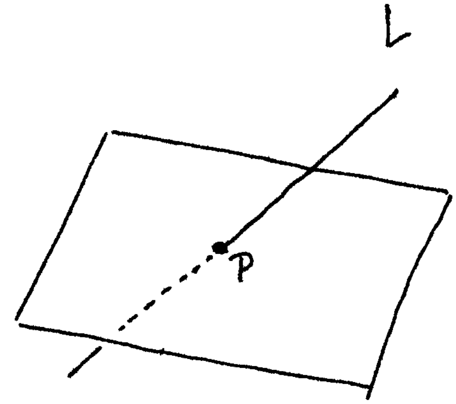
Solution to Practice Mid Term 1

1. Since the line lies on the plane $x - 2y + z = 4$ and is perpendicular to



$L: \vec{r} = \langle 1+t, 1-t, 2t \rangle$,

$\mathbb{P}: x - 2y + z = 4$

its direction vector is orthogonal to both $\langle 1,-2,1 \rangle$ and $\langle 1,-1,2 \rangle$

$$\text{Direction Vector} = \langle 1,-2,1 \rangle \times \langle 1,-1,2 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = -3\hat{i} - \hat{j} + \hat{k}$$

Let P be the point of intersection of L and $\mathbb{P}$.

$P = (1+t, 1-t, 2t)$ such that $(1+t) - 2(1-t) + 2t = 4$

$$\equiv -1 + 5t = 4$$

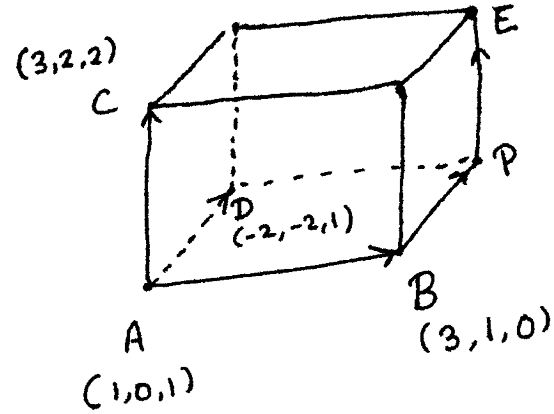
$$\equiv \boxed{t = 1}$$

$\Rightarrow P = (2, 0, 2)$

Parametric equations of line through $(2,0,2)$ parallel to $\langle -3,-1,1 \rangle$:

$$\boxed{\begin{aligned} x &= 2 - 3t \\ y &= -t \\ z &= 2 + t \end{aligned}}$$

2.



$\overrightarrow{AB} = \langle 2,1,-1\rangle$

$\overrightarrow{AC} = \langle 2,2,1\rangle$

$\overrightarrow{AD} = \langle -3,-2,0\rangle$

(a) $$\text{Volume} = \left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right| = \begin{vmatrix} 2 & 1 & -1 \\ 2 & 2 & 1 \\ -3 & -2 & 0 \end{vmatrix}$$

$$= \left|2(0-(-2)) - 1(0-(-3)) - 1(-2-(-6))\right|$$

$$= |4-3-4| = 3.$$

(b) $$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BP} + \overrightarrow{PE} = \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AC}$$

$$= \langle 1,1,0\rangle$$

$$\Rightarrow \quad E = (2,1,1)$$

(c) $$\cos(\angle EAB) = \frac{\overrightarrow{AE}\cdot\overrightarrow{AB}}{|\overrightarrow{AE}|\cdot|\overrightarrow{AB}|} = \frac{\langle 1,1,0\rangle\cdot\langle 2,1,-1\rangle}{\sqrt{1^2+1^2+0^2}\,\sqrt{2^2+1^2+(-1)^2}}$$

$$= \frac{3}{\sqrt{2}\,\sqrt{6}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \angle EAB = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

3. $\mathbb{P}: \quad x + y - 3z = 1 \qquad P = (2, 1, 0)$

(a) Line perpendicular to $\mathbb{P}$ must be along $\langle 1, 1, -3 \rangle$.

