Worksheet #2 Solutions

① $\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 =$

$$(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) + (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) =$$

$$\vec{v}\cdot\vec{v} + \cancel{2\vec{v}\cdot\vec{w}} + \vec{w}\cdot\vec{w} + \vec{v}\cdot\vec{v} - \cancel{2\vec{v}\cdot\vec{w}} + \vec{w}\cdot\vec{w} =$$

$$2\vec{v}\cdot\vec{v} + 2\vec{w}\cdot\vec{w} =$$

$$2\|\vec{v}\|^2 + 2\|\vec{w}\|^2 =$$

$$2(\|\vec{v}\|^2 + \|\vec{w}\|^2) \checkmark$$

② $(\vec{v}\times\vec{w})\cdot(\vec{v}\times\vec{w}) =$

$\|\vec{v}\times\vec{w}\|\cdot\|\vec{v}\times\vec{w}\| =$ (using ~~Ha~~ $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2 = \|\vec{a}\|\cdot\|\vec{a}\|$)

$$(\|\vec{v}\|\|\vec{w}\|\sin\theta)(\|\vec{v}\|\|\vec{w}\|\sin\theta) =$$

$$\|\vec{v}\|^2\|w\|^2\sin^2\theta =$$

$$\|\vec{v}\|^2\|\vec{w}\|^2(1-\cos^2\theta) =$$

$$\|\vec{v}\|^2\|\vec{w}\|^2 - (\|\vec{v}\|\|\vec{w}\|\cos\theta)(\|\vec{v}\|\|\vec{w}\|\cos\theta) =$$

$$(\vec{v}\cdot\vec{v})(\vec{w}\cdot\vec{w}) - (\vec{v}\cdot\vec{w})(\vec{v}\cdot\vec{w}) =$$

$$(\vec{v}\cdot\vec{v})(\vec{w}\cdot\vec{w}) - (\vec{v}\cdot\vec{w})^2$$

③ $A(3,1,-1)$, $B(2,0,1)$, $C(1,-2,0)$

$\triangle ABC$ is half of the parallelogram spanned by $\vec{AB}$ and $\vec{AC}$, whose area is the magnitude of $\vec{AB}\times\vec{AC}$.

$\|\vec{AB}\times\vec{AC}\| = \|\vec{AB}\|\cdot\|\vec{AC}\|\cdot\sin\theta$, $\quad \|\vec{AB}\| = \|\langle -1,-1,2\rangle\| = \sqrt{6}$

$\|\vec{AC}\| = \|\langle -2,-3,1\rangle\| = \sqrt{14}$

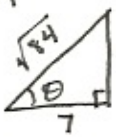
To find $\sin\theta$, we use the dot product:

$$\vec{AB}\cdot\vec{AC} = \|\vec{AB}\|\cdot\|\vec{AC}\|\cos\theta$$

$$2+3+2 = \sqrt{6}\cdot\sqrt{14}\cdot\cos\theta$$

$$7 = \sqrt{84}\cdot\cos\theta$$

$$7/\sqrt{84} = \cos\theta$$



$\sqrt{84-49} = \sqrt{35}$, so $\sin\theta = \sqrt{\frac{35}{84}} = \sqrt{\frac{5}{12}}$

$$\|\vec{AB}\times\vec{AC}\| = \sqrt{6}\cdot\sqrt{14}\cdot\sqrt{\frac{5}{12}}$$

$$= \sqrt{\frac{6\cdot 14\cdot 5}{12}}$$

$$= \sqrt{35}.$$

Since $\triangle ABC$ is half of the parallelogram,

$\boxed{\text{area of } \triangle ABC = \tfrac{1}{2}\sqrt{35}}$

Worksheet #1 Solutions

⑤ (a) $\vec{i}\cdot\vec{j} = \|\vec{i}\|\cdot\|\vec{j}\|\cdot\cos\theta, \quad \theta = 90^\circ, \ \|\vec{i}\| = 1, \ \|\vec{j}\| = 1$

$$= 1\cdot 1\cdot 0$$
$$= 0$$

Similarly, $\vec{i}\cdot\vec{k} = 0$, $\vec{j}\cdot\vec{k} = 0$.

$\vec{i}\cdot\vec{i} = \|\vec{i}\|\cdot\|\vec{i}\|\cdot\cos\theta, \quad \theta = 0^\circ$

$$= 1\cdot 1\cdot 1$$
$$= 1$$

Similarly, $\vec{j}\cdot\vec{j} = 1$, $\vec{k}\cdot\vec{k} = 1$.

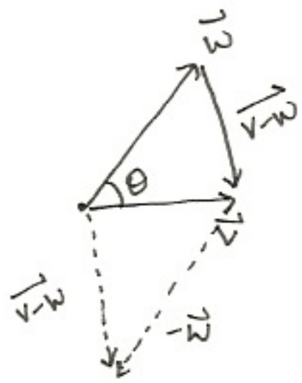
(b) $\vec{v}\cdot\vec{w} = (v_1\vec{i} + v_2\vec{j} + v_3\vec{k})\cdot(w_1\vec{i} + w_2\vec{j} + w_3\vec{k})$

$= v_1w_1(\vec{i}\cdot\vec{i}) + v_1w_2(\vec{i}\cdot\vec{j}) + v_1w_3(\vec{i}\cdot\vec{k}) + v_2w_1(\vec{j}\cdot\vec{i}) + v_2w_2(\vec{j}\cdot\vec{j})$
$+ v_2w_3(\vec{j}\cdot\vec{k}) + v_3w_1(\vec{k}\cdot\vec{i}) + v_3w_2(\vec{k}\cdot\vec{j}) + v_3w_3(\vec{k}\cdot\vec{k})$

$= v_1w_1\cdot 1 + 0 + 0 + 0 + v_2w_2\cdot 1 + 0 + 0 + 0 + v_3w_3\cdot 1$

$= v_1w_1 + v_2w_2 + v_3w_3$ ✓

⑥ (a)



(b) Law of Cosines:

$c^2 = a^2 + b^2 - 2ab\cos\theta$

$\|\vec{v-w}\| = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta$

(c) $\|\vec{v}-\vec{w}\|^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$, $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$, and $\|\vec{w}\|^2 = \vec{w}\cdot\vec{w}$, so the equation from (b) becomes:

$(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2\|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$

$\cancel{\vec{v}\cdot\vec{v}} - \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{v} + \cancel{\vec{w}\cdot\vec{w}} = \cancel{\vec{v}\cdot\vec{v}} + \cancel{\vec{w}\cdot\vec{w}} - 2\|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$

$-\cancel{2}\,\vec{v}\cdot\vec{w} = -\cancel{2}\,\|\vec{v}\|\cdot\|\vec{w}\|\cos\theta$

$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$ ✓

⑦ $\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3$ is easier when we are given coordinates, while the other is easier if we know the angle between them.