Worksheet #7 Solutions

$$\vec{x}(t) = \langle e^t \cos t,\ e^t \sin t,\ e^t \rangle$$

$$\vec{x}\,'(t) = \langle e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ e^t \rangle$$

$$\|\vec{x}\,'(t)\| = \sqrt{(e^t\cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + (e^t)^2}$$

$$= \sqrt{e^{2t}\cos^2 t - 2e^{2t}\cancel{\sin t\cos t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t + 2e^{2t}\cancel{\sin t \cos t} \cdots + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{e^{2t}(\cos^2 t + \sin^2 t + \cos^2 t + \sin^2 t) + e^{2t}}$$

$$= \sqrt{2e^{2t} + e^{2t}}$$

$$= \sqrt{3e^{2t}}$$

$$= \sqrt{3}\, e^t$$

(a) & (b) $\vec{T}(t) = \dfrac{\vec{x}\,'(t)}{\|\vec{x}\,'(t)\|} = \dfrac{\langle e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ e^t \rangle}{\sqrt{3}\, e^t}$

$$\boxed{\vec{T}(t) = \left\langle \frac{\cos t - \sin t}{\sqrt{3}},\ \frac{\sin t + \cos t}{\sqrt{3}},\ \frac{1}{\sqrt{3}} \right\rangle}$$

$$\frac{d\vec{T}}{dt} = \left\langle \frac{-\sin t - \cos t}{\sqrt{3}},\ \frac{\cos t - \sin t}{\sqrt{3}},\ 0 \right\rangle$$

$$\vec{\kappa}(t) = \frac{\left\langle \frac{-\sin t - \cos t}{\sqrt{3}},\ \frac{\cos t - \sin t}{\sqrt{3}},\ 0 \right\rangle}{\sqrt{3}\, e^t} = \boxed{\left\langle \frac{-\sin t - \cos t}{3e^t},\ \frac{\cos t - \sin t}{3e^t},\ 0 \right\rangle}$$

$$\kappa = \|\vec{\kappa}(t)\| = \sqrt{\left(\frac{-\sin t - \cos t}{3e^t}\right)^2 + \left(\frac{\cos t - \sin t}{3t}\right)^2 + 0}$$

$$= \sqrt{\frac{\sin^2 t + 2\sin t\cancel{\cos t} + \cos^2 t + \cos^2 t - 2\sin t\cancel{\cos t} + \sin^2 t}{9e^{2t}}}$$

$$= \sqrt{\frac{2}{9e^{2t}}} = \boxed{\frac{\sqrt{2}}{3e^t} = \kappa}$$

$$\vec{N}(t) = \frac{\vec{\kappa}(t)}{\|\vec{\kappa}(t)\|} = \frac{\left\langle \frac{-\sin t - \cos t}{3e^t}, \frac{\cos t - \sin t}{3e^t}, 0 \right\rangle}{\frac{\sqrt{2}}{3e^t}}$$

$$\boxed{\vec{N}(t) = \left\langle \frac{-\sin t - \cos t}{\sqrt{2}}, \frac{\cos t - \sin t}{\sqrt{2}}, 0 \right\rangle}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$= \begin{vmatrix} \vec{i} & \frac{\cos t - \sin t}{\sqrt{3}} & \frac{-\sin t - \cos t}{\sqrt{2}} \\ \vec{j} & \frac{\cos t + \sin t}{\sqrt{3}} & \frac{\cos t - \sin t}{\sqrt{2}} \\ \vec{k} & \frac{1}{\sqrt{3}} & 0 \end{vmatrix}$$

$$= \vec{k}\left(\frac{\cos t - \sin t}{\sqrt{3}} \cdot \frac{\cos t - \sin t}{\sqrt{2}} - \frac{\cos t + \sin t}{\sqrt{3}} \cdot \frac{-\sin t - \cos t}{\sqrt{2}} \right)$$

$$- \vec{j} \cdot \left(0 - \frac{1}{\sqrt{3}} \cdot \frac{-\sin t - \cos t}{\sqrt{2}} \right) + \vec{i}\left(0 - \frac{1}{\sqrt{3}} \cdot \frac{\cos t - \sin t}{\sqrt{2}} \right)$$

$$= \left\langle \frac{-\cos t + \sin t}{\sqrt{6}}, \frac{-\sin t - \cos t}{\sqrt{6}}, \frac{\cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t}{\sqrt{6}} + \frac{\cancel{\sin t \cos t} + \cos^2 t + \sin^2 t + \cancel{\sin t \cos t}}{\sqrt{6}} \right\rangle$$

$$\boxed{\vec{B}(t) = \left\langle \frac{-\cos t + \sin t}{\sqrt{6}}, \frac{-\sin t - \cos t}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle}$$

