① Consider $F(\beta)$. Since $\beta \in F(\alpha)$, we have a tower of extensions $F \le F(\beta) \le F(\alpha)$. Therefore $|F(\alpha):F| = |F(\beta):F| \cdot |F(\alpha):F(\beta)|$, so $|F(\beta):F|$ divides $|F(\alpha):F|$. Thus $\deg(\beta, F)$ divides $\deg(\alpha, F)$.

② Let $\alpha \in E \setminus F$. Then we have a tower of extensions $F \le F(\alpha) \le E$, & so $|F(\alpha):F|$ divides $|E:F|$. But $|E:F|$ is prime, so either $|F(\alpha):F| = 1$ or $|F(\alpha):F| = |E:F|$. The first case is only possible if $\alpha \in F$, which contradicts our choice of α. Thus $|F(\alpha):F| = |E:F|$. It follows that $|E:F(\alpha)| = 1$, & so $E = F(\alpha)$.

③ First, we will find $\operatorname{irr}(\sqrt{2}+\sqrt{5}, \mathbb{Q})$:

$$x = \sqrt{2}+\sqrt{5}$$
$$x^2 = 2 + 2\sqrt{10} + 5$$
$$x^2 - 7 = 2\sqrt{10}$$
$$x^4 - 14x^2 + 49 = 40$$
$$x^4 - 14x^2 + 9 = 0$$

$f(x) = x^4 - 14x^2 + 9$ has no linear factors (because $\pm 1, \pm 3, \pm 9$ are not zeros of $f(x)$).

Suppose $f(x) = x^4 - 14x^2 + 9 = (x^2 + bx + c)(x^2 + dx + e)$.

Then $x^4 - 14x^2 + 9 = x^4 + (d+b)x^3 + (c+e+bd)x^2 + (cd+eb)x + ec$, so

$$\begin{cases} d+b=0 & \to d=-b \\ c+e+bd=-14 & \\ cd+eb=0 & \to eb=cb \to e=c \text{ or } b=0 \\ ec=9 & \end{cases}$$

If $e=c$, then $e=c=3$ or $e=c=-3$, so $6-b^2=-14 \Rightarrow b^2=20$ or $-6-b^2=-14 \Rightarrow b^2=8$. In either case there is no $b \in \mathbb{Z}$ satisfying this.

If $b=0$, then $c+e=-14$ & $ec=9$. Testing all combinations of $e, c \in \mathbb{Z}$ s.t. $ec=9$, we see that none satisfy $c+e=-14$.

Therefore $f(x)$ is irreducible over $\mathbb{Q}$.

Thus $|\mathbb{Q}(\sqrt{2}+\sqrt{5}):\mathbb{Q}| = 4$, while $|\mathbb{Q}(\sqrt{2}):\mathbb{Q}| = 2$, so $\sqrt{2}+\sqrt{5} \notin \mathbb{Q}(\sqrt{2})$.
Since $\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, it follows that $\sqrt{5} \notin \mathbb{Q}(\sqrt{2})$, so $|\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}(\sqrt{2})| > 1$.
Now, $\operatorname{irred}(\sqrt{5}, \mathbb{Q}) = x^2 - 5$ & $x^2 - 5 \in \mathbb{Q}(\sqrt{2})[x]$, so $|\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}(\sqrt{2})| \le 2$.
Combining these inequalities, we see that $|\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}(\sqrt{2})| = 2$.

Since $\sqrt{2}+\sqrt{5} \in \mathbb{Q}(\sqrt{2},\sqrt{5})$, we know $\mathbb{Q}(\sqrt{2}+\sqrt{5}) \le \mathbb{Q}(\sqrt{2},\sqrt{5})$.
Thus we have the following diagram:

$$\begin{tikzcd} & \mathbb{Q}(\sqrt{2},\sqrt{5}) \arrow[dl, dash, "2"'] \arrow[d, dash] \\ \mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & \mathbb{Q}(\sqrt{2}+\sqrt{5}) \arrow[d, dash, "4"] \\ & \mathbb{Q} \end{tikzcd}$$

From the diagram we see that:

$$|\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}| = |\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}(\sqrt{2})|\cdot|\mathbb{Q}(\sqrt{2}),\mathbb{Q}|$$
$$= 2\cdot 2 = 4.$$