$$x = 2 + t, \quad y = 1 + t \quad z = -3t$$

(b) Q = point of intersection of this line and $\mathbb{P}$:

$$(2+t) + (1+t) - 3(-3t) = 1$$

$$\equiv \quad 3 + 11t = 1 \quad \equiv \quad \boxed{t = \frac{-2}{11}}$$

$$Q = \left(2 - \frac{2}{11},\ 1 - \frac{2}{11},\ -3\left(\frac{-2}{11}\right)\right) = \left(\frac{20}{11}, \frac{9}{11}, \frac{6}{11}\right)$$

(c) Distance between P and Q =

$$\sqrt{\left(2 - \frac{20}{11}\right)^2 + \left(1 - \frac{9}{11}\right)^2 + \left(0 - \frac{6}{11}\right)^2} = \sqrt{\frac{2^2 + 2^2 + (-6)^2}{(11)^2}}$$

$$= \frac{\sqrt{44}}{11} = \frac{2}{\sqrt{11}}.$$

(d) Distance between $(2, 1, 0)$ and $x + y - 3z - 1 = 0$:

$$\frac{|2 + 1 - 3(0) - 1|}{\sqrt{1^2 + 1^2 + (-3)^2}} = \frac{2}{\sqrt{11}}.$$

4. (a) False.

Counterexample. Let $\vec{v} = \langle 1, 0, 0 \rangle$ $\vec{w} = \langle 0, 1, 0 \rangle$.

Then $\vec{v} \cdot \vec{w} = 0$ but neither $\vec{v}$ nor $\vec{w}$ is $\vec{0}$.

(b) True.

Proof. $|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\phi = 0$

$\equiv$ $\vec{v} = 0$ or $\vec{w} = 0$ or $\phi = 0, \pi$

Either case $\vec{v}$ and $\vec{w}$ are parallel.

(c) False.

A pair of lines could be skew. e.g.

L_1: $x = 1 + 2t$, $y = -2 + 6t$, $z = 4 - 2t$

L_2: $x = 2s$, $y = 3 + s$, $z = -3 + 4s$

are not parallel (since $\langle 2, 6, -2 \rangle$ and $\langle 2, 1, 4 \rangle$ are not scalar multiple of each other)

and do not intersect:

$$\begin{aligned} 1 + 2t &= 2s \\ -2 + 6t &= s + 3 \end{aligned} \Rightarrow t = \frac{11}{10}, \quad s = \frac{16}{10} \text{ does not satisfy}$$

$$4 - 2t = -3 + 4s.$$

(d) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\quad \vec{b} = \langle b_1, b_2, b_3 \rangle$ $\quad \vec{c} = \langle c_1, c_2, c_3 \rangle$

$$\vec{b} \times \vec{c} = \langle b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3,\ b_1c_2 - b_2c_1 \rangle$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

Similarly $\vec{a} \times \vec{b} = \langle a_2b_3 - b_3a_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$

and $(\vec{a} \times \vec{b}) \cdot \vec{c} = (a_2b_3 - b_2a_3)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$

$$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) \qquad \text{TRUE.}$$

(e) False.

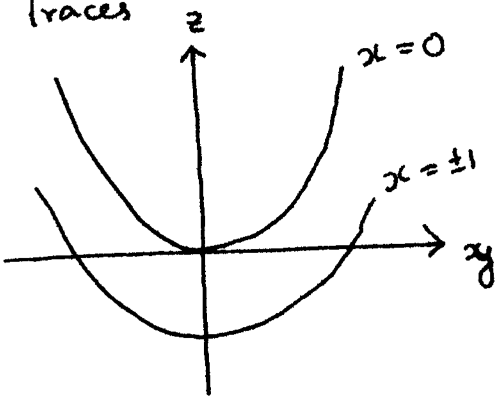
eg. let $\vec{a} = \langle 1, 0, 0 \rangle$ $\quad \vec{b} = \langle -1, 0, 0 \rangle$

