Calculus III: Practice Midterm II

Name: ______________________________

- Write your solutions in the space provided. Continue on the back if you need more space.
- You must show your work. Only writing the final answer will receive little credit.
- Partial credit will be given for incomplete work.
- The exam contains 5 problems.
- The last page is the formula sheet, which you may detatch.
- **Good luck!**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) Write true or false. No justification is needed.

(a) The curve parametrized by $\langle \sin(2t), \cos(3t), 1+t^3\rangle$ never intersects the XY plane.

Solution: False. It intersects the XY plane at $t=-1$.

True False

(b) If the acceleration vector is perpendicular to the velocity vector, the object must be going in a circle or helix.

Solution: False. Any motion with constant speed will have this property.

True False

(c) The graph of the function $f(x,y)=x^2+y^2$ is a hemisphere.

Solution: False.

True False

(d) For a vector function $\overrightarrow{r}(t)$, we have

$$\frac{d(\overrightarrow{r}(t)\cdot\overrightarrow{r}(t))}{dt}=\frac{d\overrightarrow{r}(t)}{dt}\cdot\frac{d\overrightarrow{r}(t)}{dt}.$$

Solution: False. The derivative of a (dot) product must be computed by the product rule.

True False

(e) If T, N, and B represent the unit tangent, normal, and binormal vectors, then $T=N\times B$.

Solution: True. T,N,B form a frame just like the $\mathbf{i},\mathbf{j},\mathbf{k}$.

True False

2. Let C be the intersection of the sphere of radius 2 centered at the origin and the plane $y + z = 0$.

(a) (5 points) Write parametric equations for C.

Solution: The equation for the sphere is

$$x^2 + y^2 + z^2 = 4.$$

Substituting $z = -y$ from the second equation $y + z = 0$, we get

$$x^2 + 2y^2 = 4.$$

Dividing by 4, we get

$$x^2/4 + y^2/2 = 1$$
$$\implies (x/2)^2 + (y/\sqrt{2})^2 = 1.$$

We can thus take $x/2 = \cos(t)$ and $y/\sqrt{2} = \sin(t)$. Using $z = -y$, the parametrization is

$$x = 2\cos(t), \quad y = \sqrt{2}\sin(t), \quad z = -\sqrt{2}\sin(t).$$

(b) (5 points) Choose your favorite point on C (any point will do) and write parametric equations for the tangent line to C at that point.

Solution: Taking $t = 0$, we get the point $\langle 2, 0, 0\rangle$ on C.
The derivative at this point is

$$\left\langle -2\sin(t), \sqrt{2}\cos(t), -\sqrt{2}\cos(t)\right\rangle_{t=0} = \left\langle 0, \sqrt{2}, -\sqrt{2}\right\rangle.$$

The tangent line is the line through $\langle 2, 0, 0\rangle$ in the direction of $\langle 0, 1, -1\rangle$ (which is the same as the direction of $\langle 0, \sqrt{2}, \sqrt{-2}\rangle$) Hence the parametric equations of the tangent line are

$$x = 2, y = t, z = -t.$$

3. (10 points) For which positive real number a does the curve $y^2 = x^2 + a^2$ have curvature 2 at the point $(0, a)$?

Solution: We can choose the parameterization $\langle t, \sqrt{a^2+t^2}\rangle$. This will only trace the $y > 0$ part of the curve, but that is sufficient since $(0, a)$ is on this part.

We now compute the curvature at $t = 0$.

$$\begin{aligned} r(t) &= \left\langle t, \sqrt{a^2+t^2} \right\rangle \\ r'(t) &= \left\langle 1, \frac{t}{\sqrt{a^2+t^2}} \right\rangle \\ r''(t) &= \left\langle 0, \frac{1}{\sqrt{a^2+t^2}} - \frac{t^2}{(a^2+t^2)^{3/2}} \right\rangle \\ &= \left\langle 0, \frac{a^2}{(a^2+t^2)^{3/2}} \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} r'(0) &= \langle 1, 0\rangle = \mathbf{i} \\ r''(0) &= \langle 0, 1/a\rangle = (1/a)\mathbf{j}. \end{aligned}$$

Using

$$\kappa = \frac{|r' \times r''|}{|r'|^3},$$

we get

$$\begin{aligned} \kappa &= \frac{|\mathbf{i} \times (1/a)\mathbf{j}|}{|\mathbf{i}|^3} \\ &= \frac{1}{a}. \end{aligned}$$

For $\kappa = 2$, we must have $a = 1/2$.

4. The force acting on an object of mass 2 units is given by the vector

$$\overrightarrow{F}(t) = \langle 0, 16\cos(2t), 16\sin(2t)\rangle.$$

At $t = 0$, the object is at $\langle 0,0,0\rangle$ and is travelling with velocity $\langle 3, 0, -4\rangle$.

(a) (5 points) How much distance does it travel between $t = 0$ and $t = 10$?

