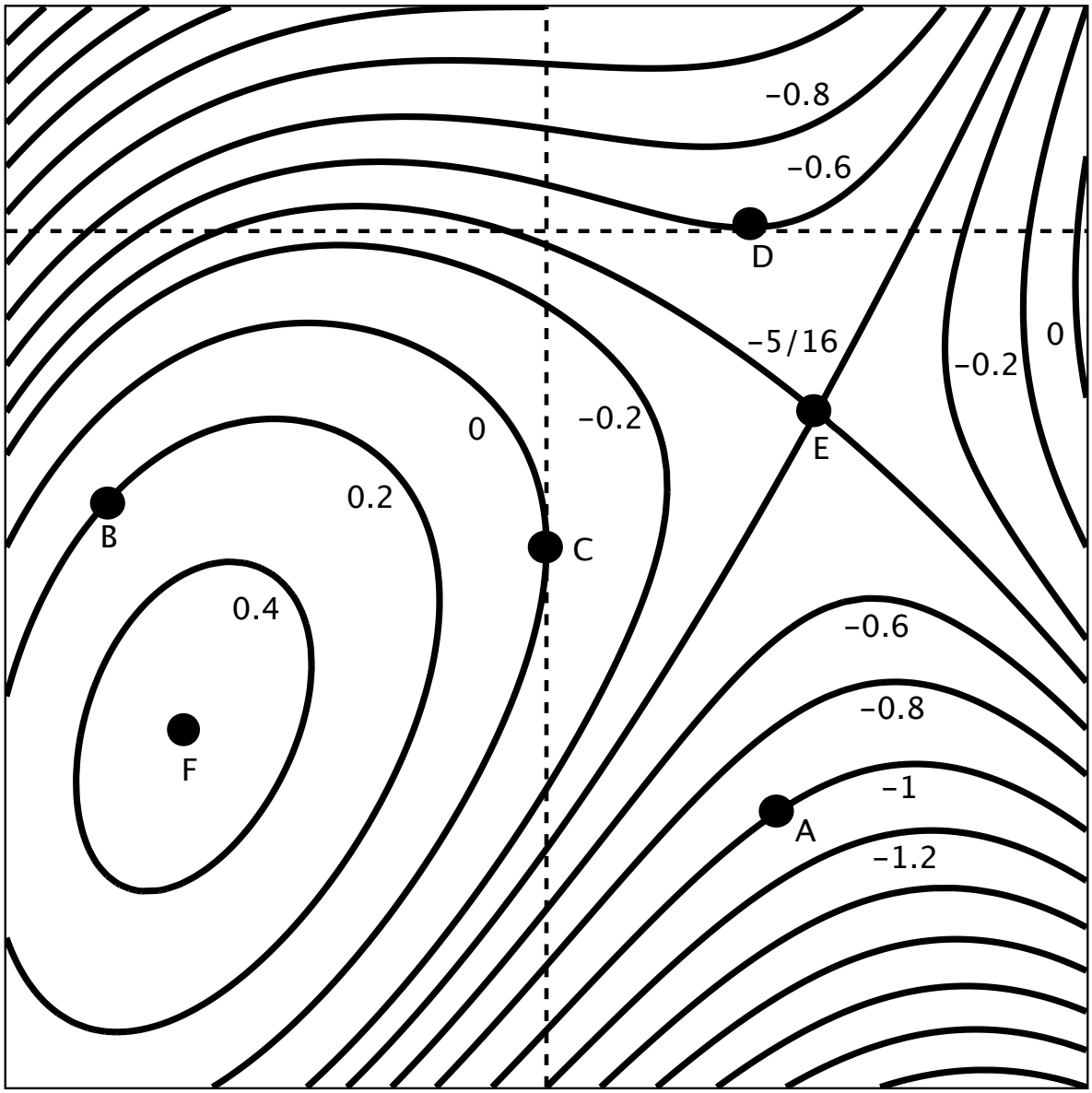
1. Let $f(x, y) = 4x^2 - 4xy + 4y^2$, and suppose $x = \cos t$ and $y = \sin t$. Find $\dfrac{\mathrm{d}f}{\mathrm{d}t}$ using the chain rule.

2. Let $f(x, y) = x^2 + y^2$. Find all the points on the level set $f(x, y) = 1$ where the gradient is parallel to $\mathbf{i} + 2\mathbf{j}$.