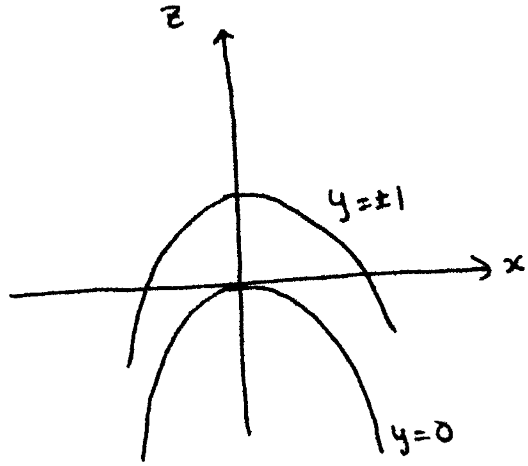
$|\vec{a} + \vec{b}| = 0$ and $|\vec{a}| + |\vec{b}| = 2$

5. $z = ax^2 + y^2$

(a) $a = -1$: $z = -x^2 + y^2$

Traces



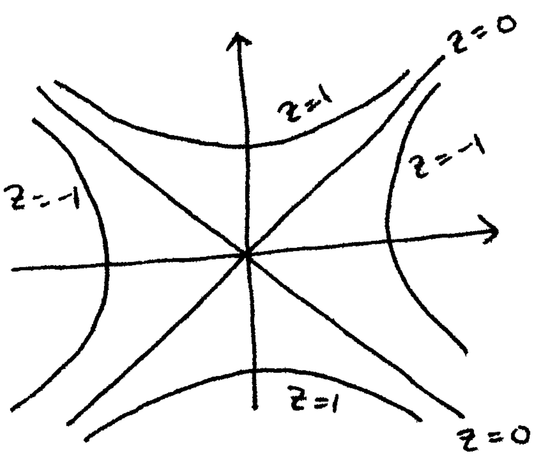


x-traces. $x = 0$: $z = y^2$

$x = \pm 1$: $z = y^2 - 1$

y-traces. $y = 0$: $z = -x^2$

$y = \pm 1$: $z = -x^2 + 1$

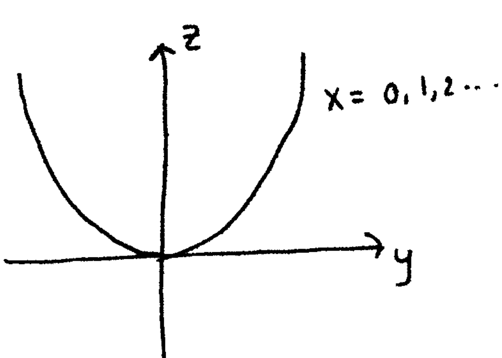


z-traces : $z = 0$ $\quad y^2 - x^2 = 0$

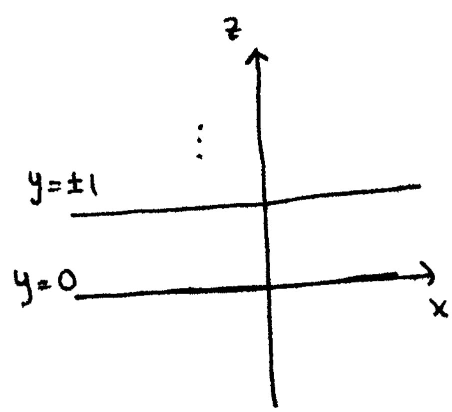
$z = +1$ $\quad y^2 - x^2 = 1$

