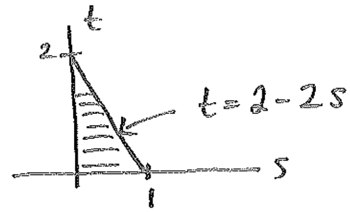
Worksheet 27 Solutions

③ $C(s,t) = \begin{pmatrix} s \\ t \\ 2-2s-t \end{pmatrix}$, $(s,t) \in$:



$$\nabla \times F = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ xz & xy & 3xz \end{vmatrix} = i(0) - j(3z-x) + k(y)$$

$$C_s = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad C_t = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad C_s \times C_t = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= i(2) - j(-1) + k(1)$$

$$\int_0^1 \int_0^{2-2s} \begin{pmatrix} 0 \\ s - 3(2-2s-t) \\ t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ +1 \\ 1 \end{pmatrix} dt\,ds$$

$$= \int_0^1 \int_0^{2-2s} (s - 6 + 6s + 3t + t)\,dt\,ds = \int_0^1 \int_0^{2-2s} (7s + 4t - 6)\,dt\,ds$$

$$= \int_0^1 7st + 2t^2 - 6t \Big|_0^{2-2s} ds = \int_0^1 (14s - 14s^2 + 2(4-8s+4s^2) - 12 + 12s)\,ds$$

$$= \int_0^1 (-6s^2 + 10s - 4)\,ds = \quad -2s^3 + 5s^2 - 4s \Big|_0^1$$

$$= -2 + 5 - 4 = \boxed{-1}$$

④ $$\nabla \times F = \begin{vmatrix} i & j & k \\ D_x & D_y & D_z \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = i(2y) - j(2x-2z) + k(2x-2y)$$

$$C(s,t) = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix}, \quad (s,t) \in [-1,1]^2 \qquad C_s \times C_t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\int_{-1}^1 \int_{-1}^1 \begin{pmatrix} 2t \\ -2s \\ 2s-2t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ds\,dt = \int_{-1}^1 \int_{-1}^1 (2s-2t)\,ds\,dt = \int_{-1}^1 s^2 - 2st \Big|_{-1}^1 dt$$

$$= \int_{-1}^1 1 - 2t - (1+2t)\,dt = \int_{-1}^1 -4t\,dt = -2t^2 \Big|_{-1}^1 = -2-(-2) = \boxed{0}$$

⑤ $\int_{\Sigma_1} \nabla \times F \cdot d\sigma = \int_{\partial \Sigma_1} F \cdot ds$

$\partial\Sigma_1 : 4x^2 + 9y^2 = 36$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\gamma(t) = \begin{pmatrix} 3\cos t \\ 2\sin t \\ 0 \end{pmatrix}, \quad \gamma'(t) = \begin{pmatrix} -3\sin t \\ 2\cos t \\ 0 \end{pmatrix}$

$t \in [0, 2\pi]$

$$\int_0^{2\pi} \begin{pmatrix} 2\sin t \\ 9\cos^2 t \\ \text{[something]} \end{pmatrix} \cdot \begin{pmatrix} -3\sin t \\ 2\cos t \\ 0 \end{pmatrix} dt = \int_0^{2\pi} (-6\sin^2 t + 18\cos^3 t)\, dt$$

$$= \int_0^{2\pi} -3 + 3\cos(2t) + \underbrace{18\cos t(1-\sin^2 t)}_{\substack{u = \sin t \\ du = \cos t\, dt}}\, dt$$

$$= -3t + \frac{3}{2}\sin(2t)\Big|_0^{2\pi} + 18\int (1-u^2)\, du$$

$$= -6\pi + 18\sin t - \frac{18}{3}\sin^3 t\Big|_0^{2\pi}$$

$$= \boxed{-6\pi}$$

⑥ Since $\nabla \times \nabla f = 0$, by Stokes' Thm, $\int_C \nabla f \cdot ds = \iint_R 0 \cdot d\sigma = \boxed{0}$

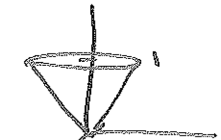
⑦ $C(s,t) = \begin{pmatrix} s \\ s^2 \\ t \end{pmatrix}, \quad \begin{matrix} s \in [-1,1] \\ t \in [0,2] \end{matrix} \qquad C_s = \begin{pmatrix} 1 \\ 2s \\ 0 \end{pmatrix} \quad C_t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$C_s \times C_t = \begin{vmatrix} i & j & k \\ s & s^2 & t \\ 0 & 0 & 1 \end{vmatrix} = i(s^2) - j(s) + k(0)$$

$$\int_{\Sigma} F \cdot d\sigma = \int_0^2 \int_{-1}^1 \begin{pmatrix} 0 \\ s^2 \\ -st \end{pmatrix} \cdot \begin{pmatrix} s^2 \\ -s \\ 0 \end{pmatrix} ds\,dt = \int_0^2 \int_{-1}^1 -s^3\, ds\,dt$$

$$= \int_0^2 -\frac{1}{4}s^4\Big|_{-1}^1 dt = \int_0^2 -\frac{1}{4} - \left(-\frac{1}{4}\right) dt = \boxed{0}$$

⑧ Closed cone: Divergence thm.



cylindrical

$$\int_{\Sigma_1} F\cdot dr = \iiint_V (y-1)\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 (r\sin\theta - 1)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 (r^2\sin\theta - r)\, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 zr^2\sin\theta - rz \Big|_r^1\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2\sin\theta - r - r^3\sin\theta + r^2)\, dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\sin\theta - \frac{r^2}{2} - \frac{r^4}{4}\sin\theta + \frac{r^3}{3}\Big|_0^1\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{3}\sin\theta - \frac{1}{2} - \frac{1}{4}\sin\theta + \frac{1}{3}\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{12}\sin\theta - \frac{1}{6}\right) d\theta$$

$$= -\frac{1}{12}\cos\theta - \frac{1}{6}\theta \Big|_0^{2\pi} = \boxed{-\frac{\pi}{3}}$$

Open Cone: Find flux over disk & subtract it from $-\pi/3$.

disk: $C(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1 \end{pmatrix}$, $C_r \times C_\theta = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$, $r\in[0,1]$, $\theta\in[0,2\pi]$

$$\int_{\text{disk}} F\cdot dr = \int_0^{2\pi}\int_0^1 \begin{pmatrix} r^2\cos\theta\sin\theta \\ 0 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} dr\,d\theta = \int_0^{2\pi}\int_0^1 -r\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{r^2}{2}\Big|_0^1\, d\theta = \int_0^{2\pi} -\frac{1}{2}\,d\theta = -\pi$$

So flux $= -\frac{\pi}{3} - (-\pi) = \boxed{\frac{2\pi}{3}}$