## 0.2 Equivalence Relations and Partitions

The notion of an equivalence relation on a set plays an important role in many constructions in algebra. As we see in this section, an equivalence relation on a set determines a partition of the set into non-overlapping pieces and, conversely, any such partition determines an equivalence relation on the set.

**0.2.1 EXAMPLE** On the set $\mathbb{Z}$ of all integers, consider the relation ~ defined by the condition $a \sim b$ if and only if $a - b$ is divisible by 5, for any $a, b \in \mathbb{Z}$. Note the following properties of ~:

(1) For any integer $a$ we have $a - a = 0$, which is divisible by 5, so $a \sim a$.

(2) For any integers $a$ and $b$, $a - b = -(b - a)$, so if $a \sim b$, meaning that $a - b$ is divisible by 5, then so is $b - a$, and we have $b \sim a$.

(3) For any integers $a$, $b$, and $c$, if $a \sim b$ and $b \sim c$, then $a - b = 5n$ and $b - c = 5m$ for some integers $n$ and $m$. But then $a - c = (a - b) + (b - c) = 5n + 5m = 5(n + m)$, and so we have $a \sim c$.

Now let us take, say, the integer 7, and find the subset $[7] = \{x \in \mathbb{Z} \mid x \sim 7\}$ of $\mathbb{Z}$ consisting of all integers $x$ such that $x \sim 7$. Note that $7 \sim 2$, and therefore if $x \sim 7$, then $x \sim 2$ by property (3). Likewise, since $2 \sim 7$, if $x \sim 2$, then $x \sim 7$. So $x \sim 7$ if and only if $x \sim 2$, which is to say if and only if $x - 2 = 5k$, or, equivalently, $x = 2 + 5k$ for some integer $k$. Thus $[7] = 2 + 5\mathbb{Z}$, the set of all integers that can be written as the sum of 2 plus a multiple of 5. ◇

**0.2.2 EXAMPLE** Let $P(\mathbb{Z})$ be the set of all subsets of $\mathbb{Z}$, and consider the relation ~ on $P(\mathbb{Z})$ defined by letting $S \sim T$ if and only if $|S| = |T|$, that is, if and only if $S$ and $T$ have the same cardinality. So $S \sim T$ if and only if there is a one-to-one, onto map $\phi: S \to T$. (See Definition 0.1.25.) Note the following properties of ~:

(1) For any $S \in P(\mathbb{Z})$, the identity map on $S$ is a one-to-one, onto map from $S$ to itself. Therefore, $S \sim S$.

(2) For any $S, T \in P(\mathbb{Z})$, if $S \sim T$, then there is a one-to-one, onto map $\phi: S \to T$. Then $\phi^{-1}: T \to S$ is one to one and onto by Theorems 0.1.20 and 0.1.24, so $T \sim S$.

(3) For any $S, T, U \in P(\mathbb{Z})$, if $S \sim T$ and $T \sim U$, then there are one-to-one, onto maps $\phi: S \to T$ and $\chi: T \to U$. Then $\chi \circ \phi: S \to U$ is one to one and onto by Theorem 0.1.24, so $S \sim U$.

In this example, if $S$ is finite, then $[S] = \{T \in P(\mathbb{Z}) \mid S \sim T\}$ consists of all subsets of $\mathbb{Z}$ that have the same number of elements as $S$. If $S$ is infinite, $[S]$ consists of all infinite subsets of $\mathbb{Z}$. ◇

**0.2.3 DEFINITION** A **relation** on a nonempty set $S$ is a subset $R$ of $S \times S$. Let $R$ be a relation on $S$ and write $aRb$ to mean that $(a, b) \in R$. Then $R$ is an **equivalence relation** on $S$ if it satisfies the following three conditions for all $a, b, c \in S$:

(1) **Reflexivity** $aRa$
(2) **Symmetry** If $aRb$, then $bRa$.
(3) **Transitivity** If $aRb$ and $bRc$, then $aRc$.

If $R$ is an equivalence relation on $S$, then for any $a \in S$, the **equivalence class** of $a$ is the set $[a] = \{b \in S \mid aRb\}$. ○

In Examples 0.2.1 and 0.2.2 the relations ~ were equivalence relations.

We prove some important properties of equivalence classes that are used frequently in algebraic constructions.

**0.2.4 THEOREM** Let ~ be an equivalence relation on a set $S$, and let $a, b \in S$ be any elements of $S$. Then

(1) $a \in [a]$.
(2) If $a \in [b]$, then $[a] = [b]$.
(3) $[a] = [b]$ if and only if $a \sim b$.
(4) Either $[a] = [b]$ or $[a] \cap [b] = \varnothing$.

**Proof** (1) Reflexivity tells us $a \sim a$ and therefore $a \in [a]$.

(2) If $a \in [b]$, then by definition of equivalence classes we have $b \sim a$, and symmetry tells us we have $a \sim b$. Now if $x \in [a]$, then $a \sim x$, and transitivity tells us $b \sim x$, so $x \in [b]$. Thus $[a] \subseteq [b]$. Similarly, if $y \in [b]$ then $b \sim y$, and transitivity tells us $a \sim y$, so $y \in [a]$. Thus $[b] \subseteq [a]$ and $[a] = [b]$.

