Midterm 2 Sample

Math UN1101: Calculus III, Section 2

Spring 2019

Instructor: Linh Truong

Name: ______________________________

Instructions:

- Print your name in the space above.
- Show your reasoning and intermediate computations.
- You have 75 minutes.
- No notes, books, calculators or any other electronic devices are allowed.
- Write answers in the space provided. If you need extra space, use the backs of pages.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. Determine whether or not the following limits exist. If it exists, evaluate the limit. Justify your answers.

   (a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{(x^2+y^2)^{3/2}}$$

Solution:

$$\text{along } x=0 \quad \Rightarrow \quad \lim_{y\to 0} \frac{y^3}{y^3} = 1$$

$$\text{along } y=x \quad \Rightarrow \quad \lim_{y\to 0} \frac{2y^3}{(2y^2)^{3/2}} = \frac{1}{\sqrt{2}}$$

Thus, the limit does NOT exist.

   (b) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x(x^2+y^4)^2}{2x^2+y^4}$$

Solution:

$$0 \le \left|\frac{x(x^2+y^4)^2}{2x^2+y^4}\right| \le \left|\frac{x(x^2+y^4)^2}{x^2+y^4}\right|$$

$$0 \le \left|\frac{x(x^2+y^4)^2}{2x^2+y^4}\right| \le |x(x^2+y^4)|$$

Clearly, $\lim_{(x,y)\to(0,0)} |x(x^2+y^4)| = 0$.

Thus, the SQUEEZE theorem implies that $\lim_{(x,y)\to(0,0)} \frac{x(x^2+y^4)^2}{2x^2+y^4} = 0$.

2. (a) (5 points) Find a vector function that represents the curve of intersection of $4x^2 + 3y^2 + 2z^2 = 16$ and the plane $y = 2$.

Solution:

$$y = 2 \quad \Longrightarrow \quad 4x^2 + 2z^2 = 16 - 12 = 4 \quad \Longrightarrow \quad x^2 + \frac{z^2}{2} = 1$$

$$x = \cos(t) \quad z = \sqrt{2}\sin(t) \quad \Longrightarrow \quad \vec{r}(t) = \langle \cos(t), 2, \sqrt{2}\sin(t) \rangle$$

(b) (5 points) Find the parametric equations of the tangent line to this curve at the point $(1, 2, 0)$.

Solution: The curve intersects the point $(1, 2, 0)$ at $t = 0$.

$$\vec{r}'(t) = \langle -\sin(t), 0, \sqrt{2}\cos(t) \rangle \quad \Rightarrow \quad \vec{r}'(0) = \langle 0, 0, \sqrt{2} \rangle$$

$$x = 1, \ y = 2, \ z = \sqrt{2}t$$

3. Write true or false. Justify your answer.

(a) (2 points) The curve parametrized by $\langle \sin(2t),\ \cos(3t),\ 1+t^3 \rangle$ never intersects the xy–plane.

Solution: False.

(b) (2 points) The curve parametrized by $\langle \sin(2t),\ t,\ \cos(2t) \rangle$ traces a helix.

Solution: True.

(c) (2 points) The graph of $f(x,y) = x^2 + y^2$ is a hemisphere.

Solution: False.

(d) (2 points) The curve parametrized by $\langle \sin(2t),\ \cos(2t),\ \sin(t) \rangle$ lies on the surface $x^2 + y^2 = 1$.

Solution: True.

(e) (2 points) For two vector functions $\vec{r}_1(t)$ and $\vec{r}_2(t)$ we have

$$\frac{d(\vec{r}_1(t) \times \vec{r}_2(t))}{dt} = \vec{r}_1'(t) \times \vec{r}_2'(t)$$

Solution: False.

4. Suppose $\vec{\mathbf{r}}(t) = \langle \sqrt{2}\cos(t),\ t - \sin(t),\ t + \sin(t) \rangle$.

(a) (5 points) Find the unit tangent vector and evaluate at $t = \pi/2$.

Solution: The tangent vector to the curve is

$$\vec{\mathbf{r}}'(t) = \langle -\sqrt{2}\sin(t),\ 1 - \cos(t),\ 1 + \cos(t) \rangle$$

and its magnitude is

$$\begin{aligned} |\vec{\mathbf{r}}'(t)| &= \sqrt{2\sin^2(t) + (1 - \cos(t))^2 + (1 + \cos(t))^2} \\ &= \sqrt{2\sin^2(t) + 2\cos^2(t) + 2} \\ &= \sqrt{2 + 2} = 2. \end{aligned}$$

The unit tangent vector is

$$\frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|} = \langle -\frac{\sqrt{2}}{2}\sin(t),\ \frac{1 - \cos(t)}{2},\ \frac{1 + \cos(t)}{2} \rangle$$

At $t = \pi/2$, the unit tangent vector is $\langle -\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2} \rangle$.

