

① Suppose  $\exists c_1, c_2 \in \mathbb{R}$  s.t.  $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) = \vec{0}$ . Then  $c_1\vec{v}_1 + c_2\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ , and  $(c_1+c_2)\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ . Since  $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$  is lin. ind., it follows that  $c_1+c_2=0 \nRightarrow c_2=0$ . Thus  $c_1=0$ , as well, as  $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2\}$  is lin. ind.

$$\textcircled{2} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad M^{100} = \begin{pmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 200 & 0 & 1 \end{pmatrix}$$

③ Suppose  $AB$  is invertible. Then  $\det(AB) \neq 0$ . Since  $\det(AB) = \det(A)\det(B)$ , it follows that  $\det(A)\det(B) \neq 0$ . Since the product of any # with 0 is 0,  $\det A \neq 0 \nRightarrow \det B \neq 0$ . Thus  $A \nparallel B$  are both invertible.

④  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is invert, but its only e-vect is  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (corresp to 1) so is not diagonalizable.

$$\textcircled{5} \quad A = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \end{pmatrix}. \quad \text{Then } BA = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3-2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = I_1.$$

$A$  cannot have a right inverse. Suppose  $\begin{pmatrix} 3 \\ 2 \end{pmatrix}(a \ b) = \begin{pmatrix} 3a & 3b \\ 2a & 2b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ .

Then  $3b=0 \Rightarrow b=0 \nRightarrow 2a=0 \Rightarrow a=0$ . But then  $3a=3(0)=0 \neq 1 \nRightarrow 2b=2(0)=0 \neq 1$ , which is a contradiction.

⑥ We want to determine if  $A^{-1} = (A^T)^{-1}$ .

$(A^T)^T = (A^T)^{-1} = A^{-1}$ , so yes  $A^{-1}$  is symmetric.

⑦ No. The matrices  $A = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \nparallel B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  are row-equivalent, but the only eigenvalue of  $B$  is 1, while 1 is not an eigenvalue of  $A$ , since  $\det(A-I) = \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$ . Therefore  $A \nparallel B$  are not similar.

⑧ For  $A$ , we manipulate the rows.

$$A \sim \begin{pmatrix} 15 & 15 & 15 & 15 & 15 \\ 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 7 \end{pmatrix} \sim 15 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 7 \end{pmatrix} \sim 15 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

So  $\det A = 15 \cdot 5^4$ , since the last matrix is upper triangular (multiply the 15 back through the first row).

$\det B = 0$  because column 1 is the sum of columns 2 through 4.

$\det C = 15 \cdot 5^4$  because it is upper triangular.

$$\textcircled{9} \quad A^T B C^T = D \Rightarrow (A^T B C^T)^T = D^T \Rightarrow C B^T A = D^T \Rightarrow B^T = C^T D^T A^T$$

⑩ (a) • closure: Let  $A, B \in V$ . Then for each  $1 \leq i \leq 4$ ,  $a_{1i} + a_{2i} + a_{3i} + a_{4i} = 0 \nparallel b_{1i} + b_{2i} + b_{3i} + b_{4i} = 0$

Similarly, for each  $1 \leq i \leq 4$ ,  $a_{ij} + a_{ij} + a_{3j} + a_{4j} = 0 \nparallel b_{ij} + b_{ij} + b_{3j} + b_{4j} = 0$ .

So  $\sum_{j=1}^4 [A+B]_{ij} = \sum_{j=1}^4 a_{ij} + \sum_{j=1}^4 b_{ij} = 0+0=0$  for each  $1 \leq i \leq 4$ .

Similarly,  $\sum_{i=1}^4 [A+B]_{ij} = \sum_{i=1}^4 a_{ij} + \sum_{i=1}^4 b_{ij} = 0+0=0$  for each  $1 \leq j \leq 4$ .  $\Rightarrow A+B \in V$

Now, let  $c \in \mathbb{R}$ ,  $A \in V$ . Then  $[cA]_{ij} = ca_{ij}$ , for each  $1 \leq i \leq 4$ ,

$\sum_{j=1}^4 [cA]_{ij} = \sum_{j=1}^4 ca_{ij} = c \sum_{j=1}^4 a_{ij} = c \cdot 0 = 0$ . Similarly, for each  $1 \leq j \leq 4$ ,

$$\sum_{i=1}^3 [cA]_{ij} = c \sum_{i=1}^3 a_{ij} = c \cdot 0 = 0 \Rightarrow cA \in V$$

Since all entries in the zero vector  $A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$  are zero, all row- & column-sums are zero  $\Rightarrow \vec{0} \in V$ .

Thus  $V$  is a subspace of  $M_{4 \times 4}(\mathbb{R})$ .

(b)  $\dim V = 9$ .

$V$  is spanned by  $A_{11}, A_{12}, A_{13}, A_{21}, \dots, A_{33}$  where  $A_{ij}$  has  $a_{ij}=1$ ,  $a_{im}=-1$  &  $a_{qj}=1$ , & all other entries are 0.

⑪ No.  $h(x) = g(x) - f(x)$ .

⑫ Since the trace of the matrix is the sum of the  $\lambda$ -vals,  $z=4$ . Also, the determinant of the matrix is the product of the  $\lambda$ -vals and the determinant is  $-w$ , so  $w=-1$ . To find  $x \in y$ , we calculate the characteristic polynomial.

$$\det \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -1 & x & y & 4-\lambda \end{pmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ x & y & 4-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{vmatrix} = -\lambda \cdot (-\lambda((\lambda-4)\lambda-y)+x) + 1$$

$$= \lambda^3(\lambda-4) - \lambda^2y - \lambda x + 1$$

$$= \lambda^4 - 4\lambda^3 - y\lambda^2 - x\lambda + 1. \text{ Since this is equal to } (\lambda-1)^4 = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 \text{ we have } y = -6 \text{ & } x = 4.$$

Thus  $w = -1, x = 4, y = -6, \text{ & } z = 4$

(b)  $\lambda=1$ , the corresp.  $\lambda$ -space is  $\text{Nul}(A-I)$ .

$$A-I \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ To find a basis for } \text{Nul}(A-I), \text{ we solve } (A-I)\vec{x} = \vec{0}: \quad \vec{x} = \begin{pmatrix} x_4 \\ x_4 \\ x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Since the  $\lambda$ -space has  $\dim 1$  while the multiplicity of  $\lambda=1$  is 4,  $A$  is not diagonalizable.

⑬  $X^2 = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}$ .  $X^2$  is diagonalizable, w/  $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \notin D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ .

Then if  $X = P \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 3 \end{pmatrix} P^{-1}$ , then  $X^2 = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} P^{-1}$ . Any combination of signs will work, so there is more than one square root.