The following figure is a level set plot for a differentiable function $f(x, y)$. The horizontal axis (not shown) is the x-axis, and the vertical axis (not shown) is the y-axis. The dashed vertical line is the tangent line to the level set $f = 0$ at C. The dashed horizontal line is the tangent line to the level set $f = -6$ at D.



Using the figure, answer the questions on the following page.

1. Determine whether $f'_x(A)$ is positive, zero, or negative. Do the same for $f'_y(A)$.

2. Determine whether $f'_x(B)$ is positive, zero, or negative. Do the same for $f'_y(B)$.

3. The dashed vertical line is the tangent line to the level set $f = 0$ at C.

   (a) Determine whether $f'_x(C)$ is positive, zero, or negative.

   (b) Suppose you are standing at the point C. Describe what happens to f as you move directly North or South, starting from the point C. Does f increase, decrease, or stay the same?

   (c) Determine whether $f'_y(C)$ is positive, zero, or negative.

4. The dashed horizontal line is the tangent line to the level set $f = -6$ at D.

   (a) Suppose you are standing at the point D. Describe what happens to f as you move directly West or East, starting from the point D. Does f increase, decrease, or stay the same?

   (b) Determine whether $f'_x(D)$ is positive, zero, or negative. Do the same for $f'_y(D)$.

5. (a) Suppose you are standing at the point E. Describe what happens to f as you move directly North, South, West, or East. Does f increase, decrease, or stay the same?

   (b) Based on your observations in (a), what can you say about $f'_x(E)$ and $f'_y(E)$?

6. The level set passing through the point F is not really missing from the figure. What is it? Can you figure out what $f'_x(F)$ and $f'_y(F)$ are?

7. (a) Can you locate (just eyeball it) all the points P in the xy-plane for which $f'_y(P) = 0$? Do these points fit a simple pattern in the figure? What is interesting about this pattern?

   (b) Can you locate (just eyeball it) all the points Q in the xy-plane for which $f'_x(Q) = 0$? Do these points fit a simple pattern in the figure? How would you describe this pattern?

8. A mysterious sage from the Himalayas tells you that the range for the variable x in the figure is $0 \leq x \leq 2$, and the range for y is $0 \leq y \leq 2$. She also tells you that the function $f(x, y)$ whose level set plot is given in the figure is actually

$$f(x, y) = x^3 - 3x^2 + x + xy - y^2 + y.$$

   (a) For the sage's function $f(x, y)$, compute f'_x and f'_y.

   (b) Set $f'_y = 0$. Sketch the graph of your equation $f'_y = 0$ in the figure. Does this confirm your suspicion from 7(a)?

   (c) Set $f'_x = 0$ and see if this confirms your suspicion from 7(b).

   (d) Can we trust the sage now?

   (e) Find the exact coordinates of the points E and F ... the sage will be impressed.

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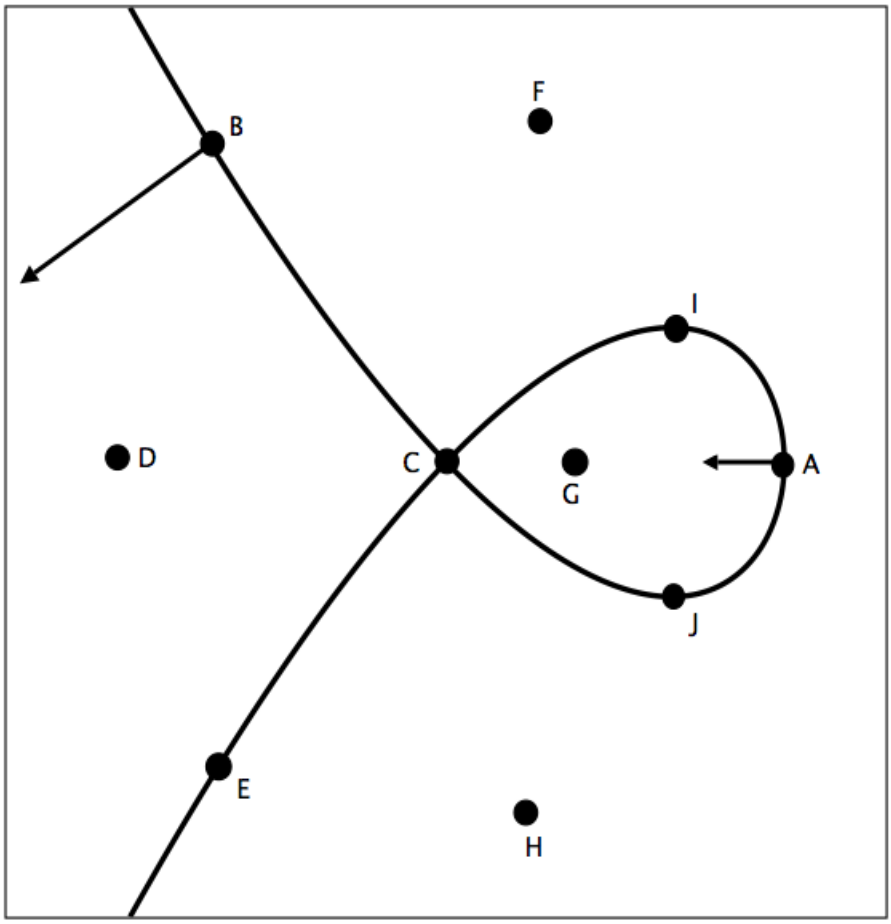
9. The curve in the following picture is the zero set of a function $f(x, y)$. We are given the gradient of f at the points A, B, as in the picture.



Answer the following questions:

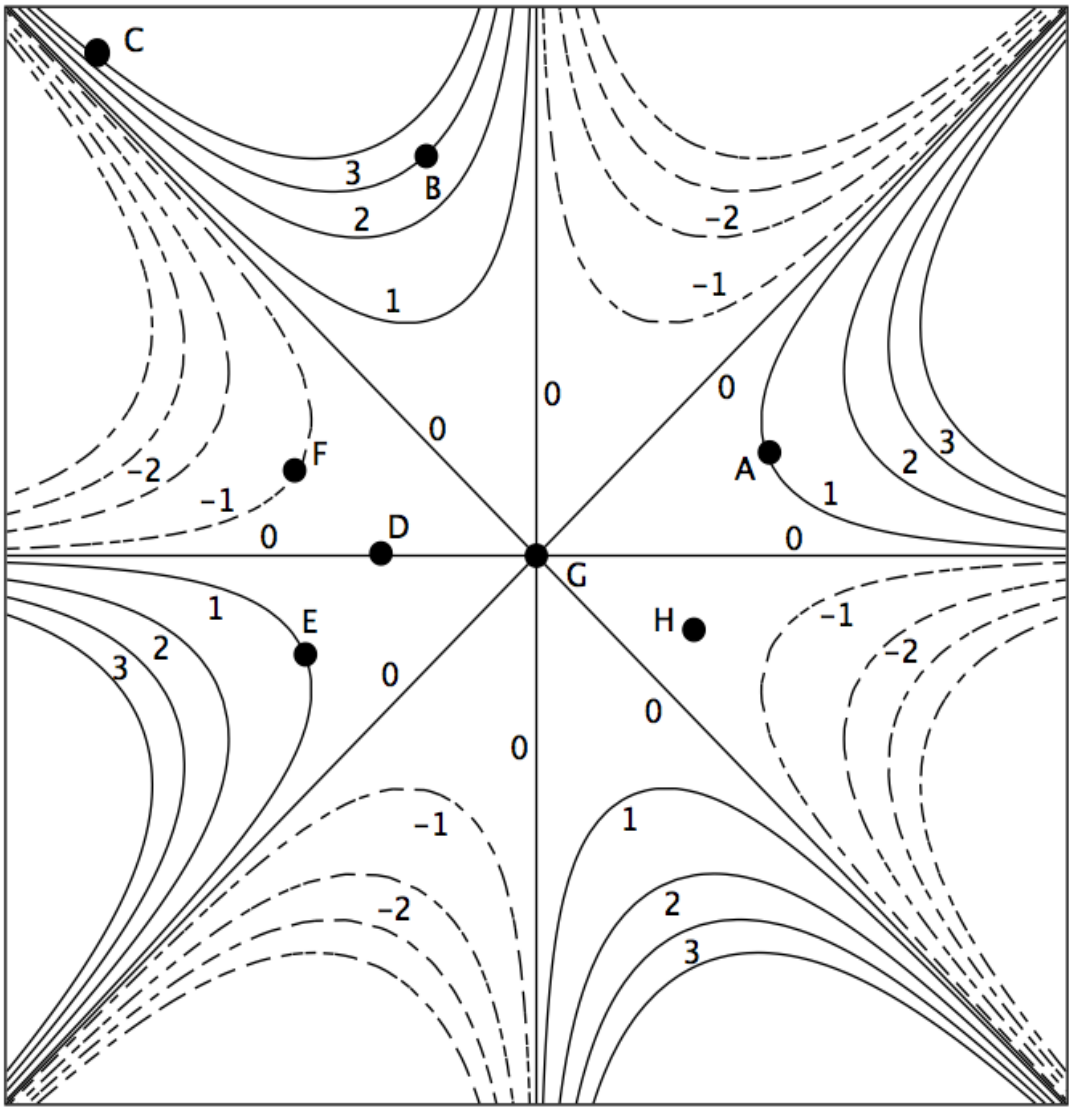
(i) What is $\mathbf{n}f(\mathrm{C})$?

(ii) Consider all ten points A through J. At which of these is f positive? Negative? Zero?

(iii) Sketch $\mathbf{n}f(\mathrm{E})$, $\mathbf{n}f(\mathrm{I})$, $\mathbf{n}f(\mathrm{J})$ in the graph.

(iv) If we were to sketch more level sets, where would you expect the level sets to be closer together, near A or near B?

10. The following picture shows an assortment of level sets for a function $f(x, y)$.



Answer the following questions:

(i) At which point in the picture is $\mathbf{n}f(x, y) = \langle 0, 0 \rangle$?

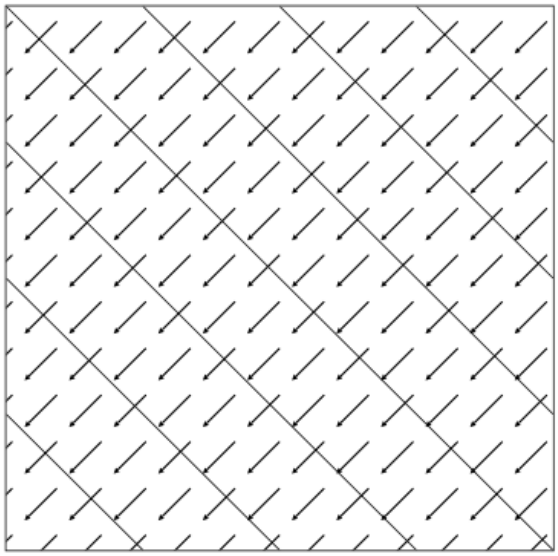
(ii) Consider the points A, B, C. At which of these three is the magnitude of $\mathbf{n}f$ largest? At which of these three is it smallest?

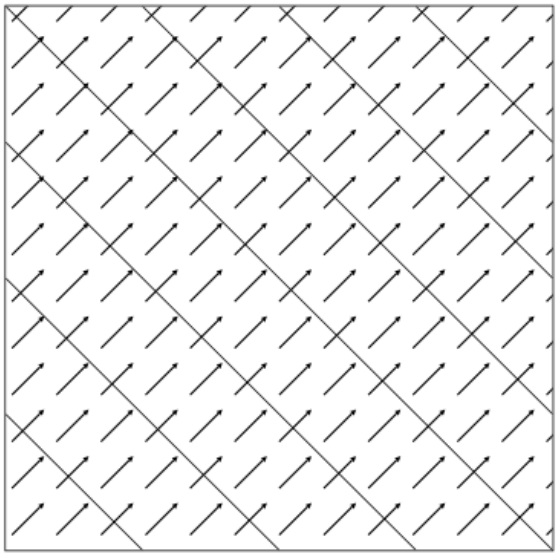
(iii) Consider the points D, E, F. At each of these points, draw the direction in which $\mathbf{n}f$ is pointing.

11. Consider the following functions:

(1) $f(x,y) = x^2 - y^2$ (2) $f(x,y) = -x^2 - y^2$ (3) $f(x,y) = y^2 - x^2$
(4) $f(x,y) = x + y$ (5) $f(x,y) = x^2 + y^2$ (6) $f(x,y) = -x - y$

Match the six functions with the six pictures below. The pictures below display an assortment of level sets, but the level sets are unmarked (and the ones at negative levels are not dotted). The pictures display the gradient of the function evaluated at various sample points.





(more pictures on the back . . .)