Exercises 13

(18) $\varphi: \mathbb{Z} \to \mathbb{Z}_{10}$ s.t. $\varphi(1) = \overline{6}$.

$|\overline{6}| = 5$, so $\varphi(5n) = (5n)\cdot\overline{6} = \overline{0} \Rightarrow \operatorname{Ker}\varphi = \{5n \mid n \in \mathbb{Z}\} = 5\mathbb{Z}$.

$$\varphi(18) = \varphi(\underbrace{1+\cdots+1}_{18}) = \underbrace{\varphi(1)+\cdots+\varphi(1)}_{18} = \underbrace{\overline{6}+\cdots+\overline{6}}_{18} = \underbrace{\overline{6}+\cdots+\overline{6}}_{15}+\overline{6}+\overline{6}+\overline{6} = \overline{0}+\overline{6}+\overline{6}+\overline{6}$$

$= \overline{8}$.

(25) 2: $\varphi: \mathbb{Z} \to \mathbb{Z}$ defined by $\varphi(1) = 1$

& $\psi: \mathbb{Z} \to \mathbb{Z}$ — " — $\varphi(1) = -1$.

(26) Infinitely many: $\alpha: \mathbb{Z} \to \mathbb{Z}$ defined by $\alpha(1) = n$ for any fixed $n \in \mathbb{Z}$.
is a homomorphism, & it's injective as long as $n \neq 0$.

(27) None. If $\varphi: G \to G'$ is injective, then $|\varphi(G)| = |G|$. But $|\mathbb{Z}| = \infty$
while $|\mathbb{Z}_2| = 2$, & $2 \neq \infty$.

(28) Let $x, y \in G$. $\varphi_g(xy) = gxy$
$\varphi_g(x)\varphi_g(y) = gx \cdot gy$

$$gxy = gxgy \Rightarrow xy = xgy \Rightarrow y = gy \Rightarrow e = g.$$

Thus this is a homomorphism $\Leftrightarrow g = e$.

(47) G a cyclic gp, $|G| = p$ prime. Let $\varphi: G \to G'$ be a homomorphism.
Let $a \in G$ be such that $\langle a \rangle = G$.
Consider $\operatorname{Ker}\varphi$, which is a subgp of G. Since all subgps of a cyclic gp are cyclic, $0 \le s \le p-1$ s.t. $\operatorname{Ker}\varphi = \langle a^s \rangle$. Then $|\operatorname{Ker}\varphi| = p/\gcd(s,p)$.
Since p is prime, for any $0 \le s \le p-1$, $\gcd(s,p) = p$ or 1.
Thus $|\operatorname{Ker}\varphi| = 1$ or p.
If $|\operatorname{Ker}\varphi| = 1$, then $\operatorname{Ker}\varphi = \{e\}$, so φ is injective.
If $|\operatorname{Ker}\varphi| = p$, then $\operatorname{Ker}\varphi = G$, so φ is trivial.

Additional Exercises:

① (a) H is a plane in $\mathbb{R}^3$.

H is a subgp: Simplify equation: $3x+2y-z=0$

- Let $(x_1,y_1,z_1),(x_2,y_2,z_2)\in H$. Then
$3(x_1+x_2)+2(y_1+y_2)-(z_1+z_2)=3x_1+2y_1-z_1+3x_2+2y_2-z_2=0+0=0$
$\Rightarrow$ H is closed.
- $3(0)+2(0)-0=0 \Rightarrow (0,0,0)\in H$
- Let $(x,y,z)\in H$. Then $3(-x)+2(-y)+z=-(3x+2y-z)=-0=0$, so $(-x,-y,-z)\in H$.

Therefore, $H\le G$.

(b) algebraic: $K=\{(na,nb,nc)\mid n\in\mathbb{Z}\}$
geometric: K is the set of points on the line through $(0,0,0)$ & (a,b,c) that are integer multiples of (a,b,c). Note: K is not the whole line!

(c) $L=\{(2t,3t,5t)\mid t\in\mathbb{R}\}$.

