Gong Solutions

① $\vec{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$, $-1 \le t \le 1$

(a) Arclength $= \int_{-1}^{1} \|\vec{r}\,'(t)\| dt$

$\vec{r}\,'(t) = \langle e^t\cos t - e^t\sin t,\ e^t\sin t + e^t\cos t,\ e^t\rangle$

$\|\vec{r}\,'(t)\|^2 = (e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2$

$= e^{2t}\cos^2 t - \cancel{2e^{2t}\sin t\cos t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t + \cancel{2e^{2t}\cos t\sin t}$
$+ e^{2t}\cos^2 t + e^{2t}$

$= e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}(\sin^2 t + \cos^2 t) + e^{2t}$

$= 3e^{2t}$

$\int_{-1}^{1}\sqrt{3e^{2t}}\,dt = \int_{-1}^{1}\sqrt{3}e^t\,dt = \sqrt{3}e^t\Big|_{-1}^{1} = \boxed{\sqrt{3}e - \sqrt{3}e^{-1}}$

② $x^2+1 = 2yz \longrightarrow x^2 - 2yz + 1 = g(x,y,z) = 0$ : Constraint

$f(x,y,z) = x^2+y^2+z^2$ : square of the distance to the origin. The minimum of this will be the minimum of the distance.

$\vec{\nabla} f = \lambda\vec{\nabla} g$:

$$\begin{cases} 2x = \lambda 2x \\ 2y = -2\lambda z \\ 2z = -2\lambda y \\ x^2 - 2yz + 1 = 0 \end{cases}$$

OR $\vec{\nabla} g = 0$:

$$\begin{cases} 2x = 0 \\ -2z = 0 \\ -2y = 0 \\ x^2 - 2yz + 1 = 0 \end{cases}$$

$\to x=0, y=0, z=0$

$x^2 - 2yz + 1 = 0 \longrightarrow 0 - 0 + 1 \neq 0$

No solution.

$\to \lambda = 1$ or $x = 0$

If $\lambda = 1$: $2y = -2z \Rightarrow y = -z$

$x^2 - 2y(-y) + 1 = 0$

$x^2 + 2y^2 \neq -1$ (LHS $= +$, RHS $= -$)

If $x = 0$: $2y = -2\lambda z$

$2z = -2\lambda y$

$2yz = 1 \to y = 1/2z$ or $z = 0$

If $y = 1/2z$ : $\begin{cases} 2(1/2z) = -2\lambda z \longrightarrow 1 = -2\lambda z^2 \\ 2z = -2\lambda(1/2z) \longrightarrow 2z^2 = -\lambda \end{cases}$

So substituting $2^{nd}$ eqn into $1^{st}$:

$1 = (2z^2)(2z^2)$

$z^2 = 1/4$

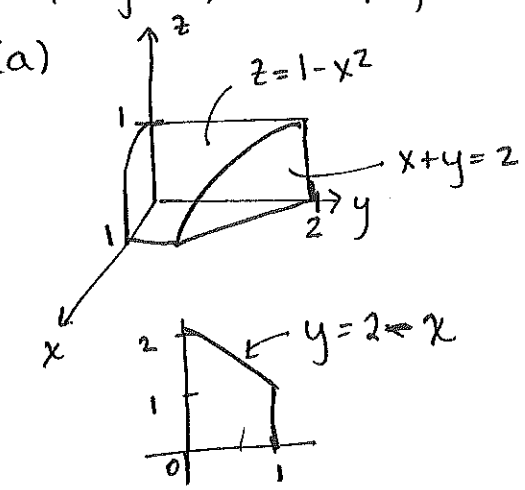
$\begin{cases} z = \pm 1/2 \\ y = \pm \frac{1}{2(1/2)} = \pm 1 \end{cases}$ $\quad (0, 1, 1/2)$ $\quad (0, -1, -1/2)$

$\to z$ & $y$ have same sign, since $y = 1/2z$

So $\boxed{(0,1,1/2) \text{ \& } (0,-1,-1/2)}$ are closest to the origin.

