Exercises 5

14. (a) No. $f(x)=3$ & $g(x)=-3$ are both elements of $\widetilde{F}$, but $(f+g)(x)=0$ so $f+g \notin \widetilde{F}$. Thus $\widetilde{F}$ is not closed.
(b) Yes. A gp is always a subgp of itself.

20. $G_2 \leq G_8 \leq G_7 \leq G_1 \leq G_4$.
$G_9 \leq G_3 \leq G_5$
$G_6 \leq G_5$

28. $\langle c \rangle = \{e, c\}$

46. Let G be a cyclic gp, & suppose $\exists\, a \in G$ s.t. $\langle a \rangle = G$. Then $\langle a^{-1} \rangle = G$ as well (this is because $\{a^n \mid n \in \mathbb{Z}\} = \{(a^{-1})^n \mid n \in \mathbb{Z}\}$. Since G has only one generator, $a = a^{-1}$; & so $a^2 = e$.
Thus $G = \langle a \rangle = \{e, a\}$, & G has 2 elements, if $a \neq e$, and one element if $a = e$.

47. Let $H = \{x \in G \mid x^2 = e\}$.
- H is closed: Let $x, y \in H$. Then we need to show $(xy)^2 = e$.
  $(xy)^2 = xyxy = xxyy = x^2y^2 = e \cdot e = e$ ✓ because G is abelian.
- $e \in H$: $e^2 = e$ ✓
- Let $x \in H$. We need to show $(x^{-1})^2 = e$.
  $(x^{-1})^2 = x^{-1}x^{-1} = (xx)^{-1} = (x^2)^{-1} = e^{-1} = e$ ✓

Therefore, $H \leq G$.

51. $H_a = \{x \in G \mid xa = ax\}$
- H_a is closed: Let $x, y \in H_a$.
  Then $xya = xay$ $(y \in H_a)$
  $= axy$ ✓ $(x \in H_a)$
- $e \in H_a$: $ea = a = ae$ ✓
- Let $x \in H_a$. Then $ax^{-1} = x^{-1}xax^{-1}$
  $= x^{-1}axx^{-1}$
  $= x^{-1}a$, so $x^{-1} \in H$ ✓

Therefore, $H_a \leq G$.

52. (a) $H_S = \{x \in G \mid xs = sx \ \ \forall s \in S\}$
(same pf as 51 works, just add $\forall a \in S$ to each line)
(b) Let $H_G = \{x \in G \mid xg = gx \ \forall g \in G\}$, & let $x, y \in H_G$.
Then $xy = yx$, because x commutes with every element of G, in particular x commutes with y.

Exercises 6:

33. D_4
34. $(\mathbb{R}, +)$
35. $\mathbb{Z}_2$
36. No example exists: Let G be an infinite cyclic gp, with generator a. Suppose $b \in G$ generates G. Then $a = b^k$ for some $k \in \mathbb{Z}$, & $b = a^m$ for

some $m \in \mathbb{Z}$. Thus $a = b^k = (a^m)^k = a^{mk}$, and so $mk = 1$.
The only pairs of integers satisfying $mk = 1$ are $m = 1$ & $k = 1$, or $m = -1$ & $k = -1$.
Therefore $b = a$ or $b = -a$ (plugging in $m = \pm 1$ to $b = a^m$), so an infinite cyclic gp can have at most 2 generators.

37. $\mathbb{Z}_{10} = \langle 1 \rangle = \langle 3 \rangle = \langle 7 \rangle = \langle 9 \rangle$, because 1, 3, 7, 9 are exactly the elements of $\mathbb{Z}_{10}$ that are relatively prime to 10.

