MATH 761-SYLLABUS- FALL 2014

- **Course** : Math 761. It meets TR at Van Vleck 131 from 9:30 am to 10:45 pm.
- **Texts**: *Lectures on Differential Geometry* by Shlomo Sternberg, Prentice-Hall Inc.; and *A comprehensive introduction to Differential Geometry*, by Michael Spivak, V. I, Publish or Perish Inc. 2005. I will follow Spivak and shift to Sternberg's occasionally for topics Spivak does not cover (like Hamiltonian mechanics, or Lie groups if we have time).
- **Lecturer** : Prof. Gloria Marí Beffa, maribeff@math.wisc.edu
- **Office**: Van Vleck Hall 309/218, office phone #: 263-1634
- **Office hours**: T 11-12 am, R 4-5 pm or by appointment.

Math 761 is an introduction to Differential Geometry and a preparatory class for the Geometry preliminary exam. The following is a rough description of lectures (more for my own discipline than your information...). You will need to come to class if you want to know what we are covering. The dates of midterms are fixed, you will be able to take them home and complete them. You will be assigned homework in class and I will collect them every 2-3 weeks. I expect students to at least try solving the problems to get computational skills and a better understanding.

Lectures

1. Manifolds: definition, examples and review of the basics.
2. Non trivial examples. Manifolds with boundaries.
3. Differentiable structures on a manifold: differentiable maps, rank, regularity, diffeomorphisms.
4. Sard's Theorem. Immersions and imbeddings.
5. Submanifolds. Partitions of Unity.
6. Tangent vectors and the differential.
7. Vector bundles. The tangent bundle.
8. Properties of the tangent bundle. Vector fields.
9. Orientation of vectors, orientable manifolds and preservation of orientation.
10. The cotangent bundle. The differential map. Tensors and tensor bundles.
11. Covariant, contravariant and mixed tensors. Contractions.
12. Integration of vector fields: existence and uniqueness. Contraction maps.
13. Integration (cont.).
14. Lie derivative. Lie brackets.
15. Lie brackets (cont.). Commuting vector fields.
16. Distributions and integral manifolds. Integrable distributions.
17. Frobenius Theorem. Foliations and global picture.
18. Alternating tensors and wedge products. Forms.
19. Differentials. A second version of Frobenius Theorem.
20. Closed and exact forms. Poincare's Lemma.
21. Hamiltonian structures.
22. Introduction to integration: the integral of n-forms over n-cubes.
23. Chains and boundaries. Closed chains.

24. Stoke's Theorem (first version on chains).
25. Orientation of a manifold with boundary. Stoke's Theorem.
26. De Rham Cohomology, examples.
27. Lie Groups, introduction and examples.
28. Invariant fields, Lie algebras and structure equations. Homogeneous spaces.

First midterm is October 13, due October 20. Second midterm is November 17, due November 24. Final exam is to be announced.

<u>9/2 - Diff. Mflds</u>

Assume M is a Hausdorff sp. (top. sp. for which separate pts have disj. nbhds) w/ a countable basis. [ex in $\mathbb{R}^n$, take $\{B_{1/n}(q) \mid q$ rat'l pt $\}$; countable]

<u>Def</u>: M is an <u>n-mfld</u> if for <u>any</u> pt $p \in M$, $\exists$ (x, U), $p \in U \subset M$ open, $x: U \to \mathbb{R}^n$ a homeomorphism (cts w/ cts inverse). n is called the <u>dimension</u> of the mfld & (x, U) are called the <u>coordinates around</u> p.

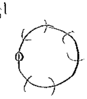
<u>Def</u>: M is an <u>n-mfld w/ boundary</u> if for any pt $p \in M$, $\exists$ (x, U) s.t. x is a homeo. to either $\mathbb{R}^n$ or $\mathbb{H}^n$, where $\mathbb{H}^n$ is the closed half-sp, $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n;\ x^n \geq 0\}$. The pts for which $\mathbb{H}^n$ is chosen form the boundary (not to be confused w/ the top. boundary - not nec. same - depends on the topology chosen)

<u>ex</u>: M^n = $n \times n$ matrices, w/ entries ≥ 0. Those w/ entries = 0 are the boundary.

<u>Ex</u>: (1) $\mathbb{R}^n$

(2) S^1 w/ top. induced by $\mathbb{R}^2$ (always need sp. & topology)
Consider the parametrization $\Phi: (-\pi, \pi) \to S^1$

$$\Phi(\theta) = (\cos\theta, \sin\theta)$$

S^1  $\xleftarrow{\Phi}$ $(-\pi, \pi) \approx \mathbb{R}$

x_1

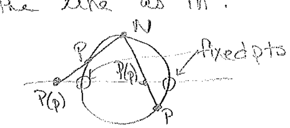
Φ is a homeo & can choose $x_1 = \Phi^{-1}$, $U_1 = S^1 - \{(-1, 0)\}$

If we use $\hat{\Phi}: (-\pi, \pi) \to \mathbb{R}$, $\hat{\Phi}(\theta) = \Phi(\theta + \pi)$, & $x_2 = \hat{\Phi}^{-1}$; $U_2 = S^1 - \{(1, 0)\}$.

We cover S^1 w/ $(x_1, U_1), (x_2, U_2)$.

* If you have a parametrization of your set, you probably have a mfld - just invert it.

<u>A 2nd coord system</u>: Define the <u>stereographic projection</u> from the North pole to be the map that assoc. to each pt on circle to one on the line as in:



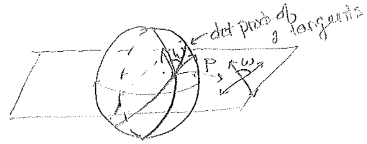
This gives coord sys for all pts but N. To get N, project from south pole. Works for $P: S^n - \{N\} \to \mathbb{R}^n$ $\forall n$.

$\boxed{\text{HW}}$ ① write down eqns (use $\vec{v}$, not $\vec{v} = (v_1, \dots, v_n)$)
↳ explicit formula for S^n & prove it is

a conformal map, i.e. it preserves angles.

That is, if we choose 2 intersecting large circles on the sphere w/ an $\measuredangle\ \omega$, the 2 str. line images of the circles via P also form an $\measuredangle\ \omega$.

← choose 2 that project to lines – those that pass through north pole.
(still well-def. w/ circle & line – use tangents)

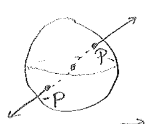


(3) M, N 2 mflds of dim's n & m. Then $M \times N$ is a mfld of dim $m+n$, w/ coordinates given by $U \times V$ & $\chi_M \times \chi_N$ a homeo. btwn $U \times V$ & $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ ⇒ product of mflds is a mfld.

Thus, $T^n = S^1 \times \cdots \times S^1$ is also a mfld.

(4) Projective space $\mathbb{RP}^n = \{$sp of lines in $\mathbb{R}^{n+1}$ through origin$\}$, i.e.
$\mathbb{RP}^n = \{[p], p \in \mathbb{R}^{n+1}, p \neq 0, [p] = [q]$ iff $p = \lambda q, \lambda \in \mathbb{R}\}$.

Topology: $\mathbb{RP}^n = S^n/\sim$, where $p \sim q \iff p = -q$



→ from top. in S^n, induced from $\mathbb{R}^{n+1}$

– Consider all sets that have p but not $-p$, & use these to generate the topology

Coordinates: If $p = (p_i)$ w/ $p_n \neq 0$, then we define the coord's around p to be given by the affine coordinates $\pi(p) = \hat{p} \in \mathbb{R}^n$, where $p/p_n = (\hat{p}, 1)$. If $p_n = 0$, some $p_i \neq 0$, & use $p/p_i = (\hat{p}_1, \dots, \hat{p}_{i-1}, 1, \hat{p}_i, \dots, \hat{p}_n)$.

(4a) $M^n = n\times n$ matrices is a mfld w/ coord's the entries

(5) Grassmanian $Gr(k,n)$ → subsp. of dim k through origin 7b1
i.e. $\mathbb{RP}^{n-1} = Gr(1,n)$ ↳ extension of $\mathbb{RP}^n$

It is the set of k-dim'l linear subsp's in $\mathbb{R}^n$, i.e. one pt in $Gr(k,n)$ is a set of k ind. vectors.

Topology: Identify a pt in $Gr(k,n)$ w/ the orthogonal proj. from $\mathbb{R}^n$ to it. [Find orth compl. to your subsp. $\{v_1,\dots,v_k\}$: $\{v_1,\dots,v_k,u_1,\dots,u_{n-k}\}$. Take pt $\in\mathbb{R}^n$ & write it in terms of this basis, then drop the last $(n-k)$-coords]. This is a linear map, so if we fix a basis in $\mathbb{R}^n$, the map is described by a matrix. Then top. is one induced by this matrix rep (as a subsp of $\mathbb{R}^{n^2}$).

$\boxed{\text{HW}}$ ② Define coords in $Gr(k,n)$ by choosing a basis for the subsp.
Hint, each $\vec{v}_i$ an n-vector, so put them as columns, & get $n\times k$ matrix. These cannot be coord's b/c not a homeo — have a lot of choices. Could have diff. basis: but it would be $(v_1 \cdots v_k)B_{k\times k}$. Try to find identity matrix by multiplying on right by sq. matrix. ↳ $k\times k$ block

Same, if $k=1$: $\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}(1/p_n) = \begin{pmatrix} \vdots \\ \boxed{1} \end{pmatrix}$. rest will be coord's.
I

What is the dim of $Gr(k,n)$?

