Exercises 0

11. $(a,1), (a,2), (a,c), (b,1), (b,2), (b,c), (c,1), (c,2), (c,3)$.

14. (a) $f: [0,1] \to [0,2]$
$f(x) = 2x$
f is well-def: Let $x,y \in [0,1]$. If $x=y$, then $2x = 2y$, so $f(x) = f(y)$.
f is injective: Let $x,y \in [0,1]$. If $f(x) = f(y)$, then $2x = 2y$. Dividing by 2 yields $x = y$.
f is surjective. Let $x \in [0,2]$. Then $\frac{x}{2} \in [0,1]$ and $f(\frac{x}{2}) = 2(\frac{x}{2}) = x$.

(b) $f: [1,3] \to [5,25]$
$f(x) = 10(x-1) + 5$
(well-defined, injective, & surjective, as in (a).)

(c) $f: [a,b] \to [c,d]$
$f(x) = \frac{d-c}{b-a}(x-a) + c$

(well-defined, injective, & surjective, as in (a))

16. a. $\mathcal{P}(\emptyset) = \emptyset$. cardinality 1.
b. $\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$. cardinality 2.
c. $\mathcal{P}(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. cardinality 4.
d. $\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. cardinality 8.

17. If $|A| = s$, then $|\mathcal{P}(A)| = 2^s$. We prove this by induction on s.
Base case: $s = 0$. Then $A = \emptyset$ and $\mathcal{P}(A) = \emptyset$, & $|\mathcal{P}(A)| = 2^0 = 1$.

Inductive assumption: Assume $|\mathcal{P}(A)| = 2^s$ for all all sets A with at most s elements.

Inductive step: Let A be a set with $s+1$ elements and fix an element $x \in A$. Let $A' = A - \{x\}$. Divide the subsets of A into 2 sets: those containing x and those not containg x.
If X is a subset containing x, then $X - \{a\}$ does not contain x, and so $X - \{a\} \subseteq A'$. On the other hand if $Y \subseteq A'$, then $Y \cup \{a\} \subseteq A$. Thus there is an injective and surjective map
$f: \mathcal{P}(A') \to \{\text{subsets of } A \text{ containing } x\}$
$f(Y) = Y \cup \{x\}$
Therefore these 2 sets have the same cardinality, which is 2^s by assumption.
If X is a subset of A not containg x, then $X \subseteq A'$. Conversely, if Y is a subset of A', then $Y \subseteq A$. Thus there is an injective and surjective map
$g: \mathcal{P}(A') \to \{\text{subsets of } A \text{ not containg } x\}$
$g(Y) = Y$
Therefore these 2 sets have the same cardinality, which is 2^s by assumption.
Since $\mathcal{P}(A) = \{\text{subsets of } A \text{ containing } x\} \cup \{\text{subsets of } A \text{ not cont. } x\}$ it follows that $|\mathcal{P}(A)| = 2^s + 2^s = 2 \cdot 2^s = 2^{s+1}$! $\square$

19. Let A be a set, and suppose there exists an injective map $f: A \to \mathcal{P}(A)$. We will show f cannot be surjective.
Let $B \in \mathcal{P}(A)$ be the subset of A containing an element $x \in A$ if and only if $x \notin f(x)$.
Suppose $\exists a \in A$ s.t. $f(a) = B$. Then if $a \in B$, then $a \notin f(a)$. But $f(a) = B$, so $a \in B$, a contradiction. If $a \notin B$, then $a \in f(a) = B$, also a contradiction. Therefore, B is not in the image of f.

No, the "set of everything" is not a logically acceptable concept. If A is such a set, the above argument shows that $\mathcal{P}(A)$ is always a strictly larger set, so A was not "everything."

20.a. $A = \{1,2\}$ $B = \{3,4,5\}$
$A \cup B = \{1,2,3,4,5\}$.
$|A| = 2$, $|B| = 3$, and $|A \cup B| = 5$, so $2+3=5$.

(i) $|\mathbb{Z}^+| = \aleph_0$. We will find the cardinality of $\mathbb{Z}^+ \cup C$, where $C = \{a,b,c\}$
Let $f: \mathbb{Z}^+ \cup C \to \mathbb{Z}^+$ be the map defined by:
$f(a) = 1$
$f(b) = 2$
$f(c) = 3$
$f(n) = n+3$ if $n \in \mathbb{Z}^+$.
f is well-defined: Clear from definition
f is injective: Let $x, y \in \mathbb{Z}^+ \cup C$, and suppose $f(x) = f(y)$.
If $f(x) = f(y) = 1, 2,$ or 3, then $x = y = a$, $x = y = b$, or $x = y = c$, respectively. If $f(x) = f(y) \geq 4$, then $x+3 = y+3$, so $x = y$.
f is surjective: Let $y \in \mathbb{Z}^+$. If $y = 1, 2,$ or 3, then $f(a) = y$, $f(b) = y$, or $f(c) = y$, respectively. If $y \geq 4$, then $y-3 \in \mathbb{Z}^+$ and $f(y-3) = y$.
Therefore $|\mathbb{Z}^+ \cup C| = |\mathbb{Z}^+| = \aleph_0$, so $3 + \aleph_0 = \aleph_0$

30 Not an equivalence relation because it fails symmetry: If $x = 2$ & $y = 3$, then $2R3$ (b/c $2 \leq 3$) but $3R2$ is not true (b/c $3 \nleq 2$).

32 Not an equivalence relation because it fails transitivity: If $x = 0$, $y = 3$, & $z = 6$, then $|0-3| \leq 3$ so $0R3$ and $|3-6| \leq 3$ so $3R6$, but $|0-6| \nleq 3$, so $0R6$ is false.