(c) $\displaystyle\int_0^{\pi} \|\vec{x}'(t)\|\, dt = \int_0^{\pi} \sqrt{3}e^t\, dt = \sqrt{3}e^t \Big|_0^{\pi} = \boxed{\sqrt{3}e^{\pi} - \sqrt{3}}$

(d) $\vec{l}(s) = \vec{t_0} + s\,\vec{v}(t_0) = \langle e^{t_0}\cos t_0, e^{t_0}\sin t_0, e^{t_0} \rangle$
$+ s\langle e^{t_0}\cos t_0 - e^{t_0}\sin t_0, e^{t_0}\sin t_0 + e^{t_0}\cos t_0, e^{t_0} \rangle$

(e) The line intersects the xy-plane when $z=0$.

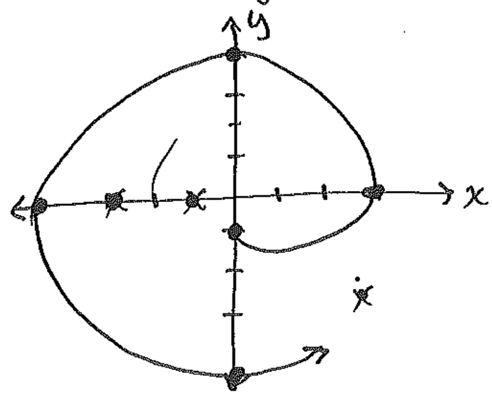
$$e^{t_0}+se^{t_0}=0$$
$$s=-1$$

So the point is $\vec{l}(-1)=\langle e^{t_0}\cos t_0 - e^{t_0}\cos t_0 + e^{t_0}\sin t_0,\ e^{t_0}\sin t_0 - e^{t_0}\sin t_0 - e^{t_0}\cos t_0, 0\rangle$

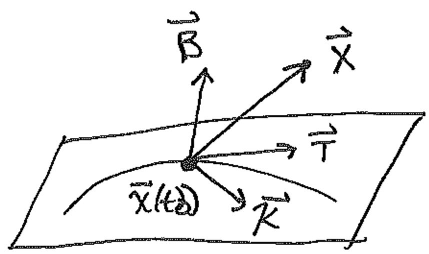
$$=\boxed{\langle e^{t_0}\sin t_0, -e^{t_0}\cos t_0, 0\rangle}$$

$\vec{y}(t_0)=\langle e^{t_0}\sin t_0, -e^{t_0}\cos t_0\rangle$. To figure out the curve, I'll plot points:

t_0	$\vec{y}(t_0)$
0	$\langle 0,-1\rangle$
$\pi/2$	$\langle e^{\pi/2}, \cancel{-e^{\pi/2}}\ 0\rangle$
π	$\langle 0, e^{\pi}\rangle$
$3\pi/2$	$\langle -e^{3\pi/2}, 0\rangle$
2π	$\langle 0, -e^{2\pi}\rangle$



(f) Osculating plane



$$\vec{B}\cdot(\vec{X}-\vec{x}(t_0))=0$$

$$\left\langle \frac{-\cos t_0+\sin t_0}{\sqrt{6}}, \frac{-\sin t_0-\cos t_0}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\rangle\cdot\langle x-e^{t_0}\cos t_0, y-e^{t_0}\sin t_0, z e^{t_0}\rangle=0$$

$$\frac{-\cos t_0+\sin t_0}{\sqrt{6}}x-\frac{\sin t_0+\cos t_0}{\sqrt{6}}y+\frac{2}{\sqrt{6}}z = +e^{t_0}\cos t_0\left(\frac{-\cos t_0+\sin t_0}{\sqrt{6}}\right)$$
$$+e^{t_0}\sin t_0\left(\frac{-\sin t_0-\cos t_0}{\sqrt{6}}\right)$$
$$=\frac{-e^{t_0}\cos^2 t_0+e^{t_0}\sin t_0\cos t_0-e^{t_0}\sin^2 t_0}{\sqrt{6}}$$
$$-\frac{e^{t_0}\sin t_0\cos t_0}{\sqrt{6}}=\frac{e^{t_0}}{\sqrt{6}}(-\cos^2 t_0-\sin^2 t_0)$$

So the equation is:

$$\boxed{\frac{-\cos t_0 + \sin t_0}{\sqrt{6}}\, x - \frac{\sin t_0 + \cos t_0}{\sqrt{6}}\, y + \frac{2}{\sqrt{6}}\, z = -\frac{e^{t_0}}{\sqrt{6}}}$$