MATH 222 (Lectures 1,2,4) **Worksheet 13** **Solutions**

Please inform your TA if you find any errors in the solutions.

1. Determine whether the following series converge. If the series depends on x, determine for which values of x it converges:

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

   (b) $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

   (c) $\sum_{n=3}^{\infty} \frac{1}{n^3+n-1}$

   (d) $\sum_{n=1}^{\infty} \left(\frac{n^3}{n!}\right)^n$

   (e) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

   (f) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

   (g) $\sum_{n=1}^{\infty} e^{-(\ln(n))^2}$ (Hints: $a^{bc} = (a^b)^c$ and $e^{-\ln(n)} = \frac{1}{n}$)

   Solution:

   (a) We can use the integral test for this. Since $\int_1^{\infty} \frac{1}{x^3}dx < \infty$ we just have to check that $\frac{1}{x^3}$ is a positive decreasing function. It is clearly positive for $x > 0$, so that is not an issue. To check that it is decreasing, take a derivative. $\frac{d}{dx}\frac{1}{x^3} = \frac{-3}{x^4} < 0$. Since the derivative is negative, the function is decreasing.

   (b) This sum diverges. We can see this by applying the n^{th} term test:

   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n^3} = \infty$$

   This limit would have to be zero for the sum to have any hope of converging.

   (c) We can do this with a limit comparison test. Call $b_n = \frac{1}{n^3}$. Then

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3+n-1} = 1$$

   Since the limit is 1, both sequences are positive, and $\sum_{n=0}^{\infty} \frac{1}{n^3} < \infty$, it follows that $\sum_{n=0}^{\infty} \frac{1}{n^3+n-1}$ converges.

   (d) This sum converges. We can see this with the root test.

   $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^3}{n!} = 0$$

   so the series converges.

   (e) This series converges for all x and we can see this with the ratio test.

   $$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{\left(\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}\right)}{\left(\frac{x^{2n+1}}{(2n+1)!}\right)}$$
   $$= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0$$

   so this sum converges for all x.

(f) We can solve this with the integral test once we check that the function $\frac{1}{x\ln(x)}$ is positive and decreasing on $(2,\infty)$. It is clearly positive, so we just need to check that it is decreasing.

$$\frac{d}{dx}\frac{1}{x\ln(x)} = -\frac{\ln(x)+1}{(x\ln(x))^2} < 0$$

for $x \in (2,\infty)$. We can now compare to the integral

$$\begin{aligned}\int_3^\infty \frac{1}{x\ln(x)}dx &= \lim_{b\to\infty}\int_3^b \frac{1}{x\ln(x)}dx \\ &= \lim_{b\to\infty}\int_{x=3}^{x=b}\frac{1}{u}du \qquad u=\ln(x) \quad du=\frac{1}{x}dx \\ &= \lim_{b\to\infty}\left[\ln|u|\right]_{x=3}^{x=b} \\ &= \lim_{b\to\infty}\left[\ln|\ln(x)|\right]_3^b = \infty\end{aligned}$$

so the sum diverges.

(g) This sum converges, which we can see by direct comparison to $\frac{1}{n^3}$ (or any power of n that converges). To see this, observe that $e^{-(\ln(n))^2} = \left(e^{-\ln(n)}\right)^{\ln(n)} = \frac{1}{n^{\ln(n)}}$ and for n large, $e^{-(\ln(n))^2}$ will be strictly less than $\frac{1}{n^3}$, since $\ln(n)\to\infty$.

2. Determine whether the following series converge. If the series depends on x, determine for which values of x it converges.

(a) $\sum_{n=0}^\infty e^{-nx}$

(b) $\sum_{n=1}^\infty \frac{1}{n^6+5n}$

(c) $\sum_{n=1}^\infty \frac{n!}{e^n}$

(d) $\sum_{n=1}^\infty \frac{\ln(n)}{n}$

(e) $\sum_{n=1}^\infty \left(1-\frac{1}{n}\right)^n$

Solution:

(a) Notice that $e^{-nx} = \left(\frac{1}{e^x}\right)^n$ and we know that this sum converges if and only if $\left|\frac{1}{e^x}\right| < 1$ which is if and only if $e^x > 1$. So this sum converges exactly for $x > 0$.

