Finding Extrema Using Contour Maps and Gradient Vector Fields

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Given a function $f(x, y)$, it is possible to identify saddle points and points where the function has a local maximum or minimum by looking at a contour map or a gradient vector field of f.

1 Contour Maps

Recall a critical point (a, b) occurs when $\nabla f(a, b) = \vec{0}$, and that $\nabla f(a, b)$ is orthogonal to the contour line containing the point (a, b). There are two different configurations of contour lines which indicate that $\nabla f(a, b) = \vec{0}$.

1. First, it is possible that a contour line is simply a point (a, b). In this case, $\nabla f(a, b) = \vec{0}$. This happens when (a, b) is in the middle of concentric rings in the contour map. The function f will always have a local maximum or minimum at such a point, depending on the value of f along the rings leading to (a, b). If the value of f is increasing as you move towards (a, b), then f has a local maximum at (a, b). On the other hand, if the value of f is decreasing as you move towards (a, b), then f has a local minimum at (a, b).

2. Second, $\nabla f(a, b) = \vec{0}$ if (a, b) is a point of intersection of two contour lines (or one contour line with itself). This is because $\nabla f(a, b)$ has to be orthogonal to both of the contour lines, and since the two lines have different tangent vectors, the only vector that is orthogonal to both is the zero vector. When this happens, the intersection of the contour lines divides a small disk around (a, b) into 4 different regions. Typically the value of f will increase in two of these regions which are opposite to each other, and f will decrease in the other two regions. (You can tell if this occurs by looking at the neighboring contour lines.) When this happens, (a, b) is a saddle point.

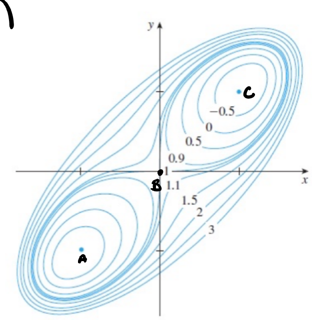
   Here are 3 examples. In all of these, if the contour map is symmetric, you should assume that function takes on symmetric values on the lines.

   (a) Points A and C are of type 1 above, and the value of f decreases as you move towards each, so f has a local minimum at each point. Point B is of type 2 above, and so it is a saddle point.

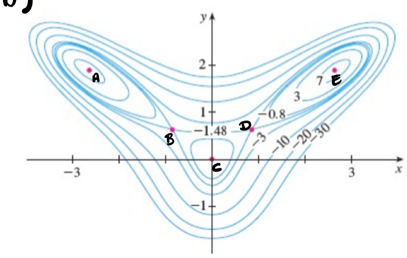
   (b) Points A, C, and E are of type 1 above, and the value of f increases as you move towards each (this does make an assumption about the value of f on the ring closest to C, since it's not labeled, so it is possible that f decreases as you move towards C – here we assume f increases), so f has a local maximum at each point. Points B and D are of type 2 above, so they are saddle points.

(c) Point B is of type 1 above, and f increases as you move towards it, so f has a local maximum at B. Point A is of type 2 above, so it is a saddle point.

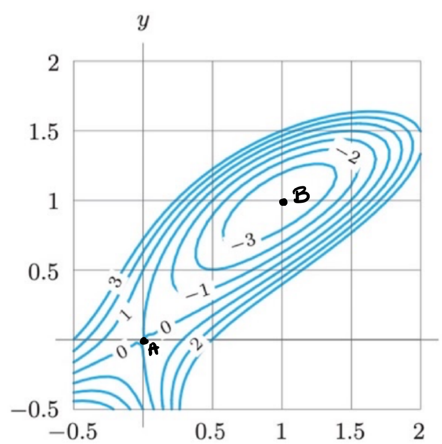
(a)



(b)



(c)



2 Gradient Vector Fields

Given a gradient vector field, the length of each vector is its magnitude. Therefore, the critical points occur where the gradient is a point. To classify a critical point (a, b), recall that $\nabla f(x, y)$ always points in the direction of greatest increase of the function. There are 3 different behaviors in a gradient vector field around critical points:

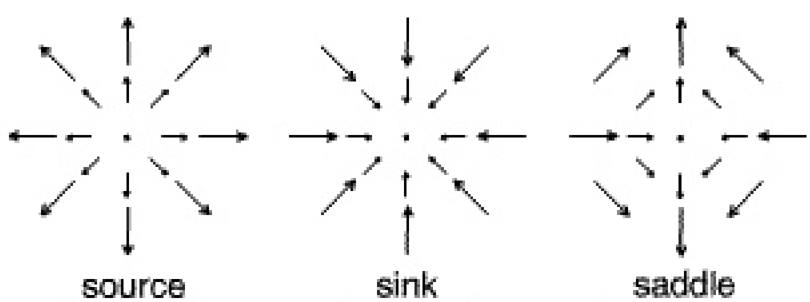
1. If all gradients near (a, b) point towards (a, b), then f has a local maximum at (a, b). This is because the gradients pointing towards (a, b) mean that no matter

how you approach the point (a, b), the value of the function is increasing. This behavior is called a *sink* (imagine (a, b) is a drain in a sink and the arrows represent the flow of the water).