Also from the diagram we see that:

$$|\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}| = |\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}(\sqrt{2}+\sqrt{5})|\cdot|\mathbb{Q}(\sqrt{2}+\sqrt{5}):\mathbb{Q}|$$

Thus, $4 = |\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}(\sqrt{2}+\sqrt{5})|\cdot 4$

$\Rightarrow |\mathbb{Q}(\sqrt{2},\sqrt{5}):\mathbb{Q}(\sqrt{2}+\sqrt{5})| = 1.$

Therefore, $\mathbb{Q}(\sqrt{2},\sqrt{5}) = \mathbb{Q}(\sqrt{2}+\sqrt{5})$.

④ We first prove $I+J$ is a subgp: I & J are subgps, so $0\in I$ & $0\in J$. Thus $0=0+0\in I+J$. If $c+d\in I+J$, then $c\in I$ & $d\in J$. Since I & J are subgps $-c\in I$ & $-d\in J$. Thus $-(c+d) = -c+-d\in I+J$. Finally, suppose $a+b, c+d\in I+J$. Then $(a+b)+(c+d) = (a+c)+(b+d)$, which is in $I+J$ since $a+c\in I$ & $b+d\in J$, again using that I & J are subgps. Therefore, $I+J$ is a subgp.

To show $I+J$ is an ideal, let $a+b\in I+J$ & let $r\in R$. Then $r(a+b) = ra+rb$. $ra\in I$ & $rb\in J$, since I & J are both ideals of R. Thus $ra+rb\in I+J$. Also, $(a+b)r = ar+br$. $ar\in I$ & $br\in J$, since I & J are both ideals of R. Thus $ar+br\in I+J$. Since $a+b$ & r were arbitrary, $I+J$ is an ideal of R.

⑤ Since $\langle a\rangle, \langle b\rangle$ are subgps, $e\in\langle a\rangle$ & $e\in\langle b\rangle$, so $e\in\langle a\rangle\cap\langle b\rangle$. Suppose $c\in\langle a\rangle\cap\langle b\rangle$ & $c\neq e$. Since $c\in\langle a\rangle$, $|c|$ divides $|a|$, & since $c\in\langle b\rangle$, $|c|$ divides $|b|$. But $\gcd(|a|,|b|)=1$, so $|c|=1$. Since the only elt of G with order 1 is e, $c=e$. Thus $\langle a\rangle\cap\langle b\rangle = \{e\}$.

⑥ Let $\varphi: R\to R'$ be a surj. ring hom. If S is a subring of R, then S is a subgp of R. Since φ is also a gp hom, we know that $\varphi(S)$ is a subgp of R'. Thus to show $\varphi(S)$ is a subring of R', we need only show it is closed under multiplication. Let $x,y\in\varphi(S)$. Then $\exists\, a,b\in S$ s.t. $x=\varphi(a)$ & $y=\varphi(b)$.
$xy = \varphi(a)\varphi(b) = \varphi(ab)$. Since S is a subring, $ab\in S$. Thus $xy\in\varphi(S)$. Therefore $\varphi(S)$ is a subring of R'.

⑦ Let $p(x)\in\mathbb{Z}[x]$ be any non-zero element. Then $\deg x\cdot p(x))$ is $\deg(x)+\deg(p(x))\geq 1$, since $\deg(x)=1$. But 1 is a constant polynomial, & so $xp(x)\neq 1$ for any non-zero $p(x)\in\mathbb{Z}[x]$. Thus x is not a unit. By the same argument, $xp(x)\neq 0$ for any non-zero $p(x)\in\mathbb{Z}[x]$, as 0 is also a constant polynomial.

(8) Suppose $(a,b)(a,b)=(1,1)$ for some $(a,b)\in \mathbb{Z}_p\times\mathbb{Z}_q$. Then $(a^2,b^2)=(1,1)$, so $a^2=1$ & $b^2=1$. Thus a & b each solve the eqn $x^2-1=0$ in $\mathbb{Z}_p$ & $\mathbb{Z}_q$, respectively. $x^2-1=(x-1)(x+1)$, which has sol'ns $a=1, p-1$ in $\mathbb{Z}_p$ & $b=1, q-1$ in $\mathbb{Z}_q$.
Thus the possibilities for (a,b) are: $(1,1), (1,q-1), (p-1,1), (p-1,q-1)$.

(9) Suppose $\text{char}(R)=n$, & n is not prime. Then $n=l\cdot m$, for some $l,m\in\mathbb{N}\setminus\{1\}$ Consider $l\cdot 1, m\cdot 1\in R$. Since $l,m<n$, $l\cdot 1$ & $m\cdot 1$ are non-zero elts of R. However, $(l\cdot 1)(m\cdot 1)=(lm)\cdot 1=n\cdot 1=0$. Thus $l\cdot 1$ & $m\cdot 1$ are zero divisors.

(10) Suppose $ab=ac$, with $a\neq 0$. Then $ab-ac=0$, & so $a(b-c)=0$, by left distribution. Since R has no zero divisors & $a\neq 0$, it must be that $b-c=0$, which implies that $b=c$.

(11) We first show $S\cap T$ is a subgp of R: Let $a,b\in S\cap T$. Then $a,b\in S$ & $a,b\in T$.
- closed under $+$: $a+b\in S$ since $a,b\in S$ & $a+b\in T$ since $a,b\in T$. Thus $a+b\in S\cap T$.
- identity: $0\in S$ & $0\in T \Rightarrow 0\in S\cap T$.
- inverses: $-a\in S$ & $-a\in T\Rightarrow -a\in S\cap T$. Thus it remains only to show that $S\cap T$ is closed under multiplication. Let $x,y\in S\cap T$. Then $x,y\in S\Rightarrow xy\in S$ & $x,y\in T\Rightarrow xy\in T$, as S & T are subrings of R. Thus $xy\in S\cap T$, & $S\cap T$ is a subring of R.

Yes, $S\cap T$ is an ideal of R. Let $r\in R$ & $x\in S\cap T$. Then $x\in S$ & $x\in T$, & since S & T are ideals of R, $rx\in S$ & $rx\in T$. Thus $rx\in S\cap T$ Similarly, $xr\in S\cap T$, & $S\cap T$ is an ideal of R.

(12) $\mathbb{Z}_6\times\mathbb{Z}_8/\langle(0,2)\rangle \simeq \mathbb{Z}_6\times\mathbb{Z}_2 \simeq \mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2$

$|\langle(0,2)\rangle| = \text{lcm}(|0|,|2|) = \text{lcm}(1,4)=4$, so $|\mathbb{Z}_6\times\mathbb{Z}_8/\langle(0,2)\rangle| = |\mathbb{Z}_6\times\mathbb{Z}_8|/4 = 48/4 = 12$.

By the FTFGAG, $\mathbb{Z}_6\times\mathbb{Z}_8/\langle(0,2)\rangle$ is isomorphic to one of the following gps: $(12=2^2\cdot 3)$

$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$

$\mathbb{Z}_4\times\mathbb{Z}_3$

We will show $\mathbb{Z}_6\times\mathbb{Z}_8/\langle(0,2)\rangle$ has no elt of order 4.

$\langle(0,2)\rangle = \{(0,0),(0,2),(0,4),(0,6)\}$, & the coset representatives are

$(0,0)$	$(0,1)$
$(1,0)$	$(1,1)$
$(2,0)$	$(2,1)$
$\vdots$	$\vdots$
$(5,0)$	$(5,1)$

Note that $m\cdot(n,0)\in\langle(0,2)\rangle \Leftrightarrow mn=0$, so $|(n,0)+\langle(0,2)\rangle| = |n|$. Since no elt of $\mathbb{Z}_6$ has order 4, no coset in the 1st column has order 4. For the 2nd column, we check directly: $2\cdot(0,1)\in\langle(0,2)\rangle$, $6\cdot(1,1)\in\langle(0,2)\rangle$, $6\cdot(2,1)\in\langle(0,2)\rangle$, $2\cdot(3,1)\in\langle(0,2)\rangle$, $6\cdot(4,1)\in\langle(0,2)\rangle$, & $6\cdot(5,1)\in\langle(0,2)\rangle$ (these are the minimal multiples which are in $\langle(0,2)\rangle$).

Thus none of the elements of $\mathbb{Z}_6\times\mathbb{Z}_8/\langle(0,2)\rangle$ has order 4, & so $\mathbb{Z}_6\times\mathbb{Z}_8/\langle(0,2)\rangle \simeq \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$.