- Yes, $L\le G$:
  - closure: $(2t_1,3t_1,5t_1)+(2t_2,3t_2,5t_2)=(2(t_1+t_2),3(t_1+t_2),5(t_1+t_2))\in L$ since $t_1+t_2\in\mathbb{R}$.
  - identity: $(0,0,0)\in L$, because it corresponds to $t=0$.
  - inverses: $-(2t,3t,5t)=(2(-t),3(-t),5(-t))\in L$ since $-t\in\mathbb{R}$.
    * Note: ↑ is the inverse written additively (like -2 is the inverse of 2 in $(\mathbb{Z},+)$).

No, L is not a cyclic subgp:
Suppose $L=\langle(a,b,c)\rangle$. Then $2t=na \Rightarrow t=\frac{na}{2}$. Thus $\mathbb{R}=\langle\frac{na}{2}\rangle$
But $\mathbb{R}$ is not cyclic, which is a contradiction.

[Solution to #2 is at end.]

③ (a) No such hom. exists. We showed in class that if $\varphi:G\to G'$ is a hom, & $g\in G$, then $|\varphi(g)|$ divides $|g|$. Suppose $\exists$ hom $\varphi:\mathbb{Z}_3\to\mathbb{Z}_5$. Let $\bar a\in\mathbb{Z}_3$. If $\bar a\ne\bar 0$, then $|\bar a|=3$. But all $\bar b\in\mathbb{Z}_5$ have order 1 or 5, & 5 does not divide 3, so $|\varphi(\bar a)|=1$. The only elt of order 1 in $\mathbb{Z}_5$ is $\bar 0$, so $\varphi(\bar a)=\bar 0$, $\forall\,\bar a\in\mathbb{Z}_3$ (b/c we know $\varphi(\bar 0)=\bar 0$, since φ a hom.). Thus φ is trivial.

(b) Define $\varphi:\mathbb{Z}_4\to\mathbb{Z}_{12}$ by $\varphi(\bar 1)=\bar 3$. Note that this is equivalent to the map $\varphi(\bar n)=\overline{3n}$.

(c) $\varphi:\mathbb{Z}_{10}\to\mathbb{Z}_8$ defined by $\varphi(\bar 1)=\bar 4$. This is the map $\varphi(\bar n)=\overline{4n}$.

④ There are 2 cases, depending on the order of a in G.
If $|a|=\infty$, then $a^i=a^j \Leftrightarrow i=j$, so φ is injective, & $\ker\varphi=\{0\}$.
If $|a|=k<\infty$, then $a^i=a^j \Leftrightarrow k$ divides $(i-j)$, or, equivalently, $a^n=e \Leftrightarrow n$ is a multiple of k. Thus $\ker\varphi=k\mathbb{Z}$.

⑤ Consider the subgp $H=\{e,\rho^2,\tau,\rho^2\tau\}\le D_4$, & the map $\varphi:D_4\to V$ defined by
$\varphi(e)=e$
$\varphi(\rho^2)=a$
$\varphi(\tau)=b$
$\varphi(\rho^2\tau)=c$.

This is a hom. To show this, we will compare gp tables

H:

	e	ρ^2	τ	$\rho^2\tau$
e	e	ρ^2	τ	$\rho^2\tau$
ρ^2	ρ^2	e	$\rho^2\tau$	τ
τ	τ	$\rho^2\tau$	e	ρ^2
$\rho^2\tau$	$\rho^2\tau$	τ	ρ^2	e

V:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	e	b	a	e

As the tables are the same after relabeling by the map φ, these are isomorphic.

⑥ $H = \langle i \rangle = \{1, i, -1, -i\}$. Consider the map $\varphi: \mathbb{Z}_4 \to H$ defined by $\varphi(\bar{1}) = i$.
We will again compare gp tables

H:

	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

$\mathbb{Z}_4$:

	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

As the tables are the same after relabeling by the map φ, these are isomorphic.

⑦ By checking the orders of the elements in Q_8, we see that $|i| = |-i| = |k| = |-k| = 4$, $|-1| = |j| = |-j| = 2$, & $|1| = 1$, so Q_8 has 4 elts of order 4. However, D_4 only has 2 elts of order 4 (ρ & ρ^3). Since any isomorphism $\varphi: Q_8 \to D_4$ must preserve the orders of the elements (i.e., $\forall g \in Q_8$, $|g| = |\varphi(g)|$), we see that there cannot be any such isomorphism.

<u>Note</u>: I found the orders of the elts in Q_8 simply by multiplying matrices.