[$Gr(k,n)$ are examples of flag mflds.]

③ Use the orthogonal complement to show that $Gr(2,3)$ is equivalent to $\mathbb{RP}^2$.

Note: The $Gr(k,n)$, S^n, $\mathbb{RP}^n$ are all examples of homogeneous sp's, those of the form G/H, G a gp, H a subgp.

ex: S^n: fix North pole. Then each pt in S^n can be id'ed w/ a rotation taking pt to N, a gp. $=G$ But there are multiple ways to rotate — they differ by a rotation that leaves N invariant $= H$.

$S^n \cong SO(n)/SO(n-1)$.

Differentiable Structures

If $f: M \to \mathbb{R}$, is f differentiable?

Def: Let $x: U \to \mathbb{R}^n$ & $y: V \to \mathbb{R}^n$ be 2 homeo's, $U, V \subseteq M$ open, $U \cap V \neq \emptyset$.
We say x & y are C^∞ related if the maps

$x \circ y^{-1}: y(U\cap V) \to x(U \cap V)$, $y \circ x^{-1}: x(U\cap V) \to y(U\cap V)$ are C^∞.
(open sets in $\mathbb{R}^n$) [could use analytic or C^k]

$$\begin{tikzcd} & U\cap V \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathbb{R}^n \arrow[rr, shift left, "y\circ x^{-1}"] & & \mathbb{R}^n \arrow[ll, shift left, "x\circ y^{-1}"] \end{tikzcd}$$

$x \circ y^{-1}$ & $y \circ x^{-1}$ are called the transition fcns (transition from x- to y-coords, or vice versa).
A family $\mathcal{A}$ of C^∞-related homeo's that cover M is called a differentiable atlas for M.

non-ex: $\mathbb{R} \to \mathbb{R}$, $x \mapsto x$ & $x \mapsto x^3$ coords. one transition fcn is $\sqrt[3]{x}$, which is not C^∞.

ex: If have S^1 w/ $(\cos\theta, \sin\theta)$ & $pt \mapsto x$, need map $\theta \mapsto x$ to be diff.

Then, if a map is diff in one coord, it will be in all others.
An elt $(x, U) \in \mathcal{A}$ is called a chart or a coord system, & $x(p)$ are the coords of p.

Def: The pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a max'l atlas (contains all possible C^∞-rel. coords), is called a C^∞ (smooth, differentiable) manifold. The max'l atlas is said to define a differentiable structure.

9/4 - Diff. Mflds

If a mfld has ∂, we say a map $f: \mathbb{H}^n \to \mathbb{R}^n$ is differentiable if we can extend it to a diff. fcn on an open nbhd of $\mathbb{H}^n$

Def: Two smooth mflds $(M, \mathcal{A})$ & $(N, \mathcal{B})$ are diffeomorphic if $\exists$ map $f: M \to N$ (called a diffeomorphism) which is 1-1 & s.t. $g \in \mathcal{B} \Leftrightarrow g \circ f \in \mathcal{A}$

ex: $(\mathbb{R}, \mathcal{A}) \xrightarrow[x \mapsto x^3]{f: \mathbb{R} \to \mathbb{R}} (\mathbb{R}, \mathcal{B})$ & any fcn that's C^∞-rel. to φ is smth C^∞-rel. to $\tilde{\varphi}$ comp. w/ f^{-1}.

$\varphi \in \mathcal{A}$, $\varphi(x) = x$; $\tilde{\varphi} \in \mathcal{B}$, $\tilde{\varphi}(x) = x^3$

Thus, f is a diffeo.

Note: f^{-1} not C^∞, so analytically not a diffeo, b/c in analysis have same diff. structure on both $\mathbb{R}$'s - here, we are allowed to change diff. structure on target $\mathbb{R}$.

Def: A fcn $f: M \to N$ is differentiable if for any (x, U) in M & (y, V) in N, the $y \circ f \circ x^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow x & & \downarrow y \\ \mathbb{R}^n & \xrightarrow{y \circ f \circ x^{-1}} & \mathbb{R}^m \end{array}$$

- What if change (y, V) to $\hat{y}$? Then have $\hat{y} \circ f \circ x^{-1} = \underbrace{\hat{y} \circ y^{-1}}_{\text{diff}} \circ \underbrace{y \circ f \circ x^{-1}}_{\text{diff}}$

↳ b/c transition fcn

So, need only to check one set of coords → i.e. one cover, & rest will be by tran. fcn's diff.

Task: Prove 1-5 in pg. 31 of Spivak, for practice

Def: Let $f: M \to \mathbb{R}$ be diff, (x, U) coords at $p \in M$. We call

$$\frac{\partial f}{\partial x^i}(p) = D_i(f \circ x^{-1})(x(p))$$

the i^{th} partial derivative of f at p.

Note: coords. are giving you directions on the mfld so can find rate of change in those directions.

$$\begin{array}{ccc} p \in M & \xrightarrow{f} & \mathbb{R} \\ \downarrow x & \nearrow_{f \circ x^{-1}} & \\ x(p) \in \mathbb{R}^n & & \end{array}$$

Prop (Change of Variable): Let (x,U) & (y,V) be 2 coord. systems on M, let $f: M \to \mathbb{R}$ be diff. Then

$$\frac{\partial f}{\partial y_i}(p) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p) \frac{\partial x_j}{\partial y_i}(p)$$

← def. above w/ x instead of f & y instead of x.

* be sure your fcns go from $\mathbb{R}^n \to \mathbb{R}^m$

Pf: $\frac{\partial f}{\partial y_i}(p) = D_i(f \circ y^{-1})(y(p)) = D_i(\underbrace{f \circ x^{-1}}_{\mathbb{R}^n \to \mathbb{R}} \circ \underbrace{x \circ y^{-1}}_{\mathbb{R}^n \to \mathbb{R}^n})(y(p))$

$$\overset{\text{chain rule in } \mathbb{R}^n}{=} \sum_{j=1}^{n} D_j(f \circ x^{-1})(x(p))\, D_i(x_j \circ y^{-1})(y(p))$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p) \frac{\partial x_j}{\partial y_i}(p) \quad \square$$

We can write this as

$$\ell_p = \left.\frac{\partial}{\partial y_i}\right|_p = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i}(p) \left.\frac{\partial}{\partial x_j}\right|_p .$$

. w/o the p, we have a diff. operator (i.e., a derivation), ℓ.

Def: Let $f: M \to N$ be a smooth fcn. We say the <u>rank</u> of f at p is equal to the rank of the $m \times n$ matrix w/ $(i,j)^{th}$ entry

$$\frac{\partial y^i \circ f}{\partial x_j}(p) = D_j(\underbrace{y^i \circ f \circ x^{-1}}_{i^{th} \text{ comp of map } \mathbb{R}^n \xrightarrow{y \circ f \circ x^{-1}} \mathbb{R}^m})(x(p))$$

where x are coords around p & y are coords around $f(p)$.
(essentially, the rank of Jacobian matrix of $y \circ f \circ x^{-1}$)
If f has rank less than $\dim N = m$ at p, we say p is a <u>critical point</u>. Otherwise p is a <u>regular point</u>. If all $p \in f^{-1}(q)$ are regular pts, q is called a <u>regular value</u>. Otherwise, q is a <u>critical value</u>.

Thm from analysis: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a C^1 fcn, [If $n<m$, then all pts are critical] : Then critical values have measure zero. (Sard's Thm).

<u>Sard's Thm</u>: Let $f: M \to N$ be a C^1 map. Assume M has at most countably many c'd components. Then the critical values of f have measure 0 in N.

- When does a subset of N have measure 0?

1. A subset of $\mathbb{R}^n$ has measure 0 if for any ε, $\exists$ a countable set of Balls $\{B_i\}$ covering the sets w/ $\sum_i V(B_i) < \varepsilon$.
2. A subset A of a mfld N has measure zero if $\exists$ a countable sequence of coords (x_i, U_i) s.t. $A \subset \bigcup_i U_i$ & $x(A \cap U_i) \subseteq \mathbb{R}^n$ has measure 0.

w/ this def, Sard's thm will be true whenever it is true for $N = \mathbb{R}^m$ & $M = \mathbb{R}^n$. We will assume this is the case.

- What does a fcn look like in coordinates depending on the rank? It depends on whether it has constant rank around a pt or not.

<u>Thm</u>: Let $f: M \to N$ be a smooth fcn.

1. If f has rank k at p, then $\exists$ coordinates (x, U) & (y, V), $p \in U$, $f(p) \in V$ s.t.
$$y \circ f \circ x^{-1}(a_1, \ldots, a_n) = (a_1, \ldots, a_k, \phi^{k+1}(a), \ldots, \phi^m(a))$$
where ϕ^i are smooth & other than order y can be any coord sys.
2. If the rank is k in a nbhd of p, then coords exist s.t.
$$y \circ f \circ x^{-1}(a_1, \ldots, a_n) = (a_1, \ldots, a_k, 0, \ldots, 0).$$
[i.e. it is just a projection w/ the right coordinates)

<u>Pf</u>: We can assume $M = \mathbb{R}^n$ & $N = \mathbb{R}^m$, b/c this is a <u>local</u> thm.