$z = -1$ $\quad x^2 - y^2 = 1$

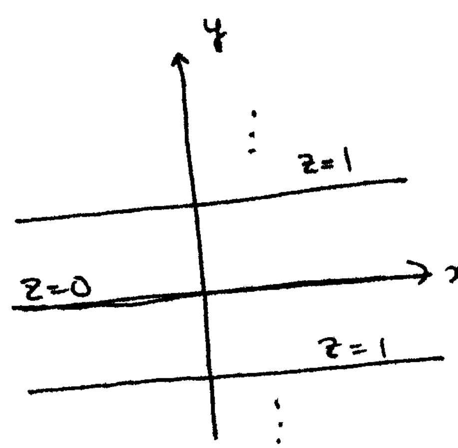
$a = 0$: $z = y^2$



x-traces

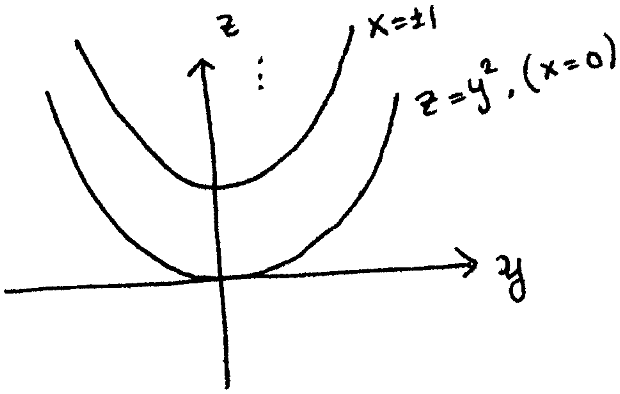


y-traces

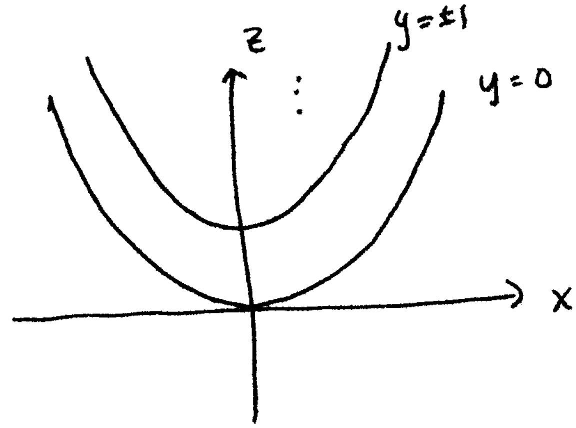


z-traces

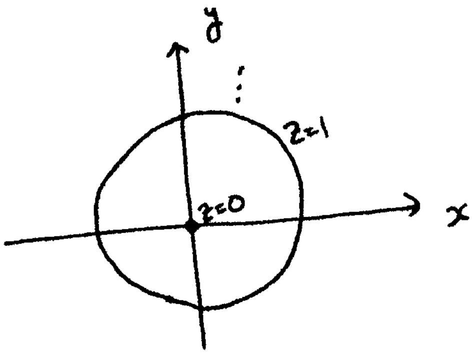
$a = 1.$ $\qquad z = x^2 + y^2$



x-traces

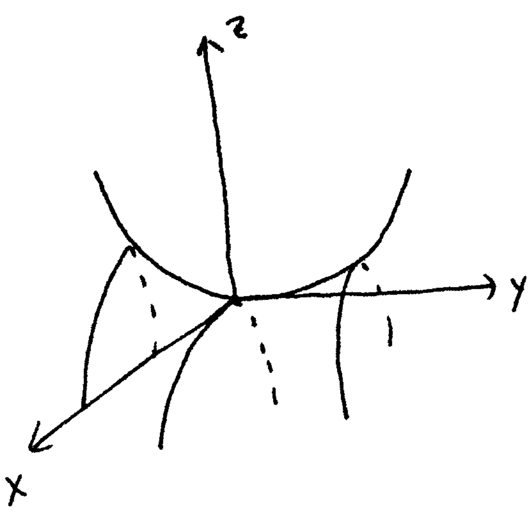


y-traces

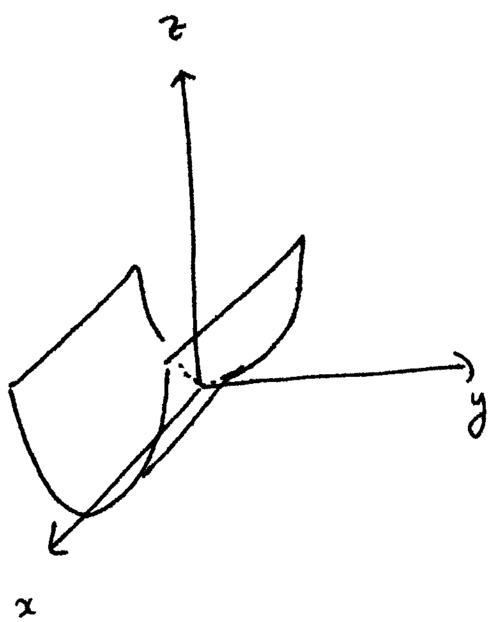


z-traces

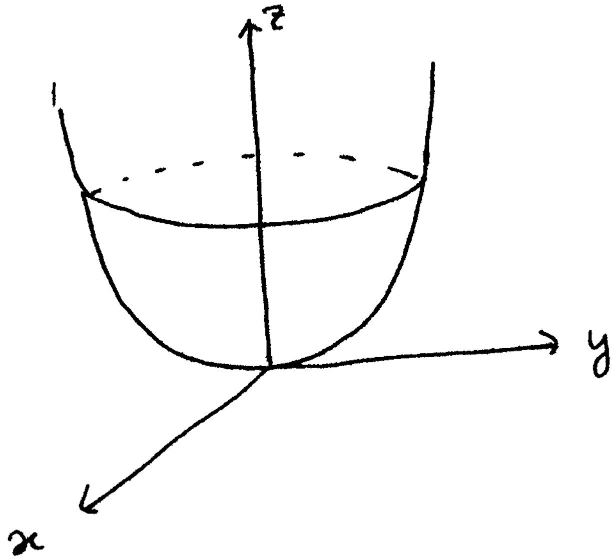
(b)



