

Problem 1. $\vec{r}(t) = \left\langle 9t, (2t)^{\frac{3}{2}}, \frac{t^2}{2} \right\rangle \quad 0 \leq t \leq 2$

$$\vec{r}'(t) = \left\langle 9, \frac{3}{2}(2t)^{\frac{1}{2}} \cdot 2, \frac{2t}{2} \right\rangle$$

$$= \left\langle 9, 3(2t)^{\frac{1}{2}}, t \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{81 + 18t + t^2} = \sqrt{(9+t)^2}$$

$$= |9+t| = 9+t \quad \text{for } 0 \leq t \leq 2.$$

$$\text{Length of the curve} = \int_0^2 (9+t) dt$$

$$= \left[9t + \frac{t^2}{2} \right]_0^2 = 9(2) + \frac{2^2}{2}$$

$$= 18 + 2 = 20.$$

Problem 2. $\vec{r}(t) = \langle t, 2t, t^2 \rangle$

$$\vec{r}'(t) = \langle 1, 2, 2t \rangle$$

$$\vec{r}''(t) = \langle 0, 0, 2 \rangle$$

$$|\vec{r}'(t)| = \sqrt{1 + 4 + 4t^2}$$

$$= \sqrt{5 + 4t^2}$$

$$a_T(t) = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$$

Tangential component
of acceleration

$$= \frac{4t}{\sqrt{5+4t^2}}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix} = \langle 4, -2, 0 \rangle$$

$$\text{Curvature } \mathcal{K}(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\sqrt{20}}{(5+4t^2)^{3/2}}$$

$$\mathcal{K}(2) = \frac{\sqrt{20}}{(8)^{3/2}} = \frac{\sqrt{20}}{(21)^{3/2}}$$

$$a_N(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|} = \frac{\sqrt{20}}{(5+4t^2)^{1/2}}$$

Final answers: $a_T(t) = \frac{4t}{\sqrt{5+4t^2}}$

$$a_N(t) = \frac{\sqrt{20}}{\sqrt{5+4t^2}}$$

$$\mathcal{K}(2) = \frac{\sqrt{20}}{(21)^{3/2}}$$

Problem 3.

$$f(x, y) = \frac{y}{x^2 + y^2}$$

(3)

Level 0: $\frac{y}{x^2 + y^2} = 0 \Rightarrow y = 0$

Level 1: $\frac{y}{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 - y = 0$
 $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

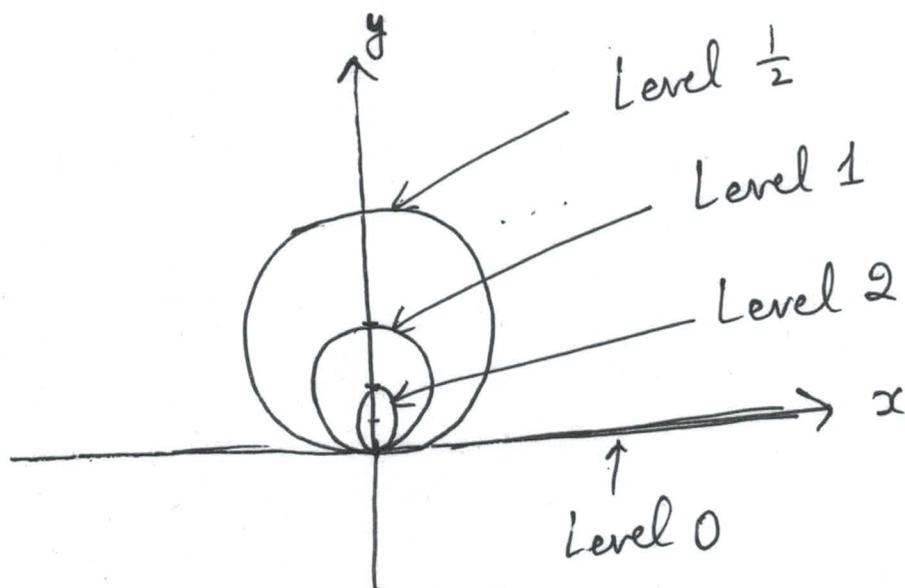
Circle of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$

Level 2: $\frac{y}{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 - \frac{y}{2} = 0$

$$x^2 + (y - \frac{1}{4})^2 = \frac{1}{16}$$

Circle of radius $\frac{1}{4}$ centered at $(0, \frac{1}{4})$

Level $\frac{1}{2}$: $\frac{y}{x^2 + y^2} = \frac{1}{2} \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y - 1)^2 = 1$
Circle of radius 1 centered at $(0, 1)$



Contour map of $f(x, y)$.