(1) Let us choose (u, U) & (y, V) around p & $f(p)$, ordered s.t. $\det\left(\frac{\partial(y^i \circ f)}{\partial u^j}\right) \neq 0$, $i,j = 1, \ldots, k$, i.e. the top left $k \times k$ block of Jacobian matrix, which has rank k, & therefore has a $k \times k$ minor w/ det $\neq 0$ $\Rightarrow$ so reorder.

Define $x^i = y^i \circ f$, $i = 1, \ldots, k$; $x^i = u^i$, $i = k+1, \ldots, n$. Is this a coord sys? Is it C^∞-rel. to u? In these coords, is our fcn what we want?

$$= \begin{vmatrix} \frac{\partial y_1 \circ f}{\partial u_1} & \frac{\partial y_1 \circ f}{\partial u_2} & \cdots \\ \vdots & & \end{vmatrix}$$

The change of variable matrix is given by

$$\left(\frac{\partial x^i}{\partial u^j}(p)\right) = \begin{pmatrix} \left(\frac{\partial y^i \circ f}{\partial u^j}(p)\right) & * \\ 0 & I \end{pmatrix} \quad \text{\& } \frac{\partial x^i}{\partial u_j} = D_j(\underbrace{x^i \circ u^{-1}}_{\text{trans. fcn}})$$

↑ Jacobian of transition fcn — it has full rank at $u(p)$ by our choice of x^i.

Thus, by inverse fcn thm, this ($x \circ u^{-1}$) a diffeomorphism (i.e. C^∞ w/ C^∞ inv). ↳ the transition fcn. So x & u are C^∞-rel. & x is a coord sys b/c $x = \underbrace{x \circ u^{-1}}_{ok} \circ \underbrace{u}_{ok}$ → i.e. homeo.

[General method - to show smth a coord system, find det of Jacobian]

In these coords, what does fcn look like?

$$y \circ f \circ x^{-1}(a^1, \dots, a^n) = y \circ f(q) \quad \text{w/ } x(q) = a, \text{ or } x^i(q) = a^i.$$

Thus $y^i \circ f(q) = a^i$ for $i = 1, \dots, k$, by def. of x^i $(= y^i \circ f)$, &

$$y \circ f \circ x^{-1}(a^1, \dots, a^n) = y \circ f(q) = (a^1, \dots, a^k, \underbrace{*, \dots, *}_{\text{can't say anything. Just smth fcns.}}).$$

(2) Choose x & v as in previous case, s.t. $v \circ f \circ x^{-1}(a) = (a^1, \dots, a^k, \phi^{k+1}(a), \dots, \phi^m(a))$.

The rank of f is the rank of the matrix

$$\begin{pmatrix} I & 0 \\ * & \left(\frac{\partial \phi^i}{\partial a^j}\right) \end{pmatrix} \quad \text{w/ } i = k+1, \dots, m, \text{ \& } j = k+1, \dots, m.$$

↑ this block must be zero, b/c know rk $f = k$. ↳ in a nbhd of p

So ϕ^i's only depend on $a^1, \dots, a^k$.

So we change v. Choose

$$y^i = v^i, \ i = 1, \dots, k, \quad y^i = v^i - \phi^i \circ v, \ i = k+1, \dots, m.$$

Check this is a coord sys: Find rk of Jacobian:

$$\left(\frac{\partial y^i}{\partial v^j}\right) = \begin{pmatrix} I & 0 \\ * & I \end{pmatrix}.$$ So full rank ⇒ y a coord sys, C^∞-rel. to v.

↑ b/c $\phi^i \circ v$ depends only on $v^1, \dots, v^k$ (as ϕ^i dep only on $a^1, \dots, a^k$)

In terms of these coords, the fcn is

$$y \circ f \circ x^{-1}(a) = y \circ v^{-1} \circ \underbrace{v \circ f \circ x^{-1}}_{\text{given above}}(a) = y \circ v^{-1}\underbrace{(a^1, \dots, a^k, \phi^{k+1}(a), \dots, \phi^m(a))}_{=q}$$

Call $v(q) = (a^1, \dots, a^k, \phi^{k+1}(a), \dots, \phi^m(a))$

Then $y^i(q) \underset{\text{def.}}{=} v^i(q) = a^i$, $i = 1, \dots, k$.

$$y^i(q) = v^i(q) - \phi^i(v(q)) = \phi^i(a) - \phi^i(a) = 0 \quad , \quad i = k+1, \dots, m$$

$v(q) \neq a$, but same 1^{st} k comp. ϕ^i dep. only on 1^{st} k comp, so $\phi^i(v(q)) = \phi^i(a)$

So, $y \circ f \circ x^{-1}(a) = (a^1, \dots, a^k, 0, \dots, 0)$. $\square$

* If full rank, then M really a subset of N, in right coords.
$(m > n)$

If $n > m$, then proj. in 1^{st} m coords.

9/9 - Diff. Mflds $\quad f: M^n \to N^m$, $p \to f(p)$

Cor: 1. If $m \le n$ & rk $f = m$ at p, then for any coord. around $f(p)$ $\exists$ one around p s.t.

$$y \circ f \circ x^{-1}(a^1, \dots, a^n) = (a^1, \dots, a^m)$$

2. If $n \le m$ & f has rk n at p, then for any coord. around p, $\exists$ one around $f(p)$ s.t.

$$y \circ f \circ x^{-1}(a^1, \dots, a^n) = (a^1, \dots, a^n, 0, \dots, 0)$$

i.e., only need to change one coord (around the one w/ higher rk) to get this form. Note: This is for max'l rk fcns.

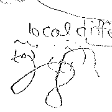
Note: Can we assume these rk's in nbhd of pt? Yes.
Can't go up b/c max'l. Can't go down – there's a minor w/ rk $\ne 0$, so $\exists$ open nbhd where that minor $\ne 0$.
In general, rk can only increase in a nbhd (always have that $\ne 0$ minor)

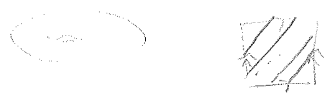
Pf: (1) Immediate from thm b/c we do not need to reorganize coords at beginning ($\frac{\partial y}{\partial x}$ already $m \times m$).

(2) The map has max'l rk in nbhd (from note). So, can find ψ & ϕ s.t. $\psi \circ f \circ \phi^{-1}(a^1, \dots, a^n) = (a^1, \dots, a^n, 0, \dots, 0)$.
But we want any x, not specific ϕ.
Assume $\psi \circ f \circ x^{-1}(a^1, \dots, a^n) = (b^1, \dots, b^n, 0, \dots, 0)$, where $x \circ \phi^{-1}(b^1, \dots, b^n) = (a^1, \dots, a^n)$.
Define $\lambda(b^1, \dots, b^m) = (x \circ \phi^{-1}(b^1, \dots, b^n), b^{n+1}, \dots, b^m)$.
Then $\lambda \circ \psi \circ f \circ x^{-1}(a^1, \dots, a^n) = \lambda \circ \psi \circ f \circ \phi^{-1}(b^1, \dots, b^n)$

$$= \lambda(b^1, \dots, b^n, 0, \dots, 0) = (a^1, \dots, a^n, 0, \dots, 0). \quad \square$$

Def: A diff. fcn $f: M \to N$ is an immersion if the rk of f is n for any pt $p \in M$.
– locally, an immersion just a proj. in right coords. locally 1-1, & locally diffeo (from inverse fcn thm) w/ image.

ex: $\mathbb{R} \to \mathbb{R}^2$ immersion.  local diffeo, only in one nbhd at a time. local diffeo w/ image, but not w/ subset of $\mathbb{R}^2$ containing image ex A flow on torus is an immersion, even if flow becomes dense.



Def: A subset $M_1 \subseteq M$ w/ a diff. structure (not nec. compatible) is called an immersed submfld of M if the inclusion $i: M_1 \to M$ is an immersion.

- Not nec. nice – may have different top. & so diff. diff. structure
  both ex's above are immersed submflds

Def: We say $f: P \to M$ is an embedding if it is a 1-1 immersion that is homeo to its image (as a subset of M)

- both ex's (2nd w/ dense flow) are not embeddings

Def: An immersed submfld is a C^∞-submfld if the inclusion is an embedding.

The prev. thm tells us that an immersed submfld is locally the level sets of some coord.

Prop: Let $f: M \to N$ have constant rk k on a nbhd of a set $f^{-1}(p)$.

ex: $f(x,y,z) = (2x+3y-z+1, y-x, 2y+z) = (\quad)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$f^{-1}(0,0,0) \leftarrow$ this dep on rk of $f \rightarrow$ rk of this

if rk $= 1$, 2-dim'l sp

$= 2$, 1-dim'l sp

$= 3$, pt

[rk + dim subsp = dim space]

Then $f^{-1}(p)$ is a closed submfld of M, w/ dim $n-k$. If p is a regular value, then $f^{-1}(p)$ is an $(n-m)$-submfld.

reg. pts have max'l rk.

$\boxed{\text{HW}}$ (3) Prove this prop. [Use the normal forms we came up w/]

Ex: Let $f: \mathbb{R}^{n+1} \to \mathbb{R}$ be $f(a) = |a|^2$.

