Then Bott Periodicity says: $\Omega^2 B\mathcal{U} = \mathbb{Z} \times B\mathcal{U}$

- Infinite orthogonal gp $\mathcal{O}$. Then Bott Periodicity says $\Omega^8 B\mathcal{O} \simeq \mathbb{Z} \times B\mathcal{O}$.
- There are results like this for all classical Lie gps from periodicity.