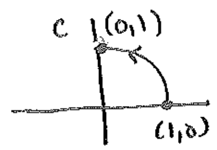
Worksheet 23 Solutions

① 

$$\oint_C -y\,dx + 2x\,dy = \int_0^{\pi/2} \langle -\sin t, 2\cos t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt$$

C: $\vec{x}(t) = \langle \cos t, \sin t\rangle$

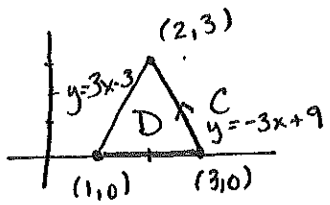
$0 \le t \le \pi/2$

$\vec{x}\,'(t) = \langle -\sin t, \cos t\rangle$

$$= \int_0^{\pi/2} \sin^2 t + 2\cos^2 t\,dt = \int_0^{\pi/2} 1 + \cos^2 t\,dt$$

$$= \int_0^{\pi/2} 1 + \tfrac{1}{2} + \frac{\cos 2t}{2}\,dt = \frac{3}{2}t + \frac{\sin 2t}{4}\Big|_0^{\pi/2}$$

$$= \boxed{\tfrac{3}{4}\pi}$$

② 

$y = 3x - 3$ $\qquad$ $y = -3x + 9$

$\frac{y+3}{3} = x$ $\qquad$ $x = \frac{9-y}{3}$

$\frac{1}{3}y + 1 = x$ $\qquad$ $x = 3 - \frac{1}{3}y$

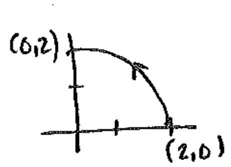
By Green's Thm, $\oint_C \overbrace{-xy}^{P}\,dx + \overbrace{x^2}^{Q}\,dy = \iint_D (Q_x - P_y)\,dA$

$$= \iint_D 2x + x\,dA = \int_0^3 \int_{\frac{1}{3}y+1}^{3-\frac{1}{3}y} 3x\,dx\,dy$$

$$= \int_0^3 \frac{3x^2}{2}\Big|_{\frac{1}{3}y+1}^{3-\frac{1}{3}y} dy = \frac{3}{2}\int_0^3 9 - 2y + \cancel{\tfrac{1}{9}y^2} - \cancel{\tfrac{1}{9}y^2} - \tfrac{2}{3}y - 1\,dy$$

$$= \frac{3}{2}\int_0^3 8 - \frac{8}{3}y\,dy = \frac{3}{2}\left(8y - \frac{4}{3}y^2\Big|_0^3\right)$$

$$= \frac{3}{2}(24 - 12) = \boxed{18}$$

③ 

$0 \le t \le \pi/2$

C: $\vec{x}(t) = \langle 2\cos t, 2\sin t\rangle$

$\vec{x}\,'(t) = \langle -2\sin t, 2\cos t\rangle$

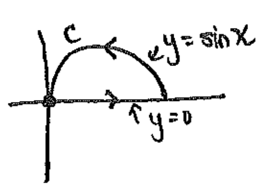
$\|\vec{x}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t}$

$= \sqrt{4} = 2$

$$\int_C xy^2\,ds = \int_0^{\pi/2} (2\cos t)(4\sin^2 t)\cdot 2\,dt$$

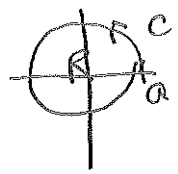
$$= 16\int_0^{\pi/2} \cos t\,\sin^2 t\,dt = 16\left(\frac{\sin^3 t}{3}\right)\Big|_0^{\pi/2}$$

$$= \boxed{\tfrac{16}{3}}$$

④ 

$\vec{F} = \langle \underbrace{3x}_{P}, \underbrace{2y}_{Q}\rangle$ is defined everywhere, & $P_y = 0 = Q_x$, so $\vec{F}$ is conservative. By the FTC for line integrals,

$\oint_C \vec{F}\cdot d\vec{x} = f(0,0) - f(0,0) = 0$, where $\vec{F} = \vec{\nabla} f$.