If $a \neq 0$, $\text{rk } f = 1$ (max'l) since $\nabla f(a) = 2a$, which is $\neq 0$ if $a \neq 0$.
Thus, 1 (or 3, or 25) is a regular value & $S^n = f^{-1}(1)$ is a submfld of $\mathbb{R}^{n+1}$.
→ If fcn has max'l rk at pt, level sets of fcn is a submfld.

$\boxed{\text{HW}}$ (4) Prove that $O(n) = \{A \in GL(n,\mathbb{R}) ;\ AA^T = I\}$ is a submfld of $GL(n,\mathbb{R}) \subseteq \mathbb{R}^{n^2}$.
[$f(A) = AA^T$. Check f has full rank at I → computational — it's not a map btwn gps]

Q: Is any mfld a submfld of $\mathbb{R}^N$ for some N? Yes! But we need a lot of stuff to answer it, particularly partitions of unity.

Preliminaries:

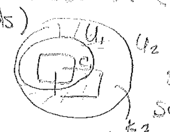
Def: Given an open cover of M, $\mathcal{O}$, we say $\mathcal{O}'$ is a refinement of $\mathcal{O}$ if $\mathcal{O}'$ is a cover of M & for any $A' \in \mathcal{O}'$, $\exists$ $A \in \mathcal{O}$ w/ $A' \subseteq A$.

[can throw away things in $\mathcal{O}$)

Def: The cover $\mathcal{O}$ is locally finite if for any pt $\exists$ nbhd that intersects finitely many members of $\mathcal{O}$.

Thm: Assume M is σ-cpt (i.e. seq of cpt sets s.t. $\bigcup = M$ — if we have a metric, it is σ-cpt) & ctd. Then, for any open cover of M, we can find a locally finite refinement.

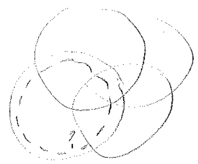
Pf: Let $C_1, \ldots, C_k, \ldots$ be cpt sets covering M. C_1 is contained in an open set w/ cpt closure U_1 (finite union of preimages of balls in $\mathbb{R}^n$? covering it - using coords)



Define U_2 to be open set w/ cpt closure containing cpt set $\overline{U}_1 \cup C_2$. Let U_i contain $\overline{U}_{i-1} \cup C_i$.
→ getting nested open sets $\{U_i\}$ w/ cpt closure & $C_i \subseteq U_i$, so they form a cover.

Assume $\exists$ cover $\mathcal{O}$. Each cpt set $K_i = \overline{U}_i - U_{i-1}$ is covered by a fin. set of elt's in $\mathcal{O}$. [Now have countable cover (fin # in each annulus, & countable # of annuli)] Take the intersection of these sets w/ $U_{i+1} - \overline{U}_{i-3}$ [controlling size of open sets]. This is a refinement, & locally the nbhd $U_{i+1} - \overline{U}_{i-3}$ will intersect only fin. many, from construction. □

<u>Thm</u> (Shrinking lemma): Let $\mathcal{O}$ be a locally fin. open cover of M.

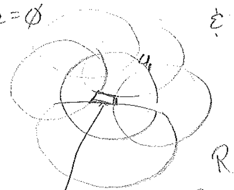


Can I shrink the cover slightly? w/ open sets whose closure is inside one of U in cover?

[Mfld has countable basis, so $\mathcal{O}$ a countable cover]

Then, for any $U \in \mathcal{O}$ we can choose U' w/ $\overline{U'} \subset U$ & s.t. $\mathcal{O}' = \{U'\}$ also forms an open cover.

<u>Pf</u>: Let $C_1 = M - (U_2 \cup \dots) = U_1 - (U_2 \cup \dots)$ which is a closed set. By Urysohn's Lemma, we can separate C_1 & U_1^c w/ 2 open sets V_1 & V_2 s.t. $V_1 \cap V_2 = \emptyset$ & $U_1^c \subset V_1$, $C_1 \subset V_2$. Therefore, $C_1 \subset V_1 \subset \overline{V_2} \subset U_1$ since $V_2 \subset V_1^c \subset U_1$ & V_1^c is closed. Let $V_2 = U_1'$. Clearly, $U_1', U_2, \dots$ is an open cover of M.



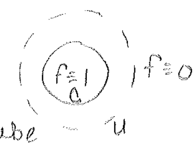
Repeat: $C_2 = U_2 - (U_1' \cup U_3 \cup \dots)$.

Separate C_2 from $= C_1$ U_2^c, & find U_2' s.t. $C_2 \subseteq U_2' \subseteq \overline{U_2'} \subseteq U_2$, & substitute U_2 w/ U_2' to still form an open cover. For finite steps, still cover. But what about ∞ steps? Still cover – or do we miss pt at end? If $p \in M$, $\exists$ max'l k s.t. $p \in U_k$, b/c $\mathcal{O}$ loc. finite, & $p \in U_1' \cup \dots \cup U_k' \cup U_{k+1} \cup U_{k+2} \cup \dots$, which is a cover, b/c only fin. many steps. Hence $p \in U_1' \cup \dots \cup U_k'$. □

9/11 - D.M.

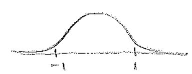
Lemma: Let $C \subseteq U$ be a cpt set contained in an open set. Then $\exists$ a C^∞ fcn $f: M \to \mathbb{R}$ w/ $f|_C \equiv 1$ & $\mathrm{supp}(f) \subseteq U$.

$[\mathrm{supp}(f) = \overline{\{p \mid f(p) \neq 0\}}]$



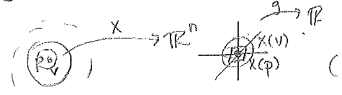
Pf: First, we claim $\exists\ g: \mathbb{R}^n \to \mathbb{R}$ which is C^∞, positive on $(-\varepsilon,\varepsilon) \times \cdots \times (-\varepsilon,\varepsilon)$ & zero outside the cube of side ε. For this, let $h: \mathbb{R} \to \mathbb{R}$ be

$$h(x) = \begin{cases} e^{-(x+1)^{-2}} e^{-(x-1)^{-2}} & x \in (-1,1) \\ 0 & x \notin (-1,1) \end{cases}$$



& define $g(x^1, \dots, x^n) = h(x^1/\varepsilon) \cdots h(x^n/\varepsilon)$.

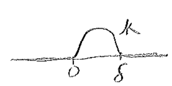
Next, for every $p \in C$, we choose coords (x, V) w/ $x \in V \subset \overline{V} \subset U$, & $x(p) = 0$



Consider $x^{-1}((-\varepsilon,\varepsilon)^n)$, for an ε s.t. $(-\varepsilon,\varepsilon)^n \subset x(V)$. Then the fcn $g \circ x$ is C^∞, positive on $x^{-1}((-\varepsilon,\varepsilon)^n)$ & zero outside. We extend it by zeros elsewhere & call it $f_p: M \to \mathbb{R}$. But C is cpt, so can cover C w/ a finite # of V's, V_{p_i}, $i = 1, \dots, k$ & consider $\hat{f} = f_{p_1} + \cdots + f_{p_k}$. $\hat{f}$ is positive on C, the $x_i^{-1}((-\varepsilon,\varepsilon)^n)$'s & $\mathrm{supp}(f_{p_i}) \subseteq U \Rightarrow \mathrm{supp}(\hat{f}) \subseteq U$. Now we need $\hat{f} = 1$ on C.

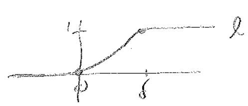
Let δ be such that $\hat{f} \geq \delta$ on C. ($\exists$ lower bound $\neq 0$ b/c C cpt)

Let $k: \mathbb{R} \to \mathbb{R}$ be positive on $(0,\delta)$ & 0 outside:



Define

$$l(x) = \frac{\int_0^x k}{\int_0^\delta k}$$



$$M \xrightarrow{\hat{f}} \mathbb{R} \xrightarrow{l} \mathbb{R}$$

l is C^∞, zero for $x \leq 0$, 1 for $x \geq \delta$. We define $f = l \circ \hat{f}$ which is 1 on C (b/c $\hat{f} \geq \delta$ on C) & zero outside U (b/c $\hat{f} = 0$ outside U) $\square$

<u>Thm</u>: Let $\mathcal{O}$ be an open loc. fin. cover of M. Then for each $U \in \mathcal{O}$, $\exists$ fcn $\phi_U : M \to [0,1]$ s.t.

(1) $\text{supp}(\phi_U) \subset U$ for each U

(2) $\sum_U \phi_U(p) = 1 \quad \forall p \in M$ (fin. sum b/c $\mathcal{O}$ loc. fin.)

<u>Pf</u>: ① Assume all $U \in \mathcal{O}$ have cpt closure. Use the shrinking lemma to obtain $U' \subseteq \overline{U'} \subseteq U$ & define $\psi_U : M \to [0,1]$ s.t. $\psi_U|_{\overline{U'}} = 1$ & $\text{supp}(\psi_U) \subseteq U$. Then

$$\sum_U \psi_U > 0 \quad \text{everywhere. (i.e, at any } p \in M\text{).}$$

It suffices to define $\phi_U = \dfrac{\psi_U}{\sum_U \psi_U}$.