Problem 4.

(a) $\frac{d}{dt} |\vec{r}(t)| = |\vec{r}'(t)|$ FALSE

Example. Let $\vec{r}(t) = \langle \cos t, \sin t \rangle$. Then

$$\left. \begin{aligned} |\vec{r}(t)| = 1 &\Rightarrow \frac{d}{dt} |\vec{r}(t)| = 0 \\ \vec{r}'(t) = \langle -\sin t, \cos t \rangle &\Rightarrow |\vec{r}'(t)| = 1 \end{aligned} \right\} \text{NOT equal.}$$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin(y)}{x^2 + y^2} = 0$ TRUE

Proof. $0 \leq x^2 \leq x^2 + y^2 \Rightarrow 0 \leq \frac{x^2}{x^2 + y^2} \leq 1$

$$\Rightarrow -|\sin(y)| \leq \frac{x^2 \sin(y)}{x^2 + y^2} \leq |\sin(y)|$$

As $y \rightarrow 0$ both $|\sin(y)|$ and $-|\sin(y)|$ tend to 0

By squeeze theorem $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin(y)}{x^2 + y^2} = 0$

(c) There is a function $f(x,y)$ such that $f_x = x + y^2$ and $f_y = x - y^2$ FALSE

If there were such a function f_{xy} will be equal to f_{yx} . But

$$f_{xy} = \frac{\partial}{\partial y} (x + y^2) = 2y \quad \text{and} \quad f_{yx} = \frac{\partial}{\partial x} (x - y^2) = 1$$

$$(d) \lim_{(x,y) \rightarrow (0,0)} \sqrt{x^2+y^2} \ln(x^2+y^2) = 0 \quad \boxed{\text{TRUE}} \quad (5)$$

Set $x = r \cos \theta$
 $y = r \sin \theta$. Then the limit in question is:

$$\lim_{r \rightarrow 0^+} r \ln(r^2) = \lim_{r \rightarrow 0^+} \frac{2 \ln(r)}{r^{-1}} \quad \left(\frac{\infty}{\infty} \text{ form} \right)$$

$$\frac{\text{L'Hospital rule}}{\lim_{r \rightarrow 0^+} \frac{2 r^{-1}}{-r^{-2}}} = \lim_{r \rightarrow 0^+} (-2r) = 0.$$

Problem 5. $f(x, y, z) = \sin^{-1}(xz) + x^2 e^{yz}$

To compute f_{xyz} . By Clairaut's theorem, it is same as

f_{yxz} .

$$f_y = \frac{\partial}{\partial y} (\sin^{-1}(xz)) + \frac{\partial}{\partial y} (x^2 e^{yz}) = 0 + x^2 e^{yz} \cdot z$$

$$f_{yx} = 2xz e^{yz}$$

$$f_{yxz} = 2xz e^{yz} \cdot y = \boxed{2xyz e^{yz}}$$

Problem 6. $\vec{r}_1(t) = \langle \cos(t), \sin(t), t \rangle$

⑥

$$\vec{r}_2(s) = \langle 1+s, s, \frac{\pi}{2}s \rangle$$

These curves intersect for t, s such that

$$\cos(t) = 1+s \text{ and } \sin(t) = s \text{ and } t = \frac{\pi}{2}s$$

Since $\cos^2(t) + \sin^2(t) = 1$ we get

$$(1+s)^2 + s^2 = 1 \Rightarrow 1 + 2s + 2s^2 = 1$$

$$\Rightarrow 2s(s+1) = 0 \Rightarrow s = 0 \text{ or } s = -1$$

$s = 0 \Rightarrow t = \frac{\pi}{2}(0) = 0$. Check: $\cos(0) = 1 = 1+0 \checkmark$
 $\sin(0) = 0 \checkmark$
 $0 = \frac{\pi}{2}(0) \checkmark$

