
Exercises 4: 3, 5, 19, 29, 31, 32, 37

Additional exercises:

1. Prove that the identity element of a group is unique. (This justifies the use of the special letter e for this element.)

2. Let G be a group and fix an element $g \in G$. Define a function $\phi_g : G \to G$ by $\phi_g(h) = ghg^{-1}$ for all $h \in G$.

   (a) Prove that ϕ_g is a bijection (i.e., injective and surjective).

   (b) Prove that $\phi_g(ab) = \phi_g(a)\phi_g(b)$ for all $a, b \in G$.

3. For $n \geq 2$, let

$$\mathbb{Z}_n^\times = \{\bar{a} \in \mathbb{Z}_n \mid \text{ there exists } \bar{c} \in \mathbb{Z}_n \text{ such that } \bar{a} \cdot_n \bar{c} = \bar{1}\}.$$

   (Note that these are the elements of $\mathbb{Z}_n$ that have a multiplicative inverse with respect to the operation $\cdot_n$. For example, recall that we mentioned in class that $\bar{2} \in \mathbb{Z}_6$ does not have a multiplicative inverse. Thus $\bar{2} \notin \mathbb{Z}_6^\times$.) Prove that $(\mathbb{Z}_n^\times, \cdot_n)$ is a group.

Challenge problem (optional, not to be turned in):

1. Using the notation in (3), show that if a and n have a common factor other than 1, then $\bar{a} \notin \mathbb{Z}_n^\times$. For an extra challenge, show that this is an "if and only if" statement.