⑤ 

$0 \le t \le 2\pi$

$\vec{x}(t) = \langle a\cos t, a\sin t\rangle$

$\vec{x}\,'(t) = \langle -a\sin t, a\cos t\rangle$

① $\oint_C \vec{F}\cdot d\vec{x} = \int_0^{2\pi} \langle -x^2y, xy^2\rangle \cdot \vec{x}\,'(t)\,dt$

$= \int_0^{2\pi} \langle -a^3\cos^2 t\sin t, a^3\cos t\sin^2 t\rangle \cdot \langle -a\sin t, a\cos t\rangle\,dt$

$= \int_0^{2\pi} a^4\cos^2 t\sin^2 t + a^4\cos^2 t\sin^2 t\,dt$

$= 2a^4\int_0^{2\pi} \cos^2 t\sin^2 t\,dt = 2a^4\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)\left(\frac{1-\cos 2t}{2}\right)dt$

$= 2a^4\int_0^{2\pi}\left(\frac{1}{4} - \frac{(\cos 2t)^2}{4}\right)dt = 2a^4\cdot\frac{1}{4}t\Big|_0^{2\pi} - \frac{2a^4}{4}\int_0^{2\pi}\frac{1+\cos 4t}{2}\,dt = a^4\pi - \frac{a^4}{2}\cdot\frac{2\pi}{2} - \frac{\sin 4t}{8}\Big|_0^{2\pi}$

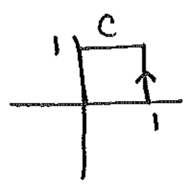
$= \boxed{\frac{a^4\pi}{2}}$

② $\iint_R (Q_x - P_y)\,dA = \int_0^{2\pi}\int_0^a (y^2+x^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^a r^3\,dr\,d\theta$

$= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^a d\theta = \int_0^{2\pi}\frac{a^4}{4}d\theta = a^4\cdot\frac{\theta}{4}\Big|_0^{2\pi} = \boxed{\frac{a^4\pi}{2}}$

$Q = xy^2 \quad P = -x^2y$

$Q_x = y^2 \quad P_y = -x^2$

⑥ 

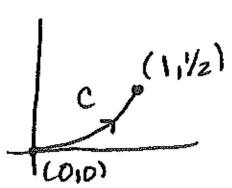
$\oint_C \langle \underbrace{x^2+4y}_{P}, \underbrace{x+y^2}_{Q}\rangle\cdot d\vec{x} = \int_0^1\int_0^1 (1-4)\,dx\,dy = -3\int_0^1\int_0^1 dx\,dy$

$= \boxed{-3}$

⑦ (same as #4)

⑧ $\int_C \frac{x+y^2}{\sqrt{1+x^2}}\,ds = \int_0^1 \frac{t+\frac{1}{4}t^4}{\sqrt{1+t^2}}\cdot\sqrt{1+t^2}\,dt = \int_0^1 t + \frac{1}{4}t^4\,dt$

$= \frac{1}{2}t^2 + \frac{1}{20}t^5\Big|_0^1 = \frac{1}{2} + \frac{1}{20} = \boxed{11/20}$



$\vec{x}(t) = \langle t, t^2/2\rangle$

$0 \le t \le 1$

$\|\vec{x}\,'(t)\| = \|\langle 1, t\rangle\| = \sqrt{1+t^2}$

⑩ $\oint_C \langle -y, x\rangle\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi}\langle -4\sin t, \cos t\rangle\cdot\langle -\sin t, 4\cos t\rangle\,dt$

$= \int_0^{2\pi} 4\sin^2 t + 4\cos^2 t\,dt = \int_0^{2\pi} 4\,dt = \boxed{8\pi}$

$\vec{r}(t) = \langle \cos t, 4\sin t\rangle$

$\vec{r}\,'(t) = \langle -\sin t, 4\cos t\rangle$

⑨ $\vec{r}(t) = \langle \sin t, \cos t, t \rangle, \; 0 \le t \le 2\pi$

*

$\vec{r}\,'(t) = \langle +\cos t, -\sin t, 1 \rangle$

$$\int_0^{2\pi} \langle t, \sin t, \cos t \rangle \cdot \langle +\cos t, -\sin t, 1 \rangle \, dt$$

$$= \int_0^{2\pi} +t\cos t - \sin^2 t + \cos t \, dt = \int_0^{2\pi} t\cos t - \frac{1-\cos 2t}{2} + \cos t \, dt$$

$$= \int_0^{2\pi} \cdot t\cos t \, dt - \left(\tfrac{1}{2}t - \frac{\sin 2t}{4} + \sin t \right)\Big|_0^{2\pi}$$

$u = t \qquad v = \sin t$

$du = dt \qquad dv = \cos t \, dt$

$$= -\left[t\sin t \Big|_0^{2\pi} - \int_0^{2\pi} \sin t \, dt \right] - \pi$$

$$= -\left(0 + \cos t \Big|_0^{2\pi} \right) - \pi = -(1-1) - \pi = \boxed{-\pi}$$

⑪ $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle, \; -2\pi \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -4\sin t, 4\cos t, 3 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = \sqrt{16+9} = 5$

$$\int_C \sqrt{x^2+y^2}\, ds = \int_{-2\pi}^{2\pi} \sqrt{16\cos^2 t + 16\sin^2 t} \cdot 5\, dt = \int_{-2\pi}^{2\pi} 20\, dt = 20(4\pi) = \boxed{80\pi}$$