② In general, we will first prove that if $C \subseteq U$ is closed, then there is $\psi_U : M \to [0,1]$ w/ $\psi_U|_C = 1$, $\text{supp}(\psi_U) \subseteq U$. For this, cover C & $M - C$ w/ open sets w/ cpt supp (i.e. cpt closure), w/ those covering C (w/ closure) inside U. Call this cover of C $\mathcal{C}$. Call whole cover $\mathcal{C}'$. Shrink $\mathcal{C}'$ (as in 1) to produce a cover $\mathcal{O}$ & define ϕ_W, $W \in \mathcal{O}$, as in case 1. Then define

$$f = \sum_{V \in \mathcal{O},\, V \subset W \in \mathcal{C}} \phi_V$$

$\{V_{p'}\}$ cov. $M - C$, $\overline{V_{p'}}$ cpt

$\mathcal{C}'$ cov. everything.

$\hookrightarrow \phi_V$ as in 1, so $\sum_{V \in \mathcal{C}'} \phi_V = 1$, so take only those in $\mathcal{C}$

$V_p \subset \overline{V_p} \subset U$, $\overline{V_p}$ cpt

$\{V_p\}_p = \mathcal{C}$

Then $f > 0$ on C, & $\text{supp}(f) \subseteq U$.

If $p \in C$, the only $V \ni p$ (loc. fin.) are those that cover C, not $M - C$, thus

for $p \in C$, $\sum_{V \in \mathcal{C}} \phi_V = \sum_{V \in \mathcal{C}'} \phi_V = 1$

(b/c all other $\phi_V = 0$, b/c $p \in V$ if $V \in \mathcal{C}' \setminus \mathcal{C}$)

Thus, $f(p) = 1$ if $p \in C$ & $\text{supp}(\phi_V) \subseteq V \subseteq U$. Then repeat pf for 1, using f. $\square$

<u>Cor</u> (Existence of partitions of unity): Let $\mathcal{O}$ be an open cover of M. Then $\exists$ a sequence of C^∞ fcns $\phi_i : M \to [0,1]$ s.t.

(1) $\{p \mid \phi_i(p) \neq 0\}$ is loc. fin.

(2) $\sum_i \phi_i(p) = 1 \quad \forall p \in M$

(3) For each i $\exists U$ w/ $\text{supp}(\phi_i) \subseteq U$.

2/3

The pf is simple: find a refinement of the cover by a loc. fin open cover & apply prev. thm.

The family $\{\varphi_i\}$ is called a $\underbrace{\text{partition of unity subordinated to } \mathcal{O}}_{\text{refers to (3).}}$

Thm: If M is a cpt C^∞ mfld, then $\exists$ an embedding $M \to \mathbb{R}^N$ for some N.

Pf: Cover M w/ coord. sys's $(x_1, U_1), \ldots, (x_r, U_r)$, shrink the cover to U_i' & let ψ_i be the fcns w/ $\text{supp}(\psi_i) \subset U_i$ & $\psi_i|_{\overline{U_i'}} = 1$. Define $f: M \to \mathbb{R}^{nr+r}$

$$f = (\underbrace{\psi_1 x_1}_{\text{pt in } \mathbb{R}^n \cdot \# \to \mathbb{R}^n}, \ldots, \underbrace{\psi_r x_r}_{\mathbb{R}^n}, \underbrace{\psi_1}_{\mathbb{R}}, \ldots, \underbrace{\psi_r}_{\mathbb{R}}) \quad \Big\} \text{ so } f: M \to \mathbb{R}^{nr+r}$$

1. f has full rk everywhere & it is an immersion b/c for each p, $\exists$ $\psi_i = 1$ in nbhd of p b/c $p \in U_i'$ for some i. So that part of f is just x_i, & that has nontriv. Jacobian $\to$ i.e. get block of dim n.

   dim Jacobian is $n \times (nr+r)$, so max rk $= n$.
   $= \begin{pmatrix} \text{Jac } \psi_1 x_1 \\ \text{Jac } \psi_2 x_2 \end{pmatrix}$

2. f is 1-1. Suppose $f(p) = f(q)$. Then $\psi_i(p) = \psi_i(q)$ $\forall i$. If $p \in U_i'$ & $q \in U_i$, b/c $\psi_i(q) = 1$ & $\text{supp}(\psi_i) \subseteq U_i$. Then look at 1st r-coords: $x_i(p) = x_i(q)$, so $p = q$ (b/c x_i is 1-1).

$\boxed{\text{HW}}$ 3. Prove f is a homeo w/ its image. Hint: Is this locally an embedding? [this is enough to say globally an emb (w/ 1 & 2)]

$\square$

$\boxed{\text{HW}}$ 1. Let $g: S^n \to \mathbb{R}P^n$ be the map $p \to [p]$. Show that $f: \mathbb{R}P^n \to M$ is C^∞ $\Leftrightarrow$ $f \circ g: S^n \to M$ is C^∞. Compare rk f w/ rk$(f \circ g)$.

Note: this means diff structure of $\mathbb{R}P^n$ comes from that of S^n

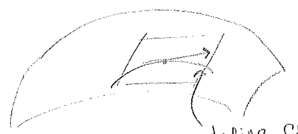
2. Let $f: \mathbb{R}P^2 \to \mathbb{R}^3$ be the map $g([x,y,z]) = (yz, xz, xy)$, whose image is the Steiner surface. Show that g is not an immersion at exactly 6 pts.

   [cannot diff wrt x, y, z, b/c they are ~~not~~ not coords. Need only 2 b/c $\mathbb{R}P^2$ is 2-dim'l]

TANGENT BUNDLE

How does one define the tangent to a mfld? If the mfld is in $\mathbb{R}^N$, it is clear how. Otherwise, it is not so clear.

Surfaces:



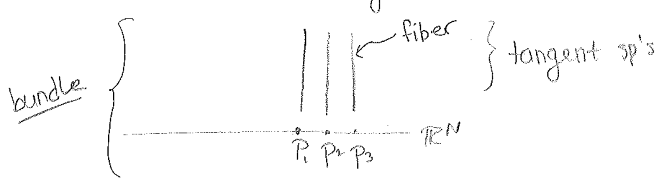
$M \overset{i}{\hookrightarrow} \mathbb{R}^N$, & then define a curve & take tangent, then (T_pM) pull it back to M.

↳ how do you do this, b/c M on left is not embedded anywhere.

Also, N not unique – what if choose diff. N?

And what is a tangent?

Start w/ $\mathbb{R}^N$, then move to mflds. We want to define the tangent "space" that contains all tangent vectors.



On M, have $p \in M$ & curve $\alpha : (-\varepsilon, \varepsilon) \to M$ s.t. $\alpha(0) = p$. Then take $\alpha'(0)$ – but what does this mean? we can diff fcns $M \to \mathbb{R}$, but what about $\mathbb{R} \to M$? Also, many diff α_i's may have same $\alpha'(0)$. So we need to define some equivalence class of curves.

<u>Def</u>: A <u>vector ^{n} bundle</u> is a quintuple $(E, B, \pi, +, \cdot)$, where E, B are mflds, $\pi : E \to B$ is C^∞ & onto, & $\pi^{-1}(p)$ (called the <u>fiber</u>) has the structure of a vector sp $+, \cdot$ of dim n. B is called the <u>base sp</u>, E the <u>total sp</u>. Also, we assume that for any pt $q \in E$, $\exists$ open set $V \ni q$, $p = \pi(q) \subseteq U$ s.t.

$V \cong U \times \mathbb{R}^n$ (<u>local triviality condition</u>)

topologically & smooth

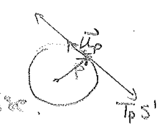
We say a bundle is <u>trivial</u> if $E = B \times \mathbb{R}^n$.

9/16 - Diff. Mflds

Def: A n-vector bundle is a family $(E, B, \pi, +, \cdot)$, where (1) E, B are top sp's,
(2) $\pi: E \to B$ is cts
(3) $\pi^{-1}(p)$ is a vect. sp of dim n, w/ operations $+, \cdot$
(4) $\exists$ a cover $\{U\}$ of B & maps $t_u: \pi^{-1}(U) \to U \times \mathbb{R}^n$, called the trivializations of the vect bundle, s.t. t_u is a homeo & $t|_{\pi^{-1}(p)}: \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ is an isomorphism (1-1 & linear)

Ex: Trivial bundle. $\begin{array}{c} B \times \mathbb{R}^n \\ \downarrow \pi \\ B \end{array}$ for any top. sp B.

Def: Two bundles over B, E_1, E_2, are equivalent if $\exists$ map (the equivalency) $f: E_1 \to E_2$ s.t. f a homeo & $f: \pi_1^{-1}(p) \to \pi_2^{-1}(p)$ is an isomorphism. If a bundle is equivalent to $B \times \mathbb{R}^n$, we say it is trivial.

Ex: (1) $TS^1 = \bigcup_{s \in S^1} T_p S^1$, where $T_p S^1$ is . Let $\vec{u}_p$ be the unit vector $\perp p$, counterclockwise.

Then $T_p S^1 = \{\lambda \vec{u}_p \mid \lambda \in \mathbb{R}\}$. Define

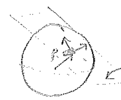
$$f: TS^1 \to S^1 \times \mathbb{R}$$
$$\vec{v}_p \longmapsto (p, \lambda), \quad \vec{v}_p = \lambda \vec{u}_p$$

- $\pi: \bigcup_p T_p S^1 \to S^1$
  $\vec{v}_p \to p$
  (cts w/ this top)

But what's the top on TS^1? Choose it s.t. f is cts, i.e. the open sets are the preimages of open sets in $S^1 \times \mathbb{R}$. With this top, f is 1-1 & a homeomorphism. This makes π cts (b/c $\pi = (\text{proj}) \circ f$, both cts).