$$s = -1 \Rightarrow t = \frac{\pi}{2}(-1) = -\frac{\pi}{2}$$

Check: $\cos\left(-\frac{\pi}{2}\right) = 0 = 1+(-1) \checkmark$

$$\sin\left(-\frac{\pi}{2}\right) = -1 \checkmark$$

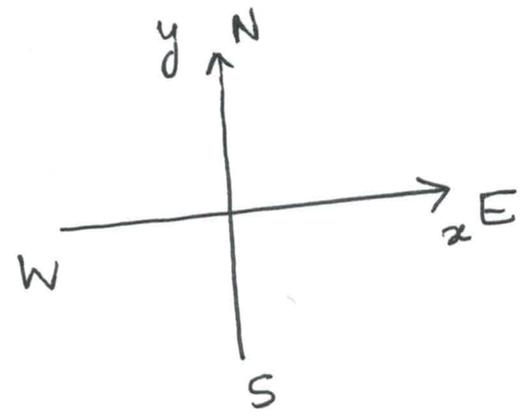
$$-\frac{\pi}{2} = \frac{\pi}{2}(-1) \checkmark$$

Points of intersection:

$$(1, 0, 0) \text{ and } \left(0, -1, -\frac{\pi}{2}\right)$$

Problem 7. Let us draw coordinates so that

- East = +ve x-axis
- North = +ve y-axis
- z-axis points upwards



Acceleration $\vec{a}(t) =$

Eastward force = 4 N.

$$\text{acceleration} = \frac{\text{Force}}{\text{mass}} = \frac{4}{0.8} \text{ m/s}^2 = 5 \text{ m/s}^2$$

Downwards (due to gravity) = $g \text{ m/s}^2 \approx 10 \text{ m/s}^2$

$$\vec{a}(t) = \langle 5, 0, -10 \rangle$$

$$\begin{aligned} \text{Initial velocity } \vec{v}(0) &= \langle 0, -30 \cos(30^\circ), 30 \sin(30^\circ) \rangle \\ &= \langle 0, -15\sqrt{3}, 15 \rangle \end{aligned}$$

$$\text{Initial position } \vec{r}(0) = \langle 0, 0, 0 \rangle$$

$$\vec{v}'(t) = \vec{a}(t) = \langle 5, 0, -10 \rangle$$

$$\Rightarrow \vec{v}(t) = \langle 5t, 0, -10t \rangle + \vec{C}$$

$$\vec{C} = \vec{v}(0) = \langle 0, -15\sqrt{3}, 15 \rangle$$

$$\begin{aligned} \text{Therefore } \vec{v}(t) &= \langle 5t, -15\sqrt{3}, 15-10t \rangle \\ &= \vec{r}'(t) \end{aligned}$$

$$\vec{r}(t) = \left\langle 5 \frac{t^2}{2}, -15\sqrt{3}t, 15t - 10 \frac{t^2}{2} \right\rangle + \vec{D} \quad (8)$$

$$\vec{D} = \vec{r}(0) = \langle 0, 0, 0 \rangle$$

Therefore $\vec{r}(t) = \left\langle 5 \frac{t^2}{2}, -15\sqrt{3}t, 15t - 10 \frac{t^2}{2} \right\rangle$.

The object is on ground for t such that $15t - 5t^2 = 0$

$$\Rightarrow t = 0 \text{ or } t = 3.$$

At $t = 3$ x -coord = $\frac{45}{2}$ y -coord = $-45\sqrt{3}$.

Hence it hits the ground $\frac{45}{2}$ m east and $45\sqrt{3}$ south of its original position.

Problem 8. $z^2 + x^2y^2 + \cos(z) = 9$.

Take derivative with respect to x to get

$$2z \frac{\partial z}{\partial x} + 2xy^2 - \sin(z) \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-2xy^2}{2z - \sin(z)}$$

Similarly $\frac{\partial}{\partial y}$ gives: $2z \frac{\partial z}{\partial y} + 2x^2y - \sin(z) \frac{\partial z}{\partial y} = 0$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-2x^2y}{2z - \sin(z)}$$