(3) ($\Rightarrow$) Suppose $[a] = [b]$. Since by (1) we have $b \in [b]$ it follows that $b \in [a]$, which by definition means $a \sim b$. ($\Leftarrow$) Suppose $a \sim b$. By definition, this means $b \in [a]$, and by (2) it follows that $[a] = [b]$.

(4) Suppose that $[a] \cap [b] \neq \varnothing$. This means there is some $c$ such that $c \in [a]$ and $c \in [b]$. By (2) it follows that $[c] = [a]$ and $[c] = [b]$, so $[a] = [b]$. □

The fact that an equivalence relation divides a set into disjoint or non-overlapping pieces, the equivalence classes, is what makes equivalence relations so useful in algebraic constructions. In the next example, instead of starting with an equivalence relation and using it to divide up a set, we start by dividing a set and use the division to define an equivalence relation.

**0.2.5 EXAMPLE** Starting with the set $\mathbb{R}$ of all real numbers, let $[1] = \{x \in \mathbb{R} \mid 0 \le x - 1 < 1\}$. In other words, [1] is the half-closed, half-open interval $[1, 2)$ in $\mathbb{R}$. Similarly, for any integer $n$ let $[n] = \{x \in \mathbb{R} \mid 0 \le x - n < 1\} = [n, n + 1)$. Note that for any distinct integers $i \neq j$ we have $[i] \cap [j] = \varnothing$, and for any real number $x \in \mathbb{R}$, $x \in [n]$ where $n$ is the greatest integer such that $n \le x$. So we have divided $\mathbb{R}$ into disjoint pieces. If now we define a relation ~ on $\mathbb{R}$ by letting $x \sim y$ if and only if $x \in [n]$ and $y \in [n]$ for the same integer $n$, then it can be checked that ~ is an equivalence relation on $\mathbb{R}$. (See Exercise 9 at the end of this section.) ◇

It is convenient to name such a division of a set into disjoint pieces.

**0.2.6 DEFINITION** Let $S$ be a nonempty set. A **partition** of $S$ consists of a collection $\{P_i\}$ of nonempty subsets of $S$ such that

(1) $S = \bigcup_i P_i$

(2) For any $P_i$, $P_j$ in the collection, either $P_i = P_j$ or $P_i \cap P_j = \varnothing$.

The subsets $P_i$ in the collection are called the **cells** of the partition. ○

We now come to the main theorem connecting equivalence relations and partitions, generalizing what we observed in Example 0.2.5.

**0.2.7 THEOREM** Let $S$ be a nonempty set.

(1) Given an equivalence relation $\sim$ on $S$, the collection of equivalence classes under $\sim$ is a partition of $S$.

(2). Given a partition $\{P_i\}$ of $S$, there is an equivalence relation on $S$ whose equivalence classes are precisely the cells of the partition.

**Proof** (1) Given an equivalence relation $\sim$, by Theorem 0.2.4, part (1), $a \in [a]$ for each $a \in S$, and therefore $S = \bigcup_a [a]$, which is the condition (1) for being a partition in Definition 0.2.6. Theorem 0.2.4, part (4), is precisely the condition (2) for being a partition in Definition 0.2.6.

(2) Given a partition $\{P_i\}$, define a relation $\sim$ by letting $a \sim b$ if and only if $a \in P_i$ and $b \in P_i$ for the same cell $P_i$. By condition (1) of Definition 0.2.6, any $a \in S$ does belong to some cell in the partition, and of course $a$ then belongs to the same cell as itself, so we have $a \sim a$. If $a \sim b$, then $a$ and $b$ belong to the same cell of the partition, which is the same as saying $b$ and $a$ belong to the same cell, and we have $b \sim a$. If $a \sim b$ and $b \sim c$, then $a$ belongs to the same cell $P_i$ in the partition as $b$ and $b$ belongs to the same cell $P_j$ in the partition as $c$. Since $b \in P_i \cap P_j$, by condition (2) of Definition 0.2.6 we must have $P_i = P_j$, and $a$ and $c$ belong to the same cell of the partition, and $a \sim c$. Finally, given $a \in S$, let $a \in P_i$. Then $x \in [a]$ if and only if $a \sim x$, hence if and only if $a$ and $x$ belong to the same cell of the partition or, in other words, if and only if $x \in P_i$. So the equivalence class of $a$ is $[a] = P_i$. □

## Exercises 0.2

In Exercises 1 through 8 determine whether the indicated relation is an equivalence relation on the indicated set and, if so, describe the equivalence classes.

**1.** In $\mathbb{R}$ $a \sim b$ if and only if $|a| = |b|$

**2.** In $\mathbb{R}$ $a \sim b$ if and only if $a \leq b$

**3.** In $\mathbb{Z}$ a ~ b if and only if $a - b$ is even

**4.** In $\mathbb{R}$ $a \sim b$ if and only if $|a - b| \leq 1$