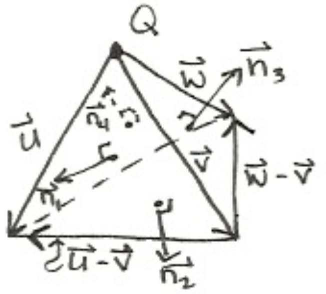
Worksheet 3 Solutions

(4)



$\vec{n_1}, \vec{n_2}, \vec{n_3}$, & $\vec{n_4}$ are the 4 normal vectors sticking out of the 4 faces. We want to show $\vec{n_1}+\vec{n_2}+\vec{n_3}+\vec{n_4}=0$

$$\left.\begin{aligned} \vec{n_1} &= \vec{u}\times\vec{v}\cdot\tfrac{1}{2} \\ \vec{n_2} &= (\vec{u}-\vec{v})\times(\vec{w}-\vec{v})\cdot\tfrac{1}{2} \\ \vec{n_3} &= \vec{v}\times\vec{w}\cdot\tfrac{1}{2} \\ \vec{n_4} &= \vec{w}\times\vec{u}\cdot\tfrac{1}{2} \end{aligned}\right\}$$

<u>Note</u>: I used the right-hand rule to determine the order of the vectors in the cross product. Be sure the normal vector points out. The $\cdot\frac{1}{2}$ in each is to ensure the correct magnitude. $\|\vec{a}\times\vec{b}\|$ is area of a parallelogram & we want our magnitude to be area of a Δ, so half of this.

$$\vec{n_1}+\vec{n_2}+\vec{n_3}+\vec{n_4} =$$

$$\tfrac{1}{2}(\vec{u}\times\vec{v})$$
$$+\tfrac{1}{2}[(\vec{u}-\vec{v})\times(\vec{w}-\vec{v})]$$
$$+\tfrac{1}{2}(\vec{v}\times\vec{w})$$
$$+\tfrac{1}{2}(\vec{w}\times\vec{u})$$

$$= \cancel{\tfrac{1}{2}(\vec{u}\times\vec{v})} + \tfrac{1}{2}(\vec{u}\times\vec{w}) - \cancel{\tfrac{1}{2}(\vec{u}\times\vec{v})} - \cancel{\tfrac{1}{2}(\vec{v}\times\vec{w})} + \tfrac{1}{2}(\vec{v}\times\vec{v}) + \cancel{\tfrac{1}{2}(\vec{v}\times\vec{w})}$$
$$+\tfrac{1}{2}(\vec{w}\times\vec{u})$$

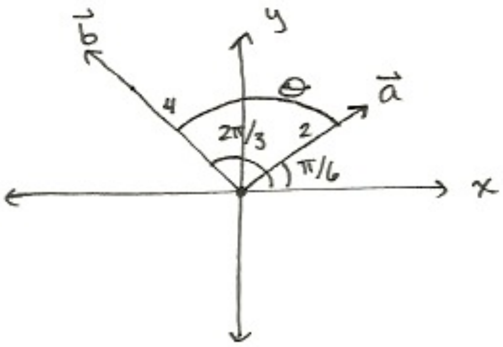
$$= \tfrac{1}{2}(\vec{u}\times\vec{w}) + \underbrace{\tfrac{1}{2}(\vec{v}\times\vec{v})}_{=0} + \underbrace{\tfrac{1}{2}(\vec{w}\times\vec{u})}_{=-\frac{1}{2}(\vec{u}\times\vec{w})}$$

$$= \tfrac{1}{2}(\vec{u}\times\vec{w}) - \tfrac{1}{2}(\vec{u}\times\vec{w})$$

$$= 0 \checkmark$$

⑤ $\|\vec{a}\| = 2$, ∡ of $\pi/6$ w/ pos. x-axis

$\|\vec{b}\| = 4$, ∡ of $2\pi/3$ w/ pos. x-axis



(a) $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$, $\theta = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$

$= 0$ (because $\vec{a}$ & $\vec{b}$ are perpendicular)

(b) $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

$= 2 \cdot 4 \cdot \sin \pi/2$

$= 2 \cdot 4 \cdot 1$

$= 8$

(c) $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & a_1 & b_1 \\ \vec{j} & a_2 & b_2 \\ \vec{k} & a_3 & b_3 \end{vmatrix}$. So we could use trig to find $\vec{a}$ & $\vec{b}$, but there's an easier way!

Since $\vec{a}$ & $\vec{b}$ lie in the xy-plane, the direction perpendicular to both $\vec{a}$ & $\vec{b}$ is the z-axis (i.e. straight out of or straight into the page). The right-hand rule tells us that $\vec{a} \times \vec{b}$ is in the positive z-axis direction. But since we know the length is 8, we see that:

$$\vec{a} \times \vec{b} = \langle 0, 0, 8 \rangle$$

(d) $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b} = \langle 0, 0, -8 \rangle$

⑥ $A(1,1,-2)$ $B(2,0,-1)$ $C(-1,1,1)$ $D(-2,2,0)$

(a) We need to show both pairs of opp. sides are parallel and have same length, i.e. opp. sides are the same vector. So $\vec{AB} = \vec{CD}$ (or $\vec{DC}$), and $\vec{BC} = \vec{AD}$ (or $\vec{DA}$)

$\vec{AB} = \langle 2-1, 0-1, -1-^-2 \rangle = \langle 1,-1,1 \rangle$

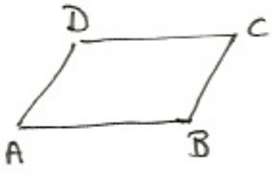
$\vec{CD} = \langle -2-^-1, 2-1, 0-1 \rangle = \langle -1,1,-1 \rangle$; $\vec{DC} = \langle 1,-1,1 \rangle$

So, $\vec{AB} = \vec{DC}$

$\vec{BC} = \langle -1-2, 1-0, 1-^-1 \rangle = \langle -3,1,2 \rangle$

$\vec{AD} = \langle -2-1, 2-1, 0-^-2 \rangle = \langle -3,1,2 \rangle$

So $\vec{BC} = \vec{AD}$



Thus, ABCD is a parallelogram.

(b) area of ABCD $= \|\vec{AB} \times \vec{AD}\|$ (or $\|\vec{BA} \times \vec{BC}\|$, etc.)

$= \|\vec{AB}\| \cdot \|\vec{AD}\| \sin\theta$. But θ isn't obvious. So we could either find θ using $\vec{AB} \cdot \vec{AD}$, or just calculate the cross product. I'll use the cross product.

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \vec{i} & 1 & -3 \\ \vec{j} & -1 & 1 \\ \vec{k} & 1 & 2 \end{vmatrix} = \vec{i}(-2-1) - \vec{j}(2+3) + \vec{k}(1+3)$$

$$= -3\vec{i} - 5\vec{j} + 4\vec{k}$$

$$= \langle -3,-5,4 \rangle$$

So, $\|\vec{AB} \times \vec{AD}\| = \|\langle -3,-5,4 \rangle\| = \sqrt{9+25+16} = \sqrt{50} = 5\sqrt{2}$.

(b) area of $\Delta ABC = \frac{1}{2} \cdot$ (area of ABCD)

$= \frac{1}{2} \cdot 5\sqrt{2} = \frac{5}{2}\sqrt{2}$.