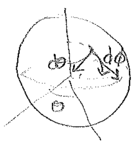
$$f: T_p(S^1) \to \{p\} \times \mathbb{R}$$
$$\vec{v}_p + \hat{\vec{v}}_p \to \lambda + \hat{\lambda} \Rightarrow f \text{ linear on } \pi^{-1}(p). \text{ (\& 1-1, etc).}$$

Thus, TS^1 is trivial.

(2) $TS^2 = \bigcup_p T_pS^2$, T_pS^2 is  , $\pi: TS^2 \to S^2$, $\vec{v}_p \to p$

This is locally trivial:
take basis at p – can move this basis smoothly to pts nearby. (i.e. choose smooth family) – 2 tan. vectors: differentiate in direction of θ & dir. of ϕ, from spherical coords, these 2 vectors are the basis.

Then $T_pS^2 = \{v_p = \lambda u_p^1 + \mu u_p^2 \mid \lambda, \mu \in \mathbb{R}\}$ $(p \in U)$

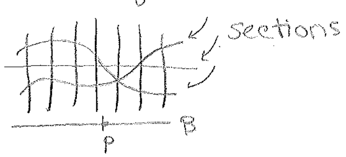


$t: \pi^{-1}(U) \to U \times \mathbb{R}^2$
$v_p \longmapsto (p, \lambda, \mu)$ $\Rightarrow$ If have a smooth family of gen's, we can always locally trivialize.

Thus TS^2 is a vector bundle.

Assume TS^2 is trivial: $TS^2 \xrightarrow{f} S^2 \times \mathbb{R}^2$. Then, if $v \neq 0$ & $v \in \mathbb{R}^2$, consider $f^{-1}(S^2 \times \{v\})$ [i.e. pulling back v to each pt of S^2] & $f^{-1}(\{p\} \times \{v\}) \neq 0$. This gives a smooth vector field on S^2, which contradicts the hairy ball thm. Thus f DNE.

Visualization of bundle:



Def A section of a vector bundle (E, B, π) is a continuous map $s: B \to E$ s.t. $\pi \circ s(p) = p$. [i.e. $s(p) \in \pi^{-1}(p)$]

*For $B = S^2$, every section of the tangent needs to vanish somewhere, to not contradict hairy ball thm.

Ex: (3) $B = \mathbb{R}^n$, $\widetilde{T\mathbb{R}^n} = \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$, where $T_p\mathbb{R}^n = \{\alpha'(0) \mid \alpha: (-\varepsilon, \varepsilon) \to \mathbb{R}^n, \alpha(0) = p\}$.

$T_p\mathbb{R}^n$ is a vector sp (can add α's, $(\alpha+\beta)'(0) = \alpha'(0) + \beta'(0)$, $(\lambda\alpha)'(0) = \lambda\alpha'(0)$)

$\widetilde{T\mathbb{R}^n} = \mathbb{R}^n \times \mathbb{R}^n$ (i.e. trivial), Top. of $\widetilde{T\mathbb{R}^n}$ is top. of $\mathbb{R}^n \times \mathbb{R}^n$

$v_p \to (p,$ coords of v_p w/ standard basis$)$ *If $f: \mathbb{R}^n \to \mathbb{R}^m$, $\exists$ an induced map $f_*: T\mathbb{R}^n \to \widetilde{T\mathbb{R}^m}$, $f_*(\alpha'(0)) = (f \circ \alpha)'(0) \in T_{f(p)}\mathbb{R}^m$. (i.e. mult α' by [illegible])

Thm: For every n-mfld M, $\exists$ an n-vector bundle TM over M (called the tangent bundle) & a bundle map (f_*, f) for any map $f: M \to N$, M, N mflds.

Def: A bundle map (f, g) from (E_1, B_1, π_1) to (E_2, B_2, π_2) is a pair of maps $f: E_1 \to E_2$, $g: B_1 \to B_2$ s.t. the diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

, and $f: \pi_1^{-1}(p) \to \pi_2^{-1}(g(p))$ is linear.

$v = \alpha'(0)$, $\alpha: (-\varepsilon, \varepsilon) \to M$

$$\begin{tikzcd} & M \arrow[dl, "y"'] \arrow[d, "x"] \\ \mathbb{R}^n & \mathbb{R}^n \end{tikzcd}$$

$(x \circ \alpha)'(0) = v$
$(y \circ \alpha)'(0) = \omega$ } vect's in $\mathbb{R}^n$

$\underbrace{\omega = D(y \circ x^{-1})(x(p))v}_{\text{want this to be our equivalence relation.}}$ b/c $(y \circ x^{-1} \circ \underbrace{x \circ \alpha}_{v})'(0) = \underbrace{(y \circ \alpha)'(0)}_{\omega}$

diff. & apply chain rule $\to$ get eqn on left

Pf: (a) We say $(x, v) \sim_p (y, \omega)$ if $\omega = D(y \circ x^{-1})(x(p))v$, w/ x & y coordinates around p & v & $\omega \in \mathbb{R}^n$. This is an equivalence rel. Let $[(x,v)]_p$ be the class of (x, v).

(b) Define $T_pM = \{[(x,v)]_p \mid x \text{ a coord}, v \in \mathbb{R}^n\}$.

$TM = \bigcup_{p \in M} T_pM$.

(c) $\pi^{-1}(p)$: $\to [(x,v)]_p + [(y,\omega)]_p$

$= [(x,v)]_p + [(x, D(x \circ y^{-1})(y(p))\omega)]_p$ } change so both are in same coord. Note: both coord. around p, so must be overlap of coords

$= [(x, v + D(x \circ y^{-1})(y(p))\omega)]$,

$\to \lambda[(x,v)]_p = [(x, \lambda v)]_p \Rightarrow \pi^{-1}(p)$ a vect. sp

(d) Trivializations: Fix (x, U) & define $t_x: \pi^{-1}(U) \to U \times \mathbb{R}^n$
$[(x,v)]_p \to (p, v)$

We want this to be cts, so let top. on TM be the preimages of open sets in $U \times \mathbb{R}^n$. As before, this implies π is cts.

So, we have a bundle.

(e) Bundle map: Let $f: M \to N$, & $f_*: TM \to TN$ def. by first fixing coords $\underset{\text{around } p}{(x,U)}$ & $\underset{\text{around } f(p)}{(y,V)}$, & then $f_*(\underset{\substack{\uparrow \\ \text{pick the rep} \\ \text{in } x}}{[\![x,v]\!]_p}) = [(y, D(y \circ f \circ x^{-1})(x(p))\,v)]_{f(p)}$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "x"'] & N \arrow[d, "y"] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^m \end{tikzcd}$$

Need to check f_* well-def: Assume we choose $(\hat{x}, \hat{U}), (\hat{y}, \hat{V})$.

$$\begin{aligned} f_*([\![x,v]\!]_p) &= f_*([(\hat{x}, D(\hat{x} \circ x^{-1})(x(p))\,v)]_p) \\ &= [(\hat{y}, D(\hat{y} \circ f \circ \hat{x}^{-1})(\hat{x}(p))\,D(\hat{x} \circ x^{-1})(x(p))\,v)]_{f(p)} \\ &= [(\hat{y}, D(\hat{y} \circ f \circ x^{-1})(x(p))\,v)]_{f(p)} \\ &= [(y, D(y \circ \hat{y}^{-1})\,D(\hat{y} \circ f \circ x^{-1})(x(p))\,v)]_{f(p)} \\ &= [(y, D(y \circ f \circ x^{-1})(x(p))\,v)]_{f(p)} \end{aligned}$$

Then check the following properties of bundle maps: $\square$

Properties of Bundle Maps:

(1) $(\mathbb{1}_M)_* = \mathbb{1}_{TM}$

(2) If $f: M \to N$ & $g: N \to \mathbb{R}$, then $(g \circ f)_* = \underset{\sim}{g_*} \circ f_*$.

HW $\to$ (3) $\exists$ bundle equivalence btwn $T\mathbb{R}^n$ & $\widetilde{T\mathbb{R}^n}$

(4) If $U \subseteq M$, then $TU \sim \underset{\substack{\| \\ \bigcup_{p \in U} T_pM}}{TM|_U}$, if U is open, & if $f: M \to N$, $(f|_U)_* = (f_*)|_{TU}$.

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Derivations at p: $D_pM = \{ \ell_p : \overbrace{C^\infty(M,\mathbb{R})}^{C^\infty \text{ maps } M\to\mathbb{R}} \to \mathbb{R} \mid \ell_p$ linear wrt mult. by #'s, i.e. $\ell_p(f+g) = \ell_p(f)+\ell_p(g)$ & $\ell_p(af) = a\ell_p(f)$, & $\ell_p(fg) = f(p)\ell_p(g) + \ell_p(f)g(p) \}$

- D_pM is a vector sp. What is $\dim D_pM$?

Note (1) $\ell_p(f)$ depends only on the value of f on a nbhd of p. We will see that if $f\equiv g$ on a nbhd of p, then $\ell_p(f) = \ell_p(g)$. First, $f\equiv 0$ on a nbhd of p. Then, $f^{-1}(0)$ contains a nbhd U of p. Let h be s.t. $\mathrm{supp}(h)\subseteq U$ & $h(p)=1$. Then $(hf)(q)=0 \ \forall q$.

