Exercises 14

⑤ $(\mathbb{Z}_2\times\mathbb{Z}_4)/\langle(1,1)\rangle$ $\quad |\langle(1,1)\rangle| = 4$ $\quad (\langle(1,1)\rangle = \{(1,1),(0,2),(1,3),(0,0)\})$

& $|\mathbb{Z}_2\times\mathbb{Z}_4| = 8$

$\Rightarrow |(\mathbb{Z}_2\times\mathbb{Z}_4)/\langle(1,1)\rangle| = 8/4 = \boxed{2}$

⑥ $(\mathbb{Z}_{12}\times\mathbb{Z}_{18})/\langle(4,3)\rangle$ $\quad$ order of 4 in $\mathbb{Z}_{12}$ is 3 $\Rightarrow$ order of $(4,3) = \text{lcm}(3,6) = 6$

order of 3 in $\mathbb{Z}_{18}$ is 6

$\Rightarrow |(\mathbb{Z}_{12}\times\mathbb{Z}_{18})/\langle(4,3)\rangle| = \frac{12\cdot 18}{6} = 2\cdot 18 = \boxed{36}$

⑦ $(\mathbb{Z}_2\times S_3)/\langle(1,\rho_1)\rangle$ $\quad$ order of 1 in $\mathbb{Z}_2$ is 2

order of $\rho_1 = (1\,2\,3)$ is S_3 is 3 $\Rightarrow$ order of $(1,\rho_1) = \text{lcm}(2,3) = 6$

$\Rightarrow |(\mathbb{Z}_2\times S_3)/\langle(1,\rho_1)\rangle| = 2\cdot 3!/6 = \boxed{2}$

⑬ $(3,1)+\langle(0,2)\rangle$ in $(\mathbb{Z}_4\times\mathbb{Z}_8)/\langle(0,2)\rangle$

$\langle(0,2)\rangle = \{(0,2),(0,4),(0,6),(0,0)\}$

$(3,1)+\langle(0,2)\rangle = \{(3,3),(3,5),(3,7),(3,1)\}$

$(2,2)+\langle(0,2)\rangle = \{(2,4),(2,6),(2,8),(2,2)\}$

$(1,3)+\langle(0,2)\rangle = \{(1,5),(1,7),(1,0),(1,3)\}$

$(0,4)+\langle(0,2)\rangle = \{(0,6),(0,0),(0,2),(0,4)\} = \langle(0,2)\rangle \Rightarrow |(3,1)+\langle(0,2)\rangle| = \boxed{4}$

<u>Note</u>: Order of $(3,1)$ in $\mathbb{Z}_4\times\mathbb{Z}_8$ is $\text{lcm}(4,8) = 8$

$|\langle 0,2\rangle| = \text{lcm}(1,2) = 2 \Rightarrow |(\mathbb{Z}_4\times\mathbb{Z}_8)/\langle(0,2)\rangle| = 32/2 = 16$

So the order of $(3,1)+\langle(0,2)\rangle$ must be a # ≤ 8 which divides 16.

This doesn't really help you solve the problem, but it's a good check.

⑭ $(3,3)+\langle(1,2)\rangle$ in $(\mathbb{Z}_4\times\mathbb{Z}_8)/\langle(1,2)\rangle$

$\langle(1,2)\rangle = \{(1,2),(2,4),(3,6),(0,0)\} \Rightarrow |(\mathbb{Z}_4\times\mathbb{Z}_8)/\langle(1,2)\rangle| = 32/4 = 8$

$(3,3)+\langle(1,2)\rangle = \{(0,5),(1,7),(2,1),(3,3)\}$

$(2,6)+\langle(1,2)\rangle = \{(3,0),(0,2),(1,4),(2,6)\}$

$(1,1)+\langle(1,2)\rangle \ne \langle(1,2)\rangle$

$(0,4)+\langle(1,2)\rangle = \{(1,6),(2,0),(3,2),(0,4)\}$

$(3,7)+\langle(1,2)\rangle \ne \langle(1,2)\rangle$

$(2,2)+\langle(1,2)\rangle \ne \langle(1,2)\rangle$

$(1,5)+\langle(1,2)\rangle \ne \langle(1,2)\rangle$

$\Rightarrow |(3,3)+\langle(1,2)\rangle| = \boxed{8}$

Here, since the order of $(3,3)+\langle(1,2)\rangle$ must divide the order of the gp, the only possibilities are 1, 2, 4, 8, which is why I could conclude immediately that certain cosets were $\ne \langle(1,2)\rangle$.

