Worksheet 4 Solutions

④ (a) The parallelepiped is a cube with sides of length 1, so its volume is 1. Thus, the volume of the tetrahedron is $\frac{1}{6}\cdot 1 = \boxed{1/6}$

(b) $A = (1,-1,2)$ $\quad \vec{AB} = \langle 1,1,-3\rangle$

$B = (2,0,-1)$ $\quad \vec{AC} = \langle 0,2,1\rangle$

$C = (1,1,3)$ $\quad \vec{AD} = \langle -3,2,-1\rangle$

$D = (-2,1,1)$

$$\text{Volume} = \frac{1}{6}\underbrace{\left(\vec{AB}\cdot(\vec{AC}\times\vec{AD})\right)}_{\text{the order doesn't matter}} = \frac{1}{6}\cdot\left|\begin{matrix} 1 & 0 & -3 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{matrix}\right| =$$

$$= \frac{1}{6}\left(1(-2-2) - 1(0+3) - 3(0+6)\right)$$

$$= \frac{1}{6}(-4-3-18) = \frac{1}{6}\cdot 25 = \boxed{\frac{25}{6}}$$

⑤ (a) FALSE.

Counterexample: $\vec{u} = \langle 1,2,1\rangle$, $\vec{w} = \langle 1,1,1\rangle$, $\vec{v} = \langle 2,1,1\rangle$

Then $\vec{u}\cdot\vec{w} = 1+2+1 = 4 = 2+1+1 = \vec{v}\cdot\vec{w}$, but $\vec{u} \neq \vec{v}$.

(b) FALSE.

Counterexample: $\vec{u} = \langle 1,0,0\rangle$, $\vec{v} = \langle 2,0,0\rangle$, $\vec{w} = \langle 3,0,0\rangle$

Then $\vec{u}\times\vec{w} = \vec{0} = \vec{v}\times\vec{w}$, but $\vec{u}\neq\vec{v}$.

(c) TRUE.

Proof: $\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w}$, so $\|\vec{u}\|\cdot\cancel{\|\vec{w}\|}\cos\theta = \|\vec{v}\|\cdot\cancel{\|\vec{w}\|}\cdot\cos\phi$,

[θ is the angle btwn $\vec{u}$ & $\vec{w}$
ϕ is the angle btwn $\vec{v}$ & $\vec{w}$]

$$\|\vec{u}\|\cos\theta = \|\vec{v}\|\cos\phi$$

$$\frac{\|\vec{u}\|}{\|\vec{v}\|} = \frac{\cos\phi}{\cos\theta}.$$

$\vec{u}\times\vec{w} = \vec{v}\times\vec{w}$, so $\|\vec{u}\times\vec{w}\| = \|\vec{v}\times\vec{w}\|$

$$\|\vec{u}\|\cdot\cancel{\|\vec{w}\|}\sin\theta = \|\vec{v}\|\cancel{\|\vec{w}\|}\sin\phi$$

$$\|\vec{u}\|\sin\theta = \|\vec{v}\|\sin\phi$$

$$\frac{\|\vec{u}\|}{\|\vec{v}\|} = \frac{\sin\phi}{\sin\theta}$$

[continued]

⑤ (c) [continued]

Putting the 2 together, we have

$$\frac{\|\vec{u}\|}{\|\vec{v}\|} = \frac{\cos\phi}{\cos\theta} = \frac{\sin\phi}{\sin\theta}, \quad \text{so} \quad \frac{\cos\phi}{\cos\theta} = \frac{\sin\phi}{\sin\theta}.$$

So, $\cos\phi \sin\theta = \sin\phi \cos\theta$

$$\cos\phi\sin\theta - \sin\phi\cos\theta = 0$$

~~Double angle form~~ Angle sum formula

$$\sin(\phi - \theta) = 0$$

$\phi - \theta = 0$ [Note: 1

$$\phi = \theta$$

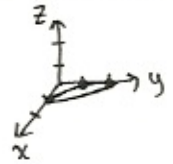
Also, that means $\|\vec{u}\| \cancel{\cos\theta} = \|\vec{v}\| \cancel{\cos\theta}$

$$\|\vec{u}\| = \|\vec{v}\|.$$

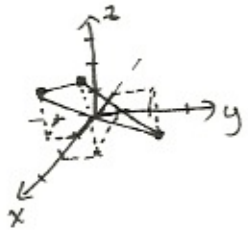
So $\vec{u}$ & $\vec{v}$ have the same length & make the same angle with $\vec{w}$. Since $\vec{u}\times\vec{w} = \vec{v}\times\vec{w}$, $\vec{u}, \vec{w}$, & $\vec{v}$ lie in the same plane. Also, the cross products being equal means that $\vec{u}$ & $\vec{v}$ must be on the same side of $\vec{w}$ (or else the cross products would be opposites). Thus, $\vec{u} = \vec{v}$.

⑥ (a) $(0,1,0), (0,2,0), (0,3,0)$. Since the 3 points are collinear, they do not determine a unique plane.

(b) $(0,1,0), (0,2,0), (1,0,0)$. Since the 3 points form a Δ, they determine a unique plane. Since all z-coords are 0, the Δ lies in the xy-plane, so the plane is the xy-plane $(z=0)$.



(c) $A(1,-1,2)$, $B(2,1,3)$, $C(-1,2,-1)$. The 3 points form a Δ, so they determine a unique plane.



$\vec{AB} = \langle 1,2,1\rangle$, $\vec{AC} = \langle -2,3,-3\rangle$, $\vec{AX} = \langle x_1 - 1, x_2 + 1, x_3 - 2\rangle$

$$\vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & 1 & -2 \\ \vec{j} & 2 & 3 \\ \vec{k} & 1 & -3 \end{vmatrix} = 3\vec{i} + \vec{j} + 7\vec{k}$$

$$0 = \vec{n}\cdot\vec{AX} = \langle 3,1,7\rangle \cdot \langle x_1 - 1, x_2 + 1, x_3 - 2\rangle = 3x_1 - 3 + x_2 + 1 + 7x_3 - 14$$

So, $3x_1 + x_2 + 7x_3 = 16$

⑥ (d) $A = (1, -1, 2)$, $B = (2, 1, 3)$, $C = (-5, -13, -4)$

$\overrightarrow{AB} = \langle 1, 2, 1 \rangle$ and $\overrightarrow{BC} = \langle -7, -14, -7 \rangle$

notice that $\overrightarrow{BC} = -7 \cdot \overrightarrow{AB}$, so they are in the same direction. Thus, the 3 points are collinear, so they don't determine a unique plane.

[Note: I didn't graph this one because the numbers in pt C seem too large to plot accurately]