Exercises 2:
37 Let $x, y \in H$. Then, by the definition of H, $x * x = x$ & $y * y = y$.
Then
$$\begin{aligned}(x*y)*(x*y) &= (y*x)*(x*y) && \text{b/c } * \text{ is commutative}\\ &= (y*(x*x))*y && \text{by associativity}\\ &= (y*x)*y && \text{b/c } x \in H\\ &= (x*y)*y && \text{b/c } * \text{ is comm.}\\ &= x*(y*y) && \text{by assoc.}\\ &= x*y && \text{b/c } y \in H.\end{aligned}$$
Therefore $x*y \in H$, & so H is closed under $*$.

Additional exercises:

1. (a) $\phi: \mathbb{R} \to \mathbb{R}$, $\phi(x) = \frac{5+3x}{2}$. Yes:

ϕ is injective: Let $x_1, x_2 \in \mathbb{R}$ s.t. $\phi(x_1) = \phi(x_2)$. Then $\frac{5+3x_1}{2} = \frac{5+3x_2}{2}$,

so
$$5+3x_1 = 5+3x_2$$
$$3x_1 = 3x_2$$
$$x_1 = x_2 \checkmark.$$

ϕ is surjective: Let $y \in \mathbb{R}$. Then $\frac{2y-5}{3} \in \mathbb{R}$, and
$$\phi\left(\frac{2y-5}{3}\right) = \frac{5+3\left(\frac{2y-5}{3}\right)}{2} = \frac{5+2y-5}{2} = \frac{2y}{2} = y. \checkmark$$

Since ϕ is inj & surj, ϕ is invertible.

(b) $\phi: \mathbb{R} \to \mathbb{R}$, $\phi(x) = x^2 - 4$. No.

ϕ is invertible if and only if ϕ is surjective and injective.
$-10 \in \mathbb{R}$, but since $x^2 - 4 \geq -4$, there is no $x \in \mathbb{R}$ s.t. $\phi(x) = -10$. Therefore ϕ is not surj. & so not invertible.

2. $\phi: A \to B$, $\chi: B \to C$, $\psi: C \to D$.

(a) Suppose $\chi \circ \phi$ is surjective. Then for every $c \in C$, there exists an element $a \in A$ s.t. $\chi \circ \phi(a) = c$. Let $b = \phi(a) \in B$. Then $\chi(b) = \chi(\phi(a)) = c$, so χ is surjective.

(b) Suppose $\chi \circ \phi$ is injective, and let $a_1, a_2 \in A$ be s.t. $\phi(a_1) = \phi(a_2)$. Since χ is well-defined, $\chi(\phi(a_1)) = \chi(\phi(a_2))$, and so $\chi \circ \phi(a_1) = \chi \circ \phi(a_2)$. Since $\chi \circ \phi$ is injective, $a_1 = a_2$, and therefore ϕ is injective.

$\phi: A \to B$, $\chi: B \to C$.

(c) Let $a \in A$. Then
$$\psi \circ (\chi \circ \phi)(a) = \psi(\chi \circ \phi(a)) = \psi(\chi(\phi(a)))$$
& $$(\psi \circ \chi) \circ \phi(a) = \psi \circ \chi(\phi(a)) = \psi(\chi(\phi(a)))$$

(d) Thus $\psi \circ (\chi \circ \phi) = (\psi \circ \chi) \circ \phi$.

Let $c \in C$. We will show that $\exists a \in A$ s.t. $\chi \circ \phi(a) = c$.
Since χ is surjective, $\exists b \in B$ s.t. $\chi(b) = c$. Since ϕ is surjective, $\exists a \in A$ s.t. $\phi(a) = b$.
Then $\chi \circ \phi(a) = \chi(\phi(a)) = \chi(b) = c$, as desired.

(e) Suppose ϕ is invertible, & let $\phi^{-1}: B \to A$ be the inverse of ϕ.
We first show ϕ is injective. Let $x, y \in A$ s.t. $\phi(x) = \phi(y)$. Then $x = \phi^{-1}(\phi(y)) = \phi^{-1} \circ \phi(y) = \mathbb{1}_A(y) = y$. Thus ϕ is inj.
We next show ϕ is surjective. Let $z \in B$. Consider $\phi^{-1}(z)$, which is an elt of A. $\phi(\phi^{-1}(z)) = \phi \circ \phi^{-1}(z) = \mathbb{1}_B(z) = z$. Thus ϕ is surj.

(f) Suppose ϕ & χ are both invertible. Then they are both bijective, so by part (d) $\chi \circ \phi$ is surj, & we showed in class that $\chi \circ \phi$ is injective. Since $\chi \circ \phi$ is thus bijective, it is invertible.

To show $(\chi \circ \phi)^{-1} = \phi^{-1} \circ \chi^{-1}$, we check:
$$(\chi \circ \phi) \circ (\phi^{-1} \circ \chi^{-1}) = (\chi \circ (\phi \circ \phi^{-1})) \circ \chi^{-1} = (\chi \circ \mathbb{1}_B) \circ \chi^{-1} = \chi \circ \chi^{-1} = \mathbb{1}_C$$
& $$(\phi^{-1} \circ \chi^{-1}) \circ (\chi \circ \phi) = (\phi^{-1} \circ (\chi^{-1} \circ \chi)) \circ \phi = (\phi^{-1} \circ \mathbb{1}_B) \circ \phi = \phi^{-1} \circ \phi = \mathbb{1}_A.$$
Thus $(\chi \circ \phi)^{-1} = \phi^{-1} \circ \chi^{-1}$.