44. $\phi: G \to G'$ an isomorphism, $G = \langle a \rangle$. Suppose $\psi: G \to G'$ an isomorphism s.t. $\phi(a) = \psi(a)$. We will show that $\forall g \in G$, $\phi(g) = \psi(g)$.
Let $g \in G$. Then $g = a^n$ for some $n \in \mathbb{Z}$, since $G = \langle a \rangle$.
Since ψ & ϕ are homomorphisms
$$\psi(g) = \psi(a^n) = \psi(a)^n = \phi(a)^n = \phi(a^n) = \phi(g). \quad ✓$$

49. The Klein 4 group (V) is not cyclic, but every proper subgp is. (you can check by checking $\langle g \rangle \neq V$ for any of the 4 elements $g \in V$; we did this in class.)

50. $\langle a \rangle = \{e, a\}$, & $\langle x \rangle \neq \{e, x\}$ for all $x \in G$ with $x \neq a$.

Consider the element xax^{-1}.
Then
$$\begin{aligned} (xax^{-1})^2 &= xax^{-1}xax^{-1} \\ &= xaax^{-1} \\ &= xa^2x^{-1} \\ &= xex^{-1} \\ &= xx^{-1} \\ &= e. \end{aligned}$$
Therefore, $\langle xax^{-1} \rangle = \{e, xax^{-1}\}$, & so $xax^{-1} = a$, by assumption.
But then $xa = ax$, as desired. ∎

55. $\mathbb{Z}_p$ is a cyclic group, & so all subgroups are cyclic. Let $1 \leq i < p$. then $\langle i \rangle = \mathbb{Z}_p$, because $\gcd(i, p) = 1$, as p is prime. Also, $\langle 0 \rangle = \{0\}$. Therefore, all subgps of $\mathbb{Z}_p$ are improper or trivial.

<u>Additional exercises:</u>

1. (a) $L = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

- Closure: $\begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ b_1 & c_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ b_2 & c_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a_1 + a_2 & 1 & 0 \\ b_1 + c_1a_2 + b_2 & c_1 + c_2 & 1 \end{pmatrix} \in L$ ✓
- The identity matrix $I \in L$ ✓
- The inverse of $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ca - b & -c & 0 \end{pmatrix} \in L$ ✓

(b) All cyclic groups are abelian, so I will find 2 cyclic subgps of L. The easiest way to show they are not isomorphic is to find 2 subgps of different cardinalities.

There are lots of choices!

Eg:

- $\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, n \in \mathbb{Z} \right\}$ is countable (and infinite)

- $\left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$ has order 2

- $\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle$ has order 3

A non-cyclic example:

$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & c & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R}^* \right\}$ is a subgp, uncountable, & abelian.

2. $\phi : G \to H$

(a) For any $g \in G$,

$\phi(g) = \phi(g e_G) = \phi(g)\phi(e_G)$, so $\phi(e_G)$ must be the identity of H.

(recall we showed in class that a right inverse is a 2-sided inverse. Or, check e_G, also.)

(b) $\phi(a)\phi(b) = \phi(ab)$

$= \phi(e_G)$ since a & b are inverses

$= e_H$ by (a)

Therefore $\phi(a)$ & $\phi(b)$ are inverses (again, we showed in class that a right inverse is a 2-sided inverse).

If $x \in G$ & $x^2 = e_G$, then $\phi(x)^2 = \phi(x^2) = \phi(e_G) \underset{\text{by (a)}}{=} e_H$, so $\phi(x)$ is its own inverse, as well.

(c) $\phi(g_2)\phi(g_1) = \phi(g_2 g_1) \underset{G \text{ is abelian}}{=} \phi(g_1 g_2) = \phi(g_1)\phi(g_2)$

This does not show H is abelian, since ϕ is not necessarily surjective.

<u>Note</u>: It is in fact possible for H to be non-abelian. Consider $G = \{I_3\}$ (the 3x3 identity matrix), & $H = GL(3, \mathbb{R})$.

Let $\phi : G \to H$ be the map $\phi(x) = x$.

This is a homomorphism, G is abelian, but H is not abelian.

(This is not part of what you needed to show – it's just for your information!)