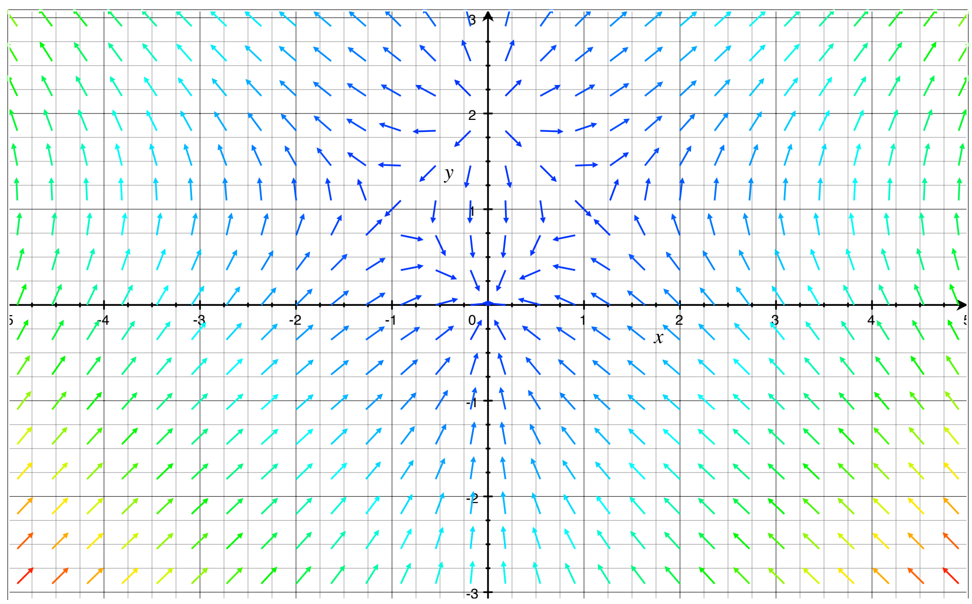
2. If all gradients near (a, b) point away from (a, b), then f has a local minimum at (a, b). This is because the gradients pointing away from (a, b) means that no matter how you move *away* from the point (a, b), the value of the function is increasing. This behavior is called a *source* (imagine that the point is a light bulb–or any light source–and the arrows represent the path of the rays of light).

3. If some gradients near (a, b) point towards (a, b) and some point away from (a, b), then (a, b) is a saddle point. Note that when you see a gradient vector field, you will not see *every* gradient vector; you will typically see those whose initial point is on a specified grid. That means that the gradients that point directly at (a, b) or directly away from (a, b) may not be shown in the gradient vector field. What you should look for is vectors on opposite sides of (a, b) pointing in opposite directions (see the figure below).

The following figure shows these three possible behaviors near critical points.

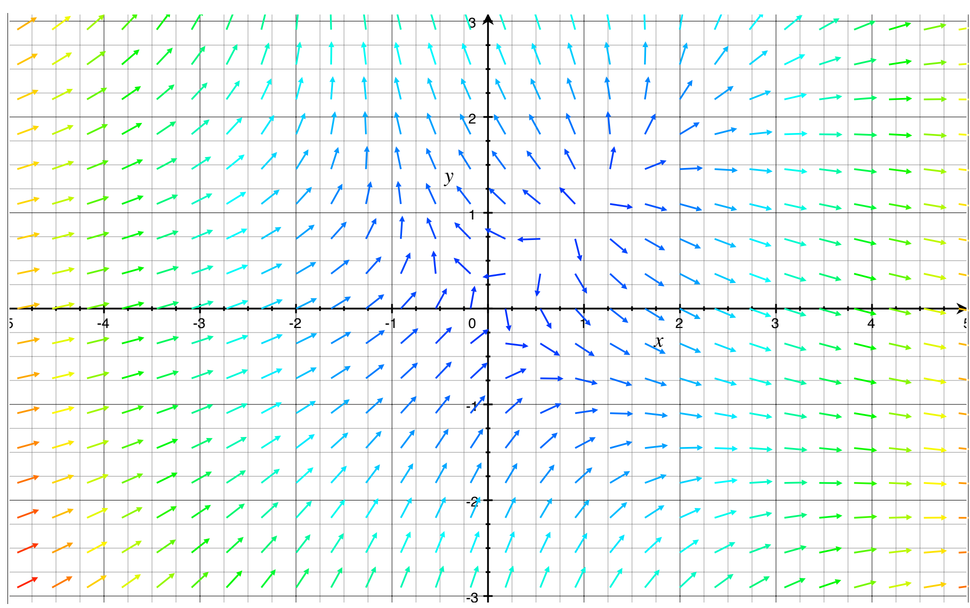


Below are three examples. Given is the gradient vector field of a function $f(x, y)$. Here, the gradient vectors are scaled to be the same length (so that it is easier to see the direction in which they point), and the color indicates the length, where the dark blue vectors are the shortest.

(a) 

This gradient vector field indicates that the function has 4 critical points, occurring at (approximately) $(0,0),(-1,1),(1,1)$, and $(0,2)$.