<u>Note</u>: This implies that the quotient gp is cyclic, & so $\cong \mathbb{Z}_8$. (This is not part of the problem; it's just to help your intuition.)

㉚ $|G/H| = |G:H| = m$. Let $a\in G$. Then $aH \in G/H$. By Lagrange's Thm, $|aH|$ divides $|G/H| = m$, so if $k = |aH|$, $\exists\, n$ s.t. $m = kn$. Thus $a^m H = (aH)^m = (aH)^{kn} = ((aH)^k)^n = (H)^n = H$. Therefore, $a^m \in H$.

(recall that $g_1H = g_2H \Leftrightarrow g_2^{-1}g_1 \in H$, & here $a^mH = H = eH$, so $a^m = a^m e^{-1} \in H$)

(31) Let $A, B \trianglelefteq G$. We have already shown $A \cap B \leq G$. Let $g \in G$ & $x \in A \cap B$. Then $x \in A$ & $A \trianglelefteq G$, so $gxg^{-1} \in A$. Also, $x \in B$ & $B \trianglelefteq G$, so $gxg^{-1} \in B$. Thus $gxg^{-1} \in A \cap B$. Since g & x were arbitrary, $g(A \cap B)g^{-1} \subseteq A \cap B$ $\forall g \in G$, so $A \cap B \trianglelefteq G$.

(34) Let G be a finite gp & suppose $H \leq G$ is the only subgp of its order. Let $g \in G$ & consider the map $i_g : G \to G$ defined by $i_g(x) = g^{-1}xg$ $\forall x \in G$. We showed that i_g is an isomorphism.
Since it is a homomorphism & $H \leq G$, $i_g(H) \leq G$. $|i_g(H)| = |H|$ b/c i_g is an isomorphism. Since H is the only subgp of its order, $i_g(H) = H$, & thus $g^{-1}Hg = H \Rightarrow Hg = gH \Rightarrow H \trianglelefteq G$.

(38) Let $H = \{g \in G \mid i_g = i_e\}$.
$i_g = i_e \Leftrightarrow \forall x \in G,\ i_g(x) = i_e(x) \Leftrightarrow gxg^{-1} = exe^{-1}$

$$gxg^{-1} = x$$
$$gx = xg \quad \forall x \in G.$$

$gx = xg$ $\forall x \in G \Leftrightarrow g$ is an element of the center of G ($Z(G)$; we discussed this subgp in class).
So $H = \{g \in G \mid gx = xg,\ \forall x \in G\}$. We showed in an earlier HW that $H \leq G$, so we need only show that $H \trianglelefteq G$.
Let $a \in G$, & $h \in H$. We will show that $aha^{-1} \in H$.
Let $x \in G$. Then

$$\begin{aligned}(aha^{-1})x &= aa^{-1}hx && \text{b/c } h \in H\\ &= hx\\ &= xh && \text{b/c } h \in H\\ &= xhaa^{-1}\\ &= x(aha^{-1}) && \text{b/c } h \in H\end{aligned}$$

$\Rightarrow aha^{-1} \in H$. Thus $H \trianglelefteq G$.

(39) Consider the map $\phi_* : G/H \to G'/H'$ defined by $\phi_*(gH) = \phi(g)H'$ $\forall gH \in G/H$. We will show that if $\phi(H) \subseteq H'$, then ϕ_* is a homomorphism.

- Well-def: Let $gH = g'H$. Then $g = g'h$ for some $h \in H$

$$\begin{aligned}\phi_*(gH) = \phi_*(g'hH) &= \phi(g'h)H'\\ &= \phi(g')\phi(h)H' && \text{b/c } \phi \text{ is a hom.}\\ &= \phi(g')H' && \text{b/c } \phi(h) \in H' \text{ by assumption}\\ &= \phi_*(g'H)\end{aligned}$$

Thus ϕ_* is well-def.

- Hom: Let $gH, g'H \in G/H$. Then,

$$\begin{aligned}\phi_*(gHg'H) = \phi_*(gg'H) = \phi(gg')H' &\overset{\phi \text{ is a hom}}{=} \phi(g)\phi(g')H' = \phi(g)H'\phi(g')H'\\ &= \phi_*(gH)\phi_*(g'H).\end{aligned}$$

Therefore, ϕ_* is a homomorphism.

Exercises 15

(3) $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$ $\quad |(1,2)| = \text{lcm}(2,2) = 2$, $|\mathbb{Z}_2 \times \mathbb{Z}_4| = 8$ $\Rightarrow |(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle| = 8/2 = 4$.
There are 2 gps of order 4: $\mathbb{Z}_2 \times \mathbb{Z}_2$ & $\mathbb{Z}_4$.
The order of $(0,1) + \langle(1,2)\rangle$ is either 1, 2, or 4, by Lagrange's Thm.
$\langle(1,2)\rangle = \{(1,2), (0,0)\}$

$(0,1)+\langle(1,2)\rangle=\{(1,3),(0,1)\} \Rightarrow |(0,1)+\langle(1,2)\rangle| \neq 1$
$(0,2)+\langle(1,2)\rangle=\{(1,0),(0,2)\} \Rightarrow |(0,1)+\langle(1,2)\rangle| \neq 2$
Thus $|(0,1)+\langle(1,2)\rangle|=4$. Since $\mathbb{Z}_2\times\mathbb{Z}_2$ does not have an elt of order 4, we conclude that $\boxed{(\mathbb{Z}_2\times\mathbb{Z}_4)/\langle(1,2)\rangle \simeq \mathbb{Z}_4.}$

④ $(\mathbb{Z}_4\times\mathbb{Z}_8)/\langle(1,2)\rangle$ $\quad |(1,2)|=\operatorname{lcm}(4,4)=4$ $\Rightarrow$ $|(\mathbb{Z}_4\times\mathbb{Z}_8)/_{\langle(1,2)\rangle}| = 32/4 = 8$
$|\mathbb{Z}_4\times\mathbb{Z}_8|=32$

There are 3 gps of order 8: $\mathbb{Z}_8$, $\mathbb{Z}_2\times\mathbb{Z}_4$, & $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$.
In problem #14 from exercises 14, we saw that $(3,3)+\langle(1,2)\rangle$ has order 8. Neither $\mathbb{Z}_2\times\mathbb{Z}_4$ nor $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ has an elt of order 8 (b/c a gp of order 8 only has an elt of order 8 if it is cyclic, & the only cyclic gp of order 8 is $\mathbb{Z}_8$), so
$\boxed{(\mathbb{Z}_4\times\mathbb{Z}_8)/\langle(1,2)\rangle \simeq \mathbb{Z}_8.}$

⑥ & ⑪ were removed from the assignment.

㉟ We have already shown that $\phi(N)\leq G'$, & since $\phi(N)\subseteq\phi(G)$, it follows that $\phi(N)\leq\phi(G)$. So we need only show that it is normal.
Let $x\in\phi(N)$ & $y\in\phi(G)$. Then $\exists$ $n\in N$ & $g\in G$ s.t. $\phi(n)=x$ & $\phi(g)=y$.
Since $N\trianglelefteq G$, $\exists n'\in N$ s.t. $gng^{-1}=n'$.
Then
$$\begin{aligned} yxy^{-1} &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= \phi(gng^{-1}) \\ &= \phi(n') \in \phi(N). \end{aligned}$$
Since x & y were arbitrary, $y\phi(N)y^{-1}\subseteq\phi(N)$ $\forall y\in\phi(G)$. Thus $\phi(N)\trianglelefteq\phi(G)$.