(b) We can do this by limit comparison. We know that $\sum_{n=1}^\infty \frac{1}{n^6}$ converges, so we just need to find the limit of the ratio of the summands in these two series.

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{1}{n^6}}{\frac{1}{n^6+5n}} &= \lim_{n\to\infty}\frac{n^6+5n}{n^6} \\ &= 1.\end{aligned}$$

Since this limit is a positive finite number and $\sum_{n=1}^\infty \frac{1}{n^6}$ converges, we know that $\sum_{n=1}^\infty \frac{1}{n^6+5n}$ converges.

(c) We can do this with the term test. Recall that $\sum_{n=0}^{\infty} a_n$ does not converges if $\lim_{n\to\infty} a_n \neq 0$ or does not exist. But $\lim_{n\to\infty} \frac{n!}{e^n} = \infty$, so this sum cannot converge.

(d) This problem can be solved with the integral test. Notice that $\frac{\ln(x)}{x} \geqslant 0$ for $x \geqslant 1$ and that $\frac{d}{dx}\frac{ln(x)}{x} = \frac{1 - x\ln(x)}{x^2} < 0$ so long as $1 - x\ln(x) < 0$, which is true for sufficiently large x. For example $\ln(x) > 1$ for $x > e$ and clearly $x > 1$ for $x > e$, so on the interval $(3, \infty)$ the function $\frac{\ln(x)}{x}$ is decreasing. We can then apply the integral test.

$$\begin{aligned}
\int_3^\infty \frac{\ln(x)}{x} dx &= \lim_{b\to\infty} \int_3^b \frac{\ln(x)}{x} dx \\
&= \lim_{b\to\infty} \int_{\ln(3)}^{\ln(b)} u du \\
&= \lim_{b\to\infty} \frac{1}{2} u^2 \Big|_{\ln(3)}^{\ln(b)} \\
&= \infty.
\end{aligned}$$

From this, we see that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ does not converge.

(e) This question may have been a little tricky. Although it looks like a good candidate for the root test, the terms here are genuinely not comparable to a geometric series (which is what the root test is checking for). Instead, we can solve this with the term test if we recall that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. From this, we see that $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \neq 0$, so the series cannot converge.

3. Determine whether the following series converge. If the series depends on x, determine for which values of x it converges.

(a) $\sum_{k=2}^{\infty} \frac{1}{k^2 - k}$

(b) $\sum_{k=100}^{\infty} \frac{1}{k\ln(k)\ln(\ln(k))}$

(c) $\sum_{k=1}^{\infty} 2^{k\ln(x)}$

Solution:

(a) This can be evaluated exactly. $\sum_{k=2}^{n} \frac{1}{k^2-k} = 1 - \frac{1}{n}$, so $\sum_{k=2}^{\infty} \frac{1}{k^2-k} = 1$ and therefore converges.

(b) This diverges by the integral test. To see this, we first check that $f(x) = \frac{1}{x\ln(x)\ln(\ln(x))}$ is positive and decreasing on $[100, \infty)$. It is positive because x, $\ln(x)$ and $\ln(\ln(x))$ are positive. To see that it is decreasing, either differentiate or note that $x\ln(x)\ln(\ln(x))$ is an increasing function on this region. The u substitution $u = \ln(\ln(x))$ has $du = \frac{dx}{x\ln(x)}$ by the chain rule, so

$$\begin{aligned}
\int_{100}^{\infty} \frac{dx}{x\ln(x)\ln(\ln(x))} &= \lim_{b\to\infty} \int_{\ln(\ln(100))}^{b} \frac{du}{u} \\
&= \lim_{b\to\infty} \left[\ln(b) - \ln(\ln(\ln(100)))\right] = \infty
\end{aligned}$$

(c) This can be done by cases. The expression only makes sense for $x > 0$. If $x = 1$, then $\ln(x) = 0$ and so for each $k, 2^{k\ln(x)} = 1$. Thus when $x = 0$, the sum diverges. We use the root test to handle the other cases:

$$\lim_{k\to\infty} \sqrt[k]{|2^{k\ln(x)}|} = \lim_{k\to\infty} 2^{\ln(x)} = 2^{\ln(x)}$$

If $0 < x < 1$ then $\ln(x) < 0$ and therefore $2^{\ln(x)} < 1$, so the series converges. If $x > 1$, then $2^{\ln(x)} > 1$, so the series diverges.