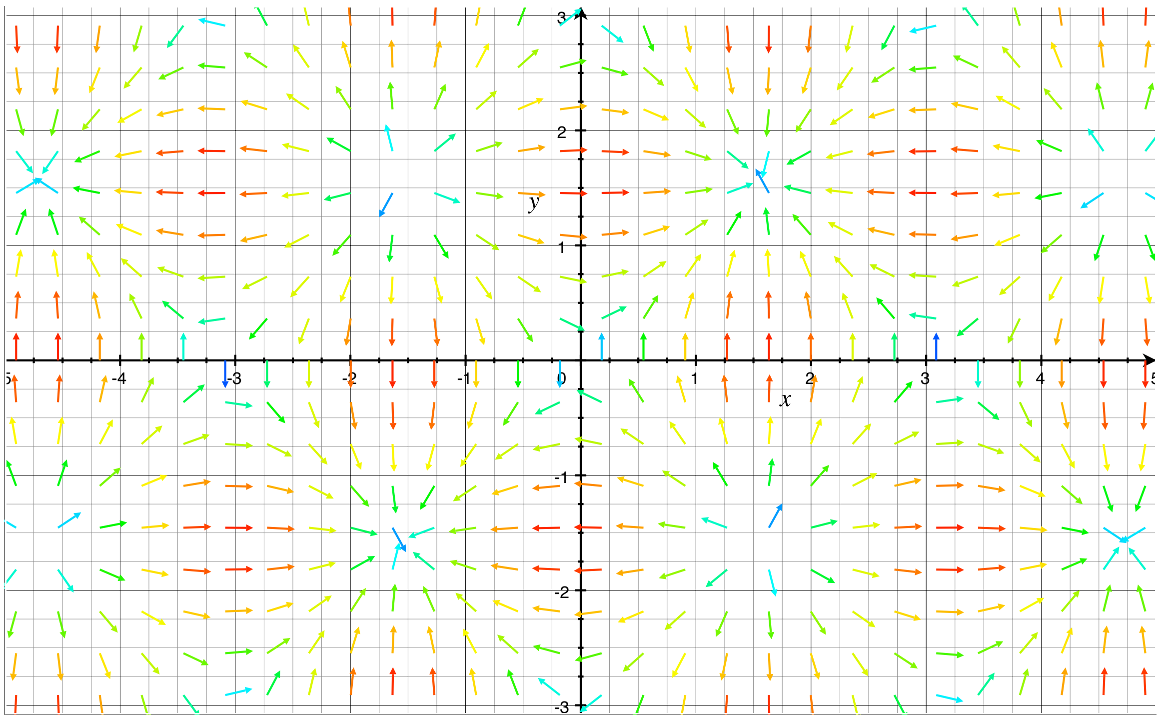
- $(0,0)$: Nearby vectors point towards $(0,0)$ so $f(0,0)$ is a local maximum.
- $(0,2)$: Nearby vectors point away from $(0,0)$ so $f(0,0)$ is a local minimum.
- $(-1,1)$ and $(1,1)$: There are nearby vectors pointing towards each of these points and nearby vectors pointing away, so these are saddle points.

(b) 

This gradient vector field indicates that the function has 2 critical points, occurring at (approximately) $(0,0)$ and $(1,1)$.

- $(0, 0)$: There are nearby vectors on opposite sides of this point which are pointing in opposite directions, so this is a saddle point. (Note the 4 vectors shown that are closest to $(0, 0)$ kind of form a "square" around the point. This is what you see in the picture of a saddle point in the figure above if you remove all the vertical and horizontal vectors.)
- $(1, 1)$: Nearby vectors point away from $(1, 1)$ so $f(1, 1)$ is a local minimum.

(c)



This gradient vector field is more complicated. There are 11 critical points shown, and the graph gives the sense that the function is periodic (which, in fact, it is). The critical points shown occur at: $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$, $(\pm\frac{3\pi}{2}, \pm\frac{\pi}{2})$, $(\pm\pi, 0)$, and $(0, 0)$. I've given the exact values of the critical points (because I know what the original function is!), but just looking at the graph, you would approximate. For example, I would approximate the point $(\frac{\pi}{2}, \frac{\pi}{2})$ as about $(1.5, 1.5)$ from the graph.

- $(\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, \frac{\pi}{2}), (-\frac{3\pi}{2}, \frac{\pi}{2}), (\frac{3\pi}{2}, -\frac{\pi}{2})$: Nearby vectors point towards these points so f has a local maximum at each.
- $(-\frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, -\frac{\pi}{2}), (-\frac{3\pi}{2}, -\frac{\pi}{2}), (\frac{3\pi}{2}, \frac{\pi}{2})$: Nearby vectors point away from these points so f has a local minimum at each.
- $(\pm\pi, 0)$, and $(0, 0)$: There are nearby vectors on opposite sides of each of these points which are pointing in opposite directions, so these are saddle points.

This is the gradient vector field of the function $f(x, y) = \sin x \sin y$. For reference, here is the graph of the function: