Review 2 Solutions

① $\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$, $\nabla f(2,1) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. The directional derivative is $D_v f = \nabla f \cdot v$, where v is a unit vector.

u isn't a unit vector, so use $v = \frac{u}{\|u\|} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

$\nabla f(2,1) \cdot v = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \frac{8}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{10}{\sqrt{5}} = \boxed{2\sqrt{5}}$. This means that if you take on step in the direction of u, the surface increases by $2\sqrt{5}$ units.

② $\nabla f = \begin{pmatrix} 2\cos(xy) - 2xy\sin(xy) - 2xe^{x^2+y^2} \\ -2x^2\sin(xy) - 2ye^{x^2+y^2} \end{pmatrix}$

③ $\frac{\partial x}{\partial s} = 2$, $\frac{\partial y}{\partial s} = t$, so when $(s,t) = (1,2)$, $\frac{\partial y}{\partial s}(1,2) = 2$.

Now $(1,2) = (s,t)$, & the inputs to ∇f are (x,y), so we need to change $(1,2)$ to (x,y)-coords:

$x = 2s + 2t \Rightarrow x = 6$

$y = st \Rightarrow y = 2$

$\nabla f(6,2) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$

Now plug all this into the chain rule:

$$\begin{aligned} \frac{\partial h}{\partial s}(1,2) &= f_x(6,2) \cdot x_s(1,2) + f_y(6,2) \cdot y_s(1,2) \\ &= (-1)(2) + (2)(2) \\ &= -2 + 4 \\ &= \boxed{2} \end{aligned}$$

④ (a) The 4 pts where the circle is tangent to the level sets.

(b) $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 2 \end{cases}$ or $\underbrace{\begin{cases} \nabla g = 0 \\ g(x,y) = 2 \end{cases}}$

$\downarrow$

$x = y = 0$, not on circle, so no solution.

$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2 + y^2 = 2 \end{cases}$ $\longrightarrow$ $\lambda = \frac{y}{2x}$ or $x = 0$.

- If $x = 0$, from ①, $y = 0$, but $0^2 + 0^2 \neq 2$, so $x \neq 0$.
- If $\lambda = \frac{y}{2x}$, then $x = 2\left(\frac{y}{2x}\right)y$

$$x^2 = y^2$$
$$x = \pm y$$

$\underline{x = y}$

$$y^2 + y^2 = 2$$
$$y^2 = 1$$
$$y = \pm 1$$

$\underline{x = -y}$

$$y^2 + y^2 = 2$$
$$y = \pm 1$$

(x,y)	$f(x,y) = xy$
$(1,1)$	1
$(-1,-1)$	1
$(1,-1)$	-1
$(-1,1)$	-1

$\boxed{\text{So global max of } 1 \text{ at } (1,1) \text{ \& } (-1,-1) \text{, and global min of } -1 \text{ at } (-1,1) \text{ and } (1,-1).}$

⑤ $\nabla f(1,1) = \begin{pmatrix} 6x^2y + 10\ln(y)x \\ 2x^3 + \frac{5x^2}{y} \end{pmatrix}\Bigg|_{(1,1)} = \begin{pmatrix} 6+0 \\ 2+5 \end{pmatrix} = \boxed{\begin{pmatrix} 6 \\ 7 \end{pmatrix}}$

If you move perpendicular to ∇f, then you are moving along a level set, so the function $f(x,y)$ remains constant. By guess & check, $\boxed{\begin{pmatrix} -7 \\ 6 \end{pmatrix}}$ is $\perp$ to ∇f. (also, $\begin{pmatrix} 7 \\ -6 \end{pmatrix}$).

⑥ $\begin{cases} 2x = 2\lambda \\ 2y = -3\lambda \\ x^2+y^2=1 \end{cases} \longrightarrow \quad x = \lambda$

$2y = -3x$

$y = -\frac{3}{2}x$

$x^2 + \left(-\frac{3}{2}x\right)^2 = 1$

$\frac{13}{4}x^2 = 1$

$x = \pm 2/\sqrt{13}$

$$\boxed{\left(2/\sqrt{13}, -3/\sqrt{13}\right) \text{ \& } \left(-2/\sqrt{13}, 3/\sqrt{13}\right)}$$

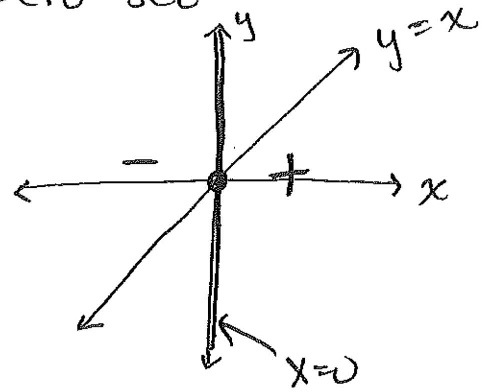
⑦ $0 = x^3 - x^2y$

$x^2(x-y) = 0$

$x^2 = 0$ or $x - y = 0$

$x = 0 \qquad y = x$

Zero set:



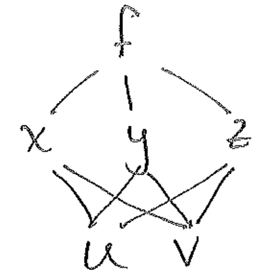
Because a level set intersects itself at (0,0), (0,0) is a critical point. By signs, $\boxed{(0,0)}$ is a $\boxed{\text{saddle pt.}}$

⑧ y is defined implicitly as a function of x when $f_y \neq 0$. $f_y = 2y$, so when $y \neq 0$, $f_y \neq 0$. The corresp. x-values are $\boxed{x \neq \pm 1}$. $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x}{2y} = -\frac{x}{y}$

The 2nd part is similar, w/ x & y switched: $f_x \neq 0$ when $x \neq 0$, so $\boxed{y \neq \pm 1}$. $dx/dy = -f_y/f_x = -y/x$.

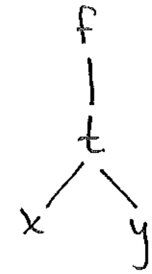
(9) (a) $g_x = f_u \cdot 3 + f_v \cdot 2 + f_z \cdot y$

$g_y = f_v \cdot 2y + f_z \cdot x$



(b) $g_x = f'(x^2 + 2xy + y^2) \cdot (2x + 2y)$

$g_y = f'(x^2 + 2xy + y^2)(2x + 2y)$



(10) $f(x,y) = x^3 + 2x^2y^2$

$f_x = 3x^2 + 4xy^2 = 0$

$f_y = 4x^2y = 0 \longrightarrow y = 0$ or $4x^2 = 0$

$x = 0$

If $y = 0$: $3x^2 = 0$

$x = 0$

If $x = 0$: y can be anything

$(0,y)$

2^{nd} derivative test

	$(0,y)$
$f_{xx} = 6x + 4y^2$	$4y^2$
$f_{yy} = 4x^2$	0
$f_{xy} = 8xy$	0
$D = f_{xx}f_{yy} - (f_{xy})^2$	$4y^2 \geq 0.$

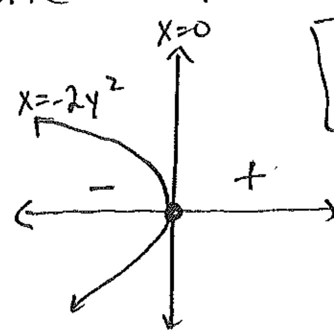
If $y \neq 0$, $4y^2 > 0$ & $f_{xx} > 0$, so $\boxed{(0,y) \text{ is a min.}}$ $(y \neq 0)$

If $y = 0$, the test is inconclusive, so we draw the zero set:

$0 = x^3 + 2x^2y^2$

$x^2(x + 2y^2) = 0$

$x = 0$ or $x = -2y^2$



$\boxed{(0,0) \text{ is a saddle}}$

(11) $f(x,y) = x^2 + 3xy + y^2$ on

Boundady

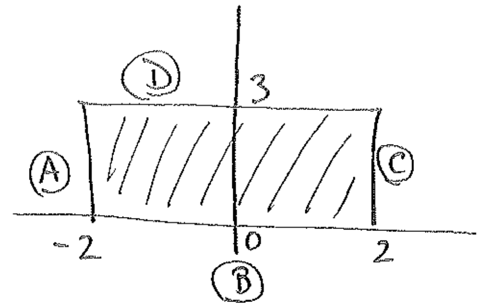
(A): $x = -2$:

$f(-2,y) = 4 - 6y + y^2$

$f'(-2,y) = -6 + 2y = 0$

$y = 3$

$(-2,3)$ (already listed)



	(x,y)	$f(x,y)$
4 corners	$(-2,0)$	4
	$(2,0)$	4
	$(2,3)$	31
	$(-2,3)$	-5
	$(0,0)$	0

$\boxed{\text{Global max of 31 at } (2,3) \text{ \& global min of } -5 \text{ at } (-2,3)}$

(B) $y = 0$:

$f(x,0) = x^2$

$f'(x,0) = 2x = 0$

$x = 0$ $\quad (0,0)$

(C) $x = 2$:

$f(2,y) = 4 + 6y + y^2$

$f'(2,y) = 6 + 2y = 0$

$y = -3$

$(2,-3)$ (not in rectangle)

(D) $y = 3$:

$f(x,3) = x^2 + 9x + 9$

$f'(x,3) = 2x + 9 = 0$

$x = -9/2$

$(-9/2, 3)$ (not in rectangle)

Interior:

$$\nabla f = \begin{pmatrix} 2x + 3y \\ 3x + 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{aligned} 2x &= -3y \\ x &= -\tfrac{3}{2}y \end{aligned}$$

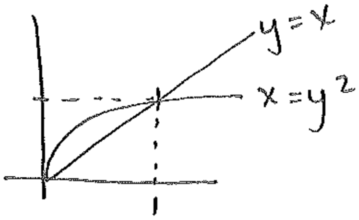
$3(-3/2\, y) + 2y = 0$

$-\frac{5}{2}y = 0$

$y = 0 \longrightarrow x = 0$ (already listed)

(12) $\int_0^3 \int_0^2 (4-y^2)\,dy\,dx = \int_0^3 \left(4y - \frac{1}{3}y^2\right)\Big|_0^2 dx = \int_0^3 \frac{16}{3}\,dx = \boxed{16}$

(13)



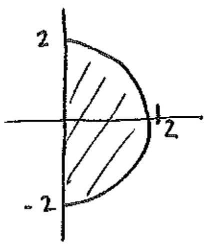
double integral: $\int_0^1 \int_{y^2}^{y} 1\,dx\,dy =$

$\int_0^1 x\Big|_{y^2}^{y} dy = \int_0^1 (y - y^2)\,dy = \frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1$

$= \frac{1}{2} - \frac{1}{3} = \boxed{1/6}$

triple integral: $\int_0^1 \int_{y^2}^{y} \int_0^1 dz\,dx\,dy$

(14)



double integral:

(1) $\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} 6x\,dx\,dy = \int_{-2}^{2} 3x^2\Big|_0^{\sqrt{4-y^2}} dy$

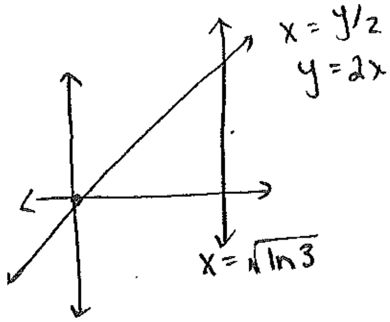
$= \int_{-2}^{2} (12 - 3y^2)\,dy = 12y - y^3\Big|_{-2}^{2} = \boxed{32}$

(2) $\int_{-\pi/2}^{\pi/2} \int_0^2 6r^2\cos\theta\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} 2r^3\cos\theta\Big|_0^2 d\theta = \int_{-\pi/2}^{\pi/2} 16\cos\theta\,d\theta$

$= 16\sin\theta\Big|_{-\pi/2}^{\pi/2} = 16 - (-16) = 32$

triple integral: $\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_0^{6x} 1\,dz\,dx\,dy$

(15)



$\int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\,dy\,dx$

(16) (see website)

(17) $f(x,y,z) = x^2+y^2+z^2$ (the square of the distance to (0,0,0))

$g(x,y,z) = x+2y+2z = 3$

$$\begin{cases} 2x = \lambda \\ 2y = 2\lambda \longrightarrow y = \lambda \text{, so} \quad \begin{matrix} 2x = y \\ x = y/2 \end{matrix} \quad \text{and} \quad \begin{matrix} 2z = 2y \\ z = y \end{matrix} \\ 2z = 2\lambda \\ x+2y+2z = 3 \end{cases}$$

$$(y/2) + 2y + 2(y) = 3$$

$$\frac{9}{2}y = 3$$

$$y = 2/3, \quad x = 1/3, \quad z = 2/3$$

$$(1/3, 2/3, 2/3)$$

$$d = \sqrt{(1/3)^2+(2/3)^2+(2/3)^2} = \boxed{\sqrt{3}}$$

(18) $f(x,y,z) = x$, $\underbrace{g(x,y,z) = x+y-z = 0, \quad h(x,y,z) = x^2+2y^2+2z^2 = 8}_{\text{2 constraints}}$

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(x,y,z) = 0 \\ h(x,y,z) = 8 \end{cases} \Rightarrow \begin{matrix} (1)\ 1 = \lambda \cdot 1 + \mu \cdot 2x \\ (2)\ 0 = \lambda \cdot 1 + \mu \cdot 4y \longrightarrow \lambda = -4\mu y \\ (3)\ 0 = \lambda \cdot (-1) + \mu \cdot (4z) \\ (4)\ x+y-z = 0 \\ (5)\ x^2+y^2+2z^2 = 8 \end{matrix}$$

(1): $1 = -4\mu y + 2\mu x$

(3) $0 = 4\mu y + 4\mu z \longrightarrow 4\mu(y+z) = 0$

$\mu \neq 0$ or $y = -z$ ↳ plug into (4) & (5)

(1): $1 = 0$ ↯

$x - 2z = 0 \to x = 2z$

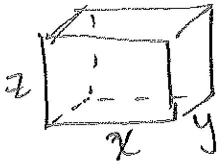
$x^2 + 3z^2 = 8 \to 7z^2 = 8$

$z^2 = 8/7$

$z = \pm\sqrt{8/7}$

(x,y,z)	$f(x,y,z) = x$	
$(2\cdot\sqrt{8/7}, -\sqrt{8/7}, \sqrt{8/7})$	$2\sqrt{8/7}$	← max
$(-2\sqrt{8/7}, \sqrt{8/7}, -\sqrt{8/7})$	$-2\sqrt{8/7}$	← min

(19)



maximize volume: $f(x,y,z) = xyz$

SA = 64: $g(x,y,z) = 2xy + 2xz + 2yz = 64$

$xy + xz + yz = 32$

$$\begin{cases} (1)\ yz = \lambda(y+z) \longrightarrow \lambda = \dfrac{yz}{y+z} & (y \neq -z \text{ because both must be positive}) \\ (2)\ xz = \lambda(x+z) \longrightarrow \lambda = \dfrac{xz}{x+z} & (x \neq -z) \\ (3)\ xy = \lambda(x+y) \longrightarrow \lambda = \dfrac{xy}{x+y} & (x \neq -y) \\ (4)\ xy + xz + yz = 32 \end{cases}$$

So, $\dfrac{yz}{y+z} = \dfrac{xz}{x+z}$

$\dfrac{y}{y+z} = \dfrac{x}{x+z}$ or $z \neq 0$ (then $f = 0$, not max)

$y(x+z) = x(y+z)$

$xy + yz = xy + xz$

$yz = xz$

$y = x$

$\dfrac{yz}{y+z} = \dfrac{xy}{x+y}$

$\dfrac{z}{y+z} = \dfrac{x}{x+y}$ ($y \neq 0$, as above)

$z(x+y) = x(y+z)$

$zx + zy = xy + xz$

$zy = xy$

$z = x$, so $x = y = z$

(4): $3x^2 = 32$

$x^2 = 32/3$ $\longrightarrow$ $\boxed{\left(\sqrt{32/3}, \sqrt{32/3}, \sqrt{32/3}\right)}$

$x = +\sqrt{32/3}$