③ $x + y = 2$, $z = 1 - x^2$, $1^{st}$ octant

(a)




$$\int_0^1 \int_0^{2-x} \int_0^{1-x^2} dz\,dy\,dx$$

$$= \int_0^1 \int_0^{2-x} 1 - x^2 \; dy\,dx$$

$$= \int_0^1 (1-x^2)(2-x)\,dx = \int_0^1 2 - x - 2x^2 + x^3\,dx$$

$$= 2x - \frac{x^2}{2} - \frac{1}{2}x^4 + \frac{1}{4}x^4 \Big|_0^1 = 2 - 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

④ $x - y + e^{xz - x} + z = 2$, $z = f(x,y)$, $f(1,1) = 1$

(a) Method 1: Use implicit differentiation.

$$\frac{\partial}{\partial x}(x - y + e^{xz-x} + z) = \frac{\partial}{\partial x}(2)$$

$$1 + e^{xz-x}\left(z + x\frac{\partial z}{\partial x} - 1\right) + \frac{\partial z}{\partial x} = 0$$

at $(1,1,1)$: $\quad 1 + 1\left(1 + \frac{\partial z}{\partial x} - 1\right) + \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -1/2$

$$\frac{\partial}{\partial y}(x - y + e^{xz-x} + z) = \frac{\partial}{\partial y}(2)$$

$$-1 + e^{xz-x}\left(x\frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial y} = 0$$

at $(1,1,1)$: $\quad -1 + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = 1/2$

So, the tangent plane is:

$$\boxed{z = 1 + (-1/2)(x-1) + 1/2(y-1)}$$

Method 2: Think of $x - y + e^{xz-x} + z = 2$ as a level set of some 4-dim'l function $g(x,y,z) = x - y + e^{xz-x} + z$. Then $\vec{\nabla} g$ is a normal vector to the tangent plane, so we can use $(\vec{\nabla} g(1,1,1)) \cdot \vec{AX} = 0$

$$\vec{\nabla} g = \langle 1 + (z-1)e^{xz-x}, -1, xe^{xz-x} + 1 \rangle$$

$$\vec{\nabla} g(1,1,1) = \langle 1, -1, 2 \rangle.$$

$A = (1,1,1)$

So, $\langle 1,-1,2 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$

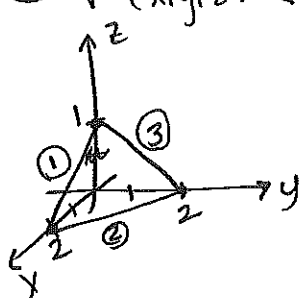
$$\boxed{(x-1) + (-1)(y-1) + 2(z-1) = 0}$$

Notice that these 2 equations are equivalent.

(b) $f$ increases most rapidly at $(1,1)$ in the direction of $\vec{\nabla} f$, so it decreases most rapidly in the direction of $-\vec{\nabla} f$.

$$-\vec{\nabla} f = -\left\langle \frac{\partial z}{\partial x}(1,1), \frac{\partial z}{\partial y}(1,1) \right\rangle = -\underbrace{\langle -1/2, 1/2 \rangle}_{\text{from (a), with } z = f(x,y)} = \boxed{\langle 1/2, -1/2 \rangle}$$

⑤ $\vec{F}(x,y,z) = \langle y, xz, x^2 \rangle$




① $\vec{x}(t) = \langle t, 0, 1-t \rangle, \quad 0 \le t \le 1$

$\vec{x}'(t) = \langle 1, 0, -1 \rangle$

$$\int_0^1 \langle 0, t - t^2, t^2 \rangle \cdot \langle 1, 0, -1 \rangle \, dt$$

$$= \int_0^1 -t^2 \, dt = -\frac{t^3}{3} \Big|_0^1 = -1/3$$

② $\vec{x}(t) = \langle 2-2t, 2t, 0 \rangle, \quad 0 \le t \le 1$

$\vec{x}'(t) = \langle -2, 2, 0 \rangle$

$$\int_0^1 \langle 2t, 0, 4 - 8t + 2t^2 \rangle \cdot \langle -2, 2, 0 \rangle \, dt = \int_0^1 -4t \, dt = -2t^2 \Big|_0^1 = -2$$

③ $\vec{x}(t) = \langle 0, 2-2t, t \rangle, \quad 0 \le t \le 1$

$\vec{x}'(t) = \langle 0, -2, 1 \rangle$

$$\int_0^1 \langle 2-2t, 0, 0 \rangle \cdot \langle 0, -2, 1 \rangle \, dt = \int_0^1 0 \, dt = 0$$

$$-1/3 - 2 + 0 = \boxed{-7/3}$$

(6) (a) $\vec{F}(x,y,z) = \langle 9x^8 e^y, x^9 e^y + 1, 1 \rangle$

(b)

For both, see Meyer, Fall 2013, #4