$a = -1$



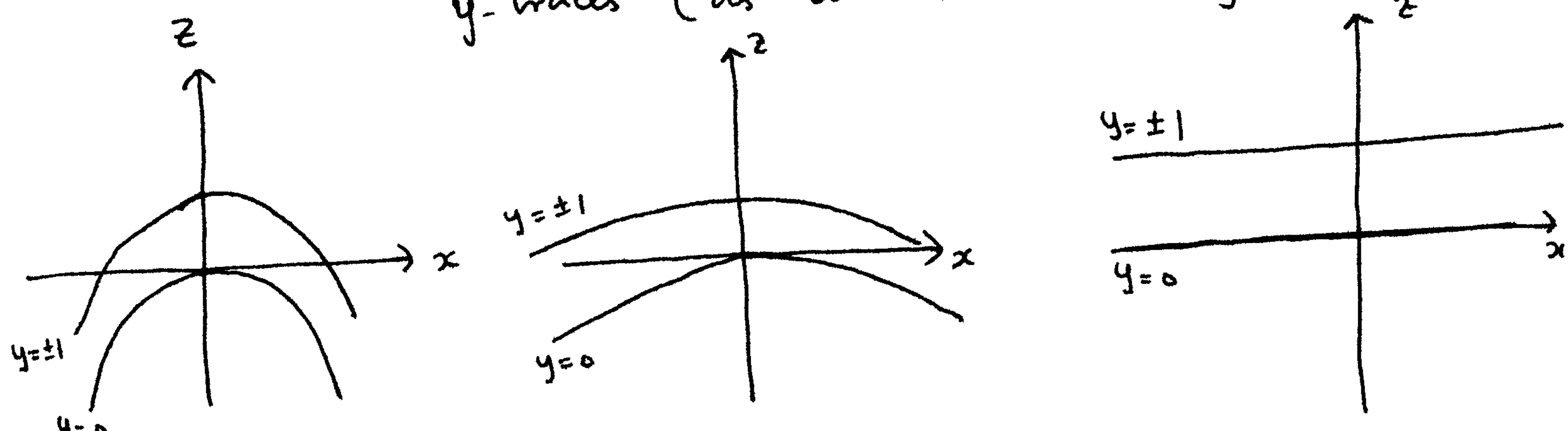
$a = 0$



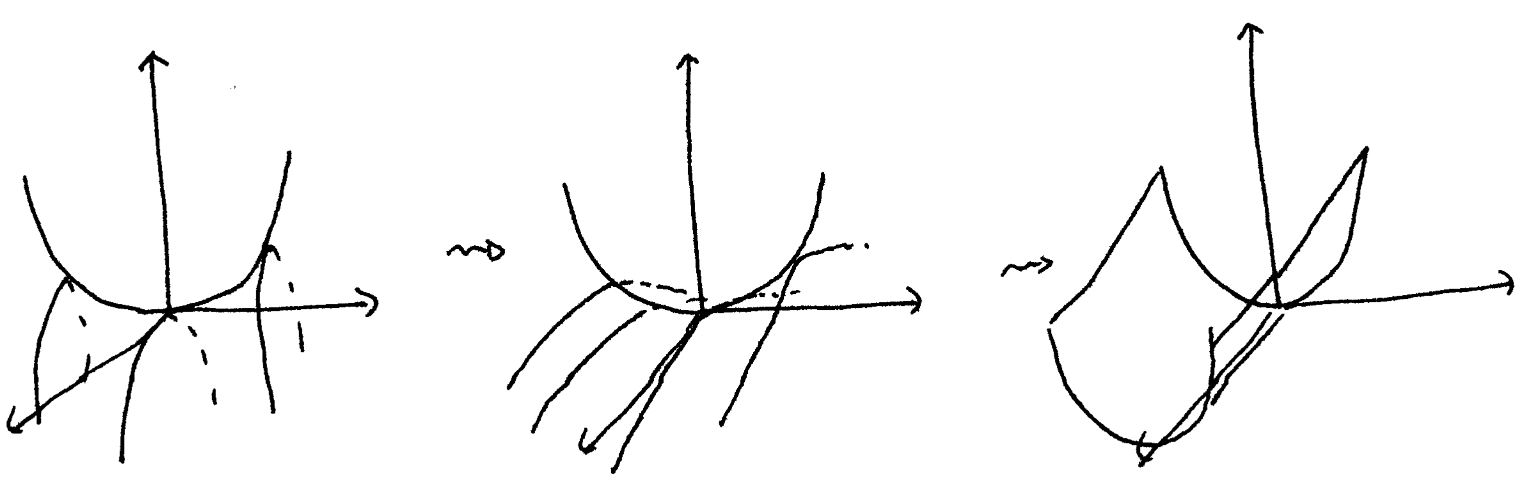
$a = 1$

(c) $a \to 0$ from left: y-traces (parabolas) become wider and wider until they become straight lines

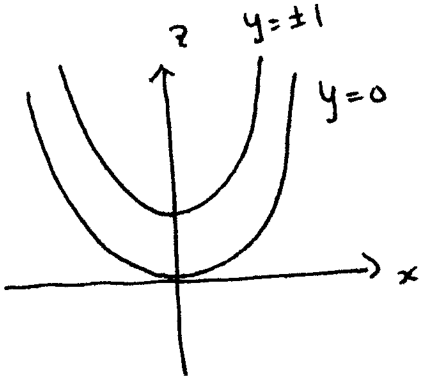
y-traces (as $a \to 0$ while being negative)



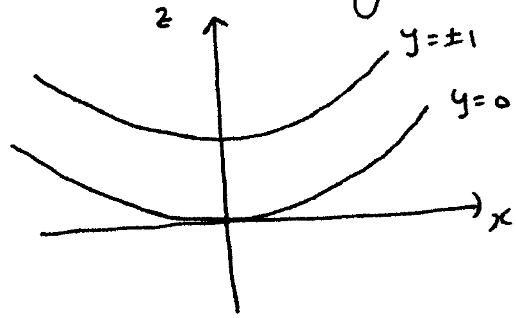
$a = -1 \rightsquigarrow a = -0.1 \rightsquigarrow a = 0$



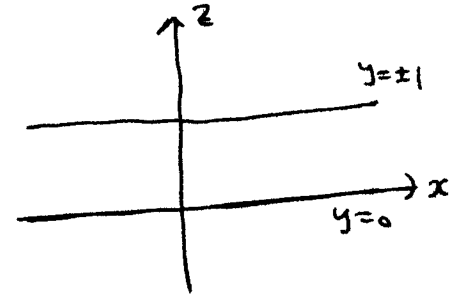
$a \to 0$ from the right: again y-traces become wider until they become lines