(b) (5 points) Find the equation of a surface that the curve lies on.

Solution: We have

$$y - z = -2\sin(t)$$

So $(y - z)^2 = 4\sin^2(t)$. Also, notice $x^2 = 2\cos^2(t)$. So

$$2x^2 + (y - z)^2 = 4\cos^2(t) + 4\sin^2(t) = 4.$$

The curve lies on the surface $2x^2 + (y - z)^2 = 4$.

Another possible answer for this question is:

$$x = \sqrt{2}\cos(\frac{y + z}{2}).$$

5. (a) (5 points) Suppose that z is defined implicitly as a function of x and y by the equation

$$ze^z = x^2 + y^2.$$

Find $\partial z/\partial x$.

(Update: A previous version of this question also asked for $\frac{\partial^2 z}{\partial x \partial y}$. Please disregard that part of the question).

Solution: To find $\partial z/\partial x$, we treat y as a constant and take the derivative of both sides of the equation with respect to x, using implicit differentiation.

$$\frac{\partial z}{\partial x} e^z + ze^z \frac{\partial z}{\partial x} = 2x$$

After rearranging this equation, we see

$$\frac{\partial z}{\partial x} = \frac{2x}{e^z + ze^z}.$$

(b) (5 points) Suppose $f(x, y, z) = \cos(x^2 y^3 z) + \arcsin(3x + y)$.
Find $\partial f/\partial z$ and f_{yz}.

Solution: To find $\partial f/\partial z$, treat x and y as constants and take the derivative with respect to z.

$$\frac{\partial f}{\partial z} = -\sin(x^2 y^3 z) \cdot x^2 y^3$$

To find f_{yz}, we will use the fact that $f_{yz} = f_{zy}$.

$$f_{yz} = (f_z)_y = \frac{\partial f_z}{\partial y} = -\cos(x^2 y^3 z) \cdot 3x^2 y^2 z \cdot x^2 y^3 - \sin(x^2 y^3 z) \cdot 3x^2 y^2$$

6. Match each of the following functions with its contour map.

(a) (4 points) $f(x, y) = \tan(x + y)$

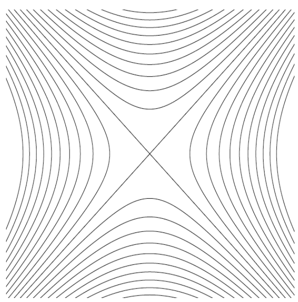
> **Solution:** Contour Map II

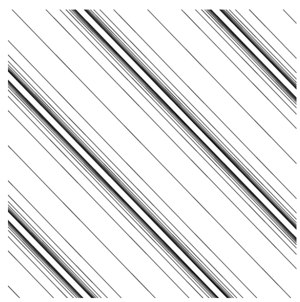
(b) (3 points) $g(x, y) = y^2 + x^2$

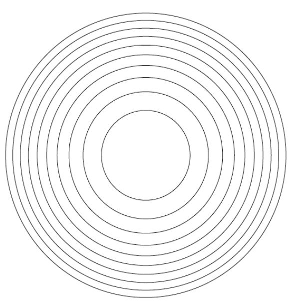
> **Solution:** Contour Map III

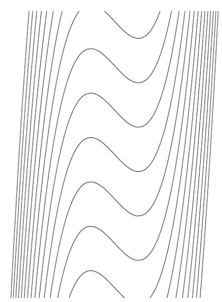
(c) (3 points) $h(x, y) = y^2 - x^2$

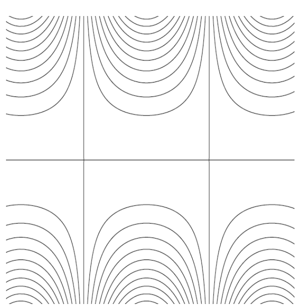
> **Solution:** Contour Map I

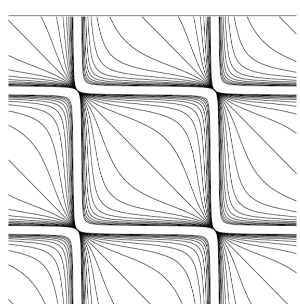

I.


II.


III.


IV.


V.


VI.