㊱ We have already shown that $\phi^{-1}(N)\leq G$, so we need only show that it is normal.
Let $g\in G$ & $n\in\phi^{-1}(N')$. Then $\phi(g)\in G'$ & $\phi(n)\in N'$. Since $N'\trianglelefteq G'$, $\exists n'\in N'$ s.t. $\phi(g)\phi(n)\phi(g)^{-1}=n'\in N'$.
Then
$$\begin{aligned} \phi(gng^{-1}) &= \phi(g)\phi(n)\phi(g)^{-1} \\ &= n' \in N', \end{aligned}$$
so $gng^{-1}\in\phi^{-1}(N')$. Since g & n were arbitrary, $g\phi^{-1}(N')g^{-1}\subseteq\phi^{-1}(N')$ $\forall g\in G$. Therefore, $\phi^{-1}(N')\trianglelefteq G$.

<u>Additional Exercises</u>:

① $SL_n(\mathbb{R})\leq GL_n(\mathbb{R})$:
- closed: Let $A,B\in SL_n(\mathbb{R})$. Then $\det(AB)=\det(A)\det(B)=1\cdot1=1 \Rightarrow AB\in SL_n(\mathbb{R})$
- identity: $\det(I_n)=1 \Rightarrow I_n\in SL_n(\mathbb{R})$.
- inverses: Let $A\in SL_n(\mathbb{R})$. Then $\det(A)\det(A^{-1})=\det(AA^{-1})=\det(I_n)=1$
  $\Rightarrow 1\cdot\det(A^{-1})=1 \Rightarrow \det(A^{-1})=1 \Rightarrow A^{-1}\in SL_n(\mathbb{R})$.

Therefore $SL_n(\mathbb{R})$ is a subgp of $GL_n(\mathbb{R})$.

(a) We will show $\forall A\in GL_n(\mathbb{R})$, $A\,SL_n(\mathbb{R})A^{-1}\subseteq SL_n(\mathbb{R})$. Let $A\in GL_n(\mathbb{R})$. Then for any $B\in SL_n(\mathbb{R})$, $\det(ABA^{-1})=\det(A)\det(B)\det(A^{-1})=\det(A)\cdot1\cdot\frac{1}{\det(A)}=1$, so $ABA^{-1}\in SL_n(\mathbb{R})$. Thus $A\,SL_n(\mathbb{R})A^{-1}\subseteq SL_n(\mathbb{R})$.
Since A was arbitrary, this is true $\forall A\in GL_n(\mathbb{R})$, & so $SL_n(\mathbb{R})\trianglelefteq GL_n(\mathbb{R})$.

(b) Consider the map $\varphi: GL_n(\mathbb{R}) \to \mathbb{R}^*$ defined by $\varphi(A) = \det A$ (recall, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, & is a gp under multiplication).

- φ is well-def: if $A = B$, $\det A = \det B \Rightarrow \varphi(A) = \varphi(B)$.
- φ is a homomorphism: Let $A, B \in GL_n(\mathbb{R})$. Then

$$\begin{aligned}\varphi(AB) &= \det(AB) \\ &= \det(A) \cdot \det(B) \\ &= \varphi(A) \cdot \varphi(B).\end{aligned}$$

- $\ker \varphi = SL_n(\mathbb{R})$: $\ker \varphi = \{A \in GL_n(\mathbb{R}) \mid \det A = \underbrace{1}_{\text{identity of } \mathbb{R}^*}\} = SL_n(\mathbb{R})$.

Therefore, $SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R})$.

② Let $G = \langle a \rangle$ by a cyclic gp & $H \trianglelefteq G$. We will show that $G/H = \langle aH \rangle$. Let $gH \in G/H$. Then $g \in G$, so $g = a^i$ for some i. Thus $gH = a^i H = (aH)^i \in \langle aH \rangle$. Since gH in G/H was arbitrary, $G/H = \langle aH \rangle$. Therefore, G/H is cyclic.