$\underset{\substack{\uparrow \\ \text{b/c linear maps} \\ \text{take 0 to 0}}}{0 =} \ell_p(hf) = h(p)\ell_p(f) + f(p)\ell_p(h) = \ell_p(f)$.

If $f\equiv g$ on a nbhd of p, then $0 = \ell_p(\overbrace{f-g}^{0 \text{ on nbhd of } p}) = \ell_p(f) - \ell_p(g)$. ✓

(2) From here, we conclude that we can apply derivations of p to fcns that are only locally defined, $f: U\to\mathbb{R}$, $p\in U$. (b/c we can extend it using h as above)

Thm: $\dim D_pM = n$. If (x,U) are local coords, then a basis for D_pM is $\left.\frac{\partial}{\partial x_i}\right|_p$, $i=1,\dots,n$; and $\ell_p = \sum_{i=1}^n \ell_p(x_i)\left.\frac{\partial}{\partial x_i}\right|_p$.

Pf: (1) $\ell_p(\overset{\text{const fcn}}{1}) = \ell_p(1\cdot 1) = \ell_p(1)+\ell_p(1) \Rightarrow \ell_p(1) = 0$. Also, $\ell_p(c) = c\,\ell_p(1) = 0$.

(2) Assume $M=\mathbb{R}^n$ & $p=0$. We write $f \overset{(a)}{=} f(0) + \sum_{i=1}^n g_i \overset{(a)}{e_i}$ ← coords in $\mathbb{R}^n$, w/ $D_i g_i(0) = D_i f(0)$. $[e_i(v_1,\dots,v_n) = v_i]$ [We can do this from analysis: $g_i \overset{(a)}{=} \int_0^1 D_i f(tu)\,dt$]

So, $\ell_p(f-f(0)) = \ell_p(f) = \ell_p(\sum g_i e_i) = \sum_i (\ell_p(g_i)\cdot \overset{e_i(p)}{0}) + \sum_i g_i(p)\,\ell_p(e_i)$

$\Rightarrow \ell_p(f) = \sum_i \ell_p(e_i)\cdot g_i(0)$, & $g_i(0) = \int_0^1 D_i f(0)\,dt = D_i f(0)$

So for $M=\mathbb{R}^n$, we have $\ell_p(f) = \sum \ell_p(e_i)\cdot \left.\frac{\partial f}{\partial e_i}\right|_0$

(3) Now consider a general mfld M, & let $\ell_p\in D_pM$. Let

$$\begin{array}{ccc} x: \underset{p}{U} & \xrightarrow{x} & \mathbb{R}^n \\ \downarrow f & \swarrow g = f\circ x^{-1} & \\ \mathbb{R} & & \end{array}$$

w/ $x(p)=0$. Define $\ell_{\mathbb{R}^n}(g) = \ell_p(g\circ x)$. $\ell_{\mathbb{R}^n}$ is a derivation, & we apply part (2).

Thus $l_p(g\circ x) = \sum l_{\mathbb{R}^n}(e_i)\frac{\partial g}{\partial e_i}\Big|_{x(p)=0}$.

$l_{\mathbb{R}^n}(e_i) = l_p(e_i\circ x) = l_p(x_i)$

If $f = g\circ x$, then $\frac{\partial g}{\partial e_i}\Big|_{x(p)} = D_i(g)(x(p)) = D_i(f\circ x^{-1})(x(p)) = \frac{\partial f}{\partial x_i}\Big|_p$,

so $l_p(f) = \sum l_p(x_i)\cdot\frac{\partial f}{\partial x_i}\Big|_p$. $\square$

What is the relationship btwn D_pM & T_pM?

- D_pM forms a bundle:

$$DM = \bigcup_p D_pM \xrightarrow{\pi} M$$
$$l_p \longrightarrow p$$

$$\bigcup_{p\in U} D_p(M) \longrightarrow U\times\mathbb{R}^n$$
$$l_p = \sum_i l_p(x_i)\frac{\partial}{\partial x_i}\Big|_p \longmapsto (p, (l_p(x_i)))$$
, linear in each fiber by construction.

Choose the topology that makes this cts, which then makes π cts.

Given $f: M\to N$, $f_*: DM \longrightarrow DN$, $(g: N\to\mathbb{R})$.

$$(f_*(l_p))(g) = l_p(g\circ f)$$

- $DM \approx TM$. Fix coordinates.

$$TM = \{[(x_i, v)]_p\}$$
$$\uparrow h$$
$$DM = \{l_p \overset{x}{=} \sum_i l_p(x_i)\frac{\partial}{\partial x_i}\Big|_p\}$$

$h(l_p) = [(x, (l_p(x_i)))]$. Linear map b/c mapping coeff's in basis to coeff's in basis, so cts a bundle equivalence.

What is $\alpha'(0)$, if $\alpha: \mathbb{R} \longrightarrow M$? [w/ $\alpha(0) = p$]

On $\mathbb{R}$, $\frac{d}{dt}$ generates the tangent (at each fiber, it's just $\mathbb{R}$).

Define $\alpha'(0)\in T_pM$ s.t. $\alpha'(0) = \alpha_*\left(\frac{d}{dt}\Big|_0\right)$.

vector at 0, so α_* gives vector at p.

- TM has a C^∞ structure:

  Choose coords for TM. Given an atlas (x,U) on M, define coords for TM given by $(\underset{\text{open sets}}{\pi^{-1}(U)}, \underset{\text{map to } \mathbb{R}^n\times\mathbb{R}^n}{(x\times 1)\circ t_x})$,
  where t_x is the local trivialization, $t_x : \pi^{-1}(U) \to U\times\mathbb{R}^n \xrightarrow{x\times 1} \mathbb{R}^n\times\mathbb{R}^n$

These form an atlas of TM, & so $\dim TM = 2n$. (n for pt & n for vector) So TM is a mfld. We'll check that transition fcns are C^∞. They are:

$$\begin{tikzcd}
(x^{-1}(q), \underset{\sum a_i \frac{\partial}{\partial x_i}|_p}{v}) & \pi^{-1}(U)\cap\pi^{-1}(V) \arrow[dl] \arrow[dr] & v\in T_pM \text{ \& } p\in U\cap V \\
(x^{-1}(q), a) \arrow[u] & U\times\mathbb{R}^n \arrow[d] & V\times\mathbb{R}^n \arrow[d] & (x^{-1}(q), \underset{\sum a_i \frac{\partial y_j}{\partial x_i}|_p}{b_i}) \\
(q,a) \arrow[u] & \mathbb{R}^n\times\mathbb{R}^n \arrow[r] & \mathbb{R}^n\times\mathbb{R}^n & (\underbrace{y\circ x^{-1}(q)}_{C^\infty}, \underbrace{\sum a_i \frac{\partial y_j}{\partial x_i}|_p}_{C^\infty \text{ b/c deriv. of } y\circ x^{-1}})
\end{tikzcd}$$

so
$$v = \sum a_i \frac{\partial}{\partial x_i}\Big|_p = \sum b_i \frac{\partial}{\partial y_i}\Big|_p = \sum a_i \sum_j \frac{\partial y_j(p)}{\partial x_i}\frac{\partial}{\partial y_j}\Big|_p$$
change of var's
$$\Rightarrow b_j = \sum a_i \frac{\partial y_j}{\partial x_i}(p)$$

So, $(q,a) \longmapsto (y\circ x^{-1}(q), \sum a_i D_i(y\circ x^{-1})(q))$ is C^∞ b/c each piece is.

<u>Def</u>: A section of TM is called a <u>vector field</u>. $\bar{X}_p$ is a derivation on $C^\infty(M,\mathbb{R})$, b/c in coords, $\bar{X}_p = \sum_i \bar{X}_p(x_i)\frac{\partial}{\partial x_i}\Big|_p$. So we represent a vector field locally on U as:

$$\bar{X} = \sum a^i \frac{\partial}{\partial x_i}, \text{ where } a^i \text{ are fcns on } U,\ a^i : U\to\mathbb{R}.$$

Since $\bar{X}_p$ is a derivation, then $\bar{X}(fg)(p) = f(p)\bar{X}_p(g) + g(p)\bar{X}_p(f)$ in the ring of fcns.

$$\bar{X} : C^\infty(M,\mathbb{R}) \to \text{Maps}(M,\mathbb{R}), \quad \bar{X}(f)(p) = \bar{X}_p(f).$$

<u>Def</u>: <u>$\bar{X}$ is C^∞</u> if $\bar{X}(f)$ is C^∞ for any f in C^∞. This def. is equiv. to saying $\bar{X}$ is C^∞ as a section, a map btwn 2 C^∞ mflds. Equiv. to locally, the a^i are C^∞.

i.e, for <u>any</u> coordinates (x,u) in the atlas, $\bar{X} = \sum a^i \frac{\partial}{\partial x_i}$, w/ a^i C^∞.

So, $\bar{X} : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$.

<u>Orientability</u>

Given a vector sp. V:

- Start w/ $\mathbb{R}^n$. basis: $\{v_1, \dots, v_n\}$. $\det(v_1,\dots,v_n) > 0$ / < 0, choose one to be positive orientation, other to be negative orientation. Then for any vector, just take det.