⑧ Let $a, b \in \varphi(G)$. Then $\exists x, y \in G$ s.t. $\varphi(x) = a$ & $\varphi(y) = b$. Then

$$ab = \varphi(x)\varphi(y) = \varphi(xy) \underset{\substack{\uparrow \\ G \text{ is abelian}}}{=} \varphi(yx) = \varphi(y)\varphi(x) = ba.$$

Thus $\varphi(G)$ is abelian.

No, this does not prove H is abelian, b/c φ may not be surjective. For example, the map $\varphi: \mathbb{Z}_4 \to D_4$ defined by $\varphi(\bar{1}) = \rho$ is a homomorphism, G is abelian, & $\varphi(G) = \langle \rho \rangle$ is abelian, but D_4 is <u>not</u> abelian.

⑨ ($\Leftarrow$) Suppose $xyx^{-1}y^{-1} \in \ker\varphi$ $\forall x, y \in G$, & let $a, b \in \varphi(G)$. Then $\exists g, h \in G$ s.t. $\varphi(g) = a$ & $\varphi(h) = b$.

Since $ghg^{-1}h^{-1} \in \ker\varphi$,

$e = \varphi(ghg^{-1}h^{-1})$
$e = \varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1}$
$e = aba^{-1}b^{-1}$
$b = aba^{-1}$
$ba = ab.$

Thus $\varphi(G)$ is abelian.

($\Rightarrow$): Suppose $\varphi(G)$ is abelian & let $g,h \in G$. Then $\varphi(g), \varphi(h) \in \varphi(G)$, so

$\varphi(g)\varphi(h) = \varphi(h)\varphi(g)$
$\varphi(g)\varphi(h)\varphi(g)^{-1} = \varphi(h)$
$\varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1} = e$
$\varphi(ghg^{-1}h^{-1}) = e$

Thus $ghg^{-1}h^{-1} \in \ker \varphi$.

② (a) Not a hom: let $x,y \in G$. Then $\varphi(xy) = (xy)^{-1} = y^{-1}x^{-1}$, while $\varphi(x)\varphi(y) = x^{-1}y^{-1}$. $y^{-1}x^{-1} \neq x^{-1}y^{-1}$ $\forall x,y \in G$ unless G is abelian, & there are non-abelian gps.

(b) Not a hom: $\varphi(0) = -1$, so φ does not send the identity to the identity. Thus φ is not a hom.

(c) Yes, a hom: let $x,y \in \mathbb{R}^*$. Then $\varphi(xy) = |xy| = |x|\cdot|y| = \varphi(x)\varphi(y)$.
$\ker\varphi = \{\pm 1\}$.

(d) Yes, a hom: let $x,y \in D_4$. There are 3 cases to consider:

- If $x=\rho^i$ & $y=\rho^j$, then $\varphi(xy) = \varphi(\rho^i\rho^j) = \varphi(\rho^{i+j}) = 0 = 0+0 = \varphi(\rho^i)+\varphi(\rho^j)$.
- If $x=\rho^i$ & $y=\rho^j\tau$, then $\varphi(xy) = \varphi(\rho^i\cdot\rho^j\tau) = \varphi(\rho^{i+j}\tau) = 1 = 0+1 = \varphi(\rho^i)\varphi(\rho^j\tau)$.

  (If $x=\rho^i\tau$ & $y=\rho^j$, then $xy = \rho^i\tau\rho^j = \rho^i\rho^{-j}\tau = \rho^{i-j}\tau$, so a similar argument works)
- If $x=\rho^i\tau$ & $y=\rho^j\tau$, then $\varphi(xy) = \varphi(\rho^i\tau\rho^j\tau) = \varphi(\rho^i\rho^{-j}\tau\tau) = \varphi(\rho^{i-j}) = 0 = 1+1 = \varphi(\rho^i\tau)+\varphi(\rho^j\tau)$.

Thus, φ is a hom.
$\ker\varphi = \langle\rho\rangle$.

(e) Not a hom: consider $\bar{5}, \bar{6} \in \mathbb{Z}_7$. Then $\varphi(\bar{5}+\bar{6}) = \varphi(\bar{4}) = \bar{0}$, but $\varphi(\bar{5})+\varphi(\bar{6}) = \bar{1}+\bar{0} = \bar{1}$, & $\bar{1} \neq \bar{0}$.