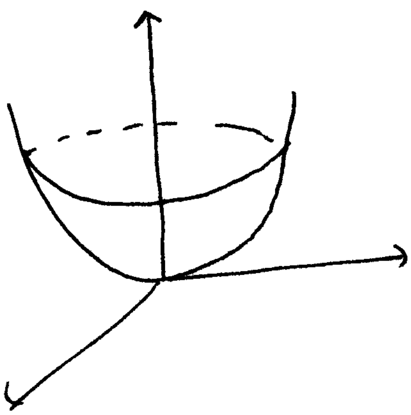
$a = 1$

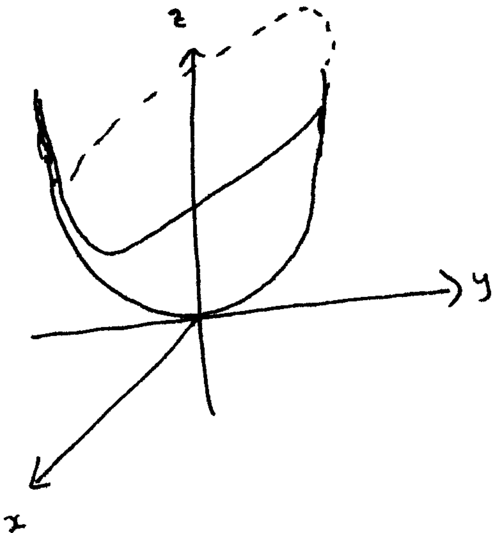


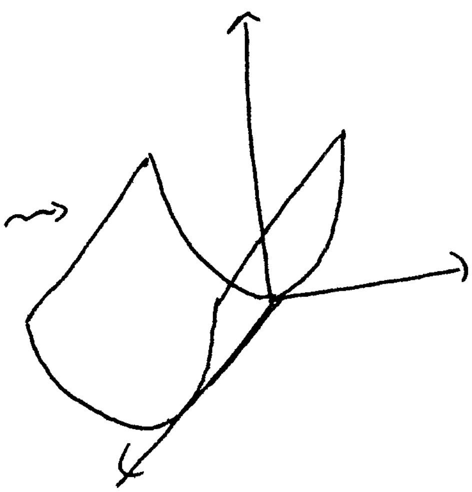
$a = \frac{1}{2}$



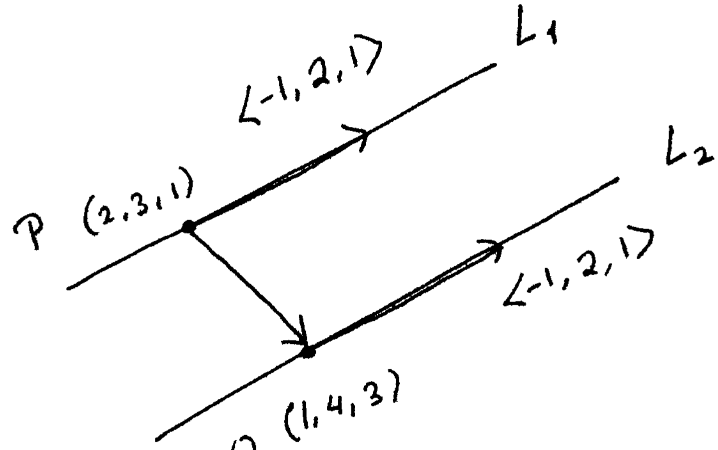
$a = 0$







6.



The plane contains both $\langle -1, 2, 1\rangle$ and $\overrightarrow{PQ} = \langle -1, 1, 2\rangle$.

$$\vec{n} = \langle -1, 2, 1\rangle \times \langle -1, 1, 2\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= 3\hat{i} + \hat{j} + \hat{k}$$

Equation of the plane $\quad 3(x-2) + 1(y-3) + 1(z-1) = 0$

$$\equiv \boxed{3x + y + z = 10}.$$

7. $$\operatorname{Proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} \qquad (\vec{v} \neq \vec{0})$$

(a) $\operatorname{Proj}_{\vec{v}}(\vec{u}) = 0 \quad \equiv \quad \vec{u}\cdot\vec{v} = 0 \quad \equiv \quad \vec{u} \perp \vec{v}$

(b) $\operatorname{Proj}_{\vec{v}}(\vec{u}) = \vec{u} \quad \equiv \quad \vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$ i.e. $\vec{u} = (\text{scalar})\,\vec{v}$

$\equiv$ $\vec{u}$ and $\vec{v}$ are parallel