Solution: Using $\overrightarrow{F} = m\overrightarrow{a}$, we get that the acceleration is

$$\overrightarrow{a}(t) = \langle 0, 8\cos(2t), 8\sin(2t)\rangle.$$

Integrating, we get the velocity

$$\overrightarrow{v}(t) = \langle 0, 4\sin(2t), -4\cos(2t)\rangle + \overrightarrow{c}.$$

Since $\overrightarrow{v}(0) = \langle 3, 0, -4\rangle$, we get $\overrightarrow{c} = \langle 3, 0, 0\rangle$. Hence

$$\overrightarrow{v}(t) = \langle 3, 4\sin(2t), -4\cos(2t)\rangle.$$

At this point, we can compute the position, but we don't need to. The speed is

$$|\overrightarrow{v}(t)| = \sqrt{3^2 + 4^2} = 5.$$

Hence in 10 seconds, the object travels 50 units.

(b) (5 points) Write an equation of the normal plane to its motion at $t = \pi$.

Solution: We must calculate the position. Integrating $\overrightarrow{v}(t)$, we get

$$\overrightarrow{r}(t) = \langle 3t, -2\cos(2t), -2\sin(2t)\rangle + \overrightarrow{c}.$$

Since $\overrightarrow{r}(0) = \langle 0,0,0\rangle$, we have $\overrightarrow{c} = \langle 0, 2, 0\rangle$. Hence

$$\overrightarrow{r}(t) = \langle 3t, 2 - 2\cos(2t), -2\sin(2t)\rangle.$$

At $t = \pi$, we have

$$\overrightarrow{r}(\pi) = \langle 3\pi, 0, 0\rangle$$
$$\overrightarrow{r}'(\pi) = \overrightarrow{v}(\pi) = \langle 3, 0, -4\rangle.$$

The normal plane is the plane through $\langle 3\pi, 0, 0\rangle$ and perperdicular to $\langle 3, 0, -4\rangle$. The equation is

$$3(x - 3\pi) - 4z = 0$$
$$\text{that is: } 3x - 4z = 9\pi.$$

5. (10 points) Let $\overrightarrow{r}(t) = \left\langle 2t, t^2, t^3/3 \right\rangle$. Find the unit tangent vector, unit normal vector, and the unit binormal vector to the curve at $t = 0$.

Solution: We have

$$\begin{aligned} \overrightarrow{r}(t) &= \langle 2t, t^2, t^3/3 \rangle \\ \overrightarrow{r}'(t) &= \langle 2, 2t, t^2 \rangle \\ |\overrightarrow{r}'(t)| &= \sqrt{4 + 4t^2 + t^4} = (t^2 + 2). \end{aligned}$$

So the unit tangent vector is

$$\overrightarrow{T} = \frac{1}{t^2 + 2} \langle 2, 2t, t^2 \rangle.$$

Differentiating, we get

$$\overrightarrow{T}' = \frac{-2t}{(t^2 + 2)^2} \langle 2, 2t, t^2 \rangle + \frac{1}{t^2 + 2} \langle 0, 2, 2t \rangle.$$

Since we are only interested at $t = 0$, we get

$$\begin{aligned} \overrightarrow{T}(0) &= \frac{1}{2} \langle 2, 0, 0 \rangle = \langle 1, 0, 0 \rangle \\ \overrightarrow{T}'(0) &= \frac{1}{2} \langle 0, 2, 0 \rangle = \langle 0, 1, 0 \rangle. \end{aligned}$$

Since $\overrightarrow{T}'(0)$ is already a unit vector, this *is* the unit normal vector. In other words,

$$T = \mathbf{i}, N = \mathbf{j}, \text{ and hence } B = T \times N = \mathbf{k}.$$

LIST OF USEFUL IDENTITIES

1. Derivatives

(1) $\frac{d}{dx}x^n = nx^{n-1}$

(2) $\frac{d}{dx}\sin x = \cos x$

(3) $\frac{d}{dx}\cos x = -\sin x$

(4) $\frac{d}{dx}\tan x = \sec^2 x$

(5) $\frac{d}{dx}\cot x = -\csc^2 x$

(6) $\frac{d}{dx}\sec x = \sec x \tan x$

(7) $\frac{d}{dx}\csc x = -\csc x \cot x$

(8) $\frac{d}{dx}e^x = e^x$

(9) $\frac{d}{dx}\ln|x| = \frac{1}{x}$

(10) $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

(11) $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$

(12) $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

2. Trigonometry

(1) $\sin^2 x + \cos^2 x = 1$

(2) $\tan^2 x + 1 = \sec^2 x$

(3) $1 + \cot^2 x = \csc^2 x$

(4) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(5) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(6) $\sin^2 x = \frac{1-\cos 2x}{2}$

(7) $\cos^2 x = \frac{1+\cos 2x}{2}$.

3. Space curves

For a parametric space curve given by $\overline{r}(t)$

(1) Curvature $\quad \kappa = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$.

(2) Tangent component of acceleration $\quad a_T = |r'(t)|' = \dfrac{r'(t) \cdot r''(t)}{|r'(t)|}$.

(3) Normal component of acceleration $\quad a_N = \kappa |r'(t)|^2 = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|}$.