  If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, $f(v) = Av$, & $\det A > 0$, we say f preserves the orientation. O/w, it reverses it.
- Do same thing in general vect sp, after fix a basis.

On a mfld, can choose an orientation for each vect. sp (b/c there's one at each pt). But how do we compare the or. for diff vect. sp's?

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Orientability

Def: A mfld M is orientable if there exists an atlas $\mathcal{A}$ s.t. if $(x,U), (y,V) \in \mathcal{A}$, then $\left(\frac{\partial y_i}{\partial x_j}(p)\right)$ has positive det. $\forall p \in U \cap V$.

$$\left(\frac{\partial y_i}{\partial x_j}(p)\right) = D_j(y_i \circ x^{-1})(x(p)) = \text{Jacobian of transition fcn}$$

Note: $y_i \circ x^{-1}$ is a diffeomorph., so the det. never changes sign, so suffices to check one $p \in U \cap V$ in each ctd comp.

b/c matrix has full rank $\Rightarrow$ det $\neq 0$ $\forall p$

* like a continuity of orientations btwn charts.

The matrix $(D_j(y_i \circ x^{-1}))$ defines the change of coord. for the tan. bundle:

$$\left.\begin{array}{l} t_x : \pi^{-1}(U) \to U \times \mathbb{R}^n \\ t_y : \pi^{-1}(V) \to V \times \mathbb{R}^n \end{array}\right\} \text{these gave the coords on } TM \text{ after composing w/ } x \times \mathbb{1} \ (y \times \mathbb{1})$$

$$t_y \circ t_x^{-1}(q,a) = (y \circ x^{-1}(q), (D_j(y_i \circ x^{-1})(q))a)$$

When we fix a fiber, this is a linear map:

$$t_x : \underbrace{\pi^{-1}(p)}_{= T_pM} \longrightarrow \{p\} \times \mathbb{R}^n.$$

This relates or. in T_pM to or in $\{p\} \times \mathbb{R}^n \xrightarrow{\cong \mathbb{R}^n}$ either preserves or reverses or, b/c t_x an iso & so invertible. (after or's are fixed in each)

* At each fiber, b/c $D_j(y_i \circ x^{-1})(q)$ has pos. det, then t_y either pres. or rev. or, in same way as t_x:

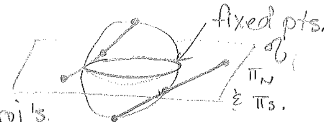
$t_y \circ t_x^{-1}$ pres or, & $t_y = t_y \circ t_x^{-1} \circ t_x$, so t_y does same thing to or as t_x

$\underline{\underline{Ex}}$: (0) $\mathbb{R}^n$ is or.

(1) S^n is or.:

Atlas from N proj. & S proj

Let π_N & π_S be the 2 stereographic proj's.



For $p \in$ equator, $\pi_N \circ \pi_S^{-1}(p) = p$. [recall, we only need to choose one pt, so we chose a fixed pt] But, we need the Jacobian; So we need to do calculations. Yuck. So we use the other def.

Consider $S^n \hookrightarrow \mathbb{R}^{n+1}$ & let's define an or. on T_pS^n.
If $\{v_1, \ldots, v_n\}$ span T_pS^n, we say they are pos. or. if $\{p, \underline{v_1, \ldots, v_n}\}$ are pos. or. in $\mathbb{R}^{n+1}$.

local choice $\rightarrow$ so not putting sm. vect. field on S^2

(2) Möbius band is non-or:

Assume we have an or. on Möbius band.

Coordinates: $f: [0, 2\pi] \times (-1,1) \longrightarrow \mathbb{R}^3$

$$(\theta, t) \longmapsto \left(2\cos\theta + t\cos\tfrac{\theta}{2}\cos\theta,\ 2\sin\theta + t\cos\tfrac{\theta}{2}\sin\theta,\ t\sin\tfrac{\theta}{2}\right)$$

Choose $t=0$, which gives a loop. (choose $t=0$ b/c easy)

$$f_\theta^{(\theta,0)} = (-2\sin\theta,\ 2\cos\theta,\ 0)$$

$$f_t(\theta,0) = \left(\cos\tfrac{\theta}{2}\cos\theta,\ \cos\tfrac{\theta}{2}\sin\theta,\ \sin\tfrac{\theta}{2}\right)$$

$$f_\theta(2\pi, 0) = f_\theta(0,0) = (0,2,0)$$

$$f_t(0,0) = (1,0,0) \quad \text{but} \quad f_t(2\pi,0) = (-1,0,0)$$

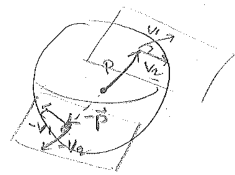
but one det is +, other det is −, so it is not possible to orient.

(3) $\mathbb{RP}^2$ is not or.

Consider $S^2 \xrightarrow{a} S^2$, $p \longmapsto -p$. $T_pS^2 \xrightarrow{a_*} T_pS^2$, $v \longmapsto -v$ & $\mathbb{R}^3 \xrightarrow{a} \mathbb{R}^3$ (extend a to $\mathbb{R}^3$)

So $a_*(\{p, v_1, v_2\}) = \{-p, -v_1, -v_2\} \Rightarrow a_*$ rev. or. of S^2.

$\{p, v_1, v_2\}$ ← gives or on S^2



$$\begin{tikzcd} S^2 \arrow[r, "\pi"] \arrow[d, "a"'] \arrow[dr, phantom, "\circ"] & \mathbb{RP}^2 \arrow[d, "\mathbb{1}"] \\ S^2 \arrow[r, "\pi"'] & \mathbb{RP}^2 \end{tikzcd}$$

← pres. or rev

a reverses

$\mathbb{1}$ ← a induces $\mathbb{1}$ on $\mathbb{RP}^2$

↑ rev. or pres.

Thus if $\mathbb{RP}^2$ or, either $\{$pres = rev ∘ pres or rev = rev ∘ rev$\}$, or $\mathbb{1}_{\mathbb{RP}^2}$ rev. or ↯.

(4) $\mathbb{RP}^3$ is or.

Follow same argument, but now $\{p, v_1, v_2, v_3\}$, & a_* actually preserves or. Its or is inherited from S^3.

Tensor Bundles

Def: V^* is the <u>dual vector sp to V</u> if

$$V^* = \{\lambda : V \to \mathbb{R} \mid \lambda \text{ linear map}\},$$

with the usual operations $+, \cdot$.

If $f: V \to W$ is linear, $f^*: W^* \to V^*$ is the <u>dual map</u> to f

$$f^*(\lambda_W)(v) = (\lambda_W \circ f)(v)$$

$$\begin{tikzcd} W \arrow[r, "\lambda_W"] & \mathbb{R} \\ V \arrow[u, "f"] \arrow[ur, "\lambda_W \circ f"'] & \end{tikzcd}$$

<u>Properties</u>:

(1) $\mathbb{1}_V^* = \mathbb{1}_{V^*}$

(2) f^* is linear

(3) $(f \circ g)^* = g^* \circ f^*$

- $\dim V = \dim V^*$. Given a basis $\{v_1, \dots, v_n\}$ for V, $\exists$ a <u>dual basis</u> $\{v_1^*, \dots, v_n^*\}$ of V^*, as the linear maps sat.

$$v_i^*(v_j) = \delta_{ij}.$$

→ V can be identified w/ V^*, but not in a natural way, i.e. we need a basis to do it. V can be naturally identified with $(V^*)^*$.

If $v \in V$, then $f_v \in (V^*)^*$ is defined by $f_v(\underset{V^*}{\underset{\cap}{\lambda}}) = \lambda(v)$

<u>Def</u>: Let $\xi = (E, B, \pi)$ be a bundle. Then the <u>dual bundle</u> is $\xi^* = (E^*, B, \pi^*)$, where $E^* := \bigcup_{p \in B} (\pi^{-1}(p))^*$. [Since $E = \bigcup_{p\in B} \underbrace{\pi^{-1}(p)}_{\text{vector sp's}}$]

& $\pi^* : E^* \to B$.
$\underset{(\pi^{-1}(p))^*}{\underset{\cap}{\lambda_p}} \longmapsto p$ <u>Note</u>: The fibers are maps on vector sp's.

<u>Why is this a bundle?</u>

- Assume we have local trivializations for ξ:

$$t : \pi^{-1}(U) \to U \times \mathbb{R}.$$

At each fiber, $t : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ is linear & an isomorphism.

So $t^* : \underbrace{\{p\} \times (\mathbb{R}^n)^*}_{\cong \, \mathbb{R}^n} \to (\pi^{-1}(p))^*$. Since this is an $\simeq$, we can invert it:

id w/ $\mathbb{R}^n$ by fixing a basis (the same one everywhere)

$(t^*)^{-1} : (\pi^{-1}(p))^* \to \{p\} \times \mathbb{R}^n$. Define $\hat{t}$, the triv. for the dual as $\hat{t} : (\pi^*)^{-1}(U) \to U \times \mathbb{R}^n$, $\hat{t}\big|_{(\pi^*)^{-1}(p)} = (t^*)^{-1}$.

- Cts b/c t is.

Set the topology in E^* that makes $\hat{t}$ continuous. Then π^* will also be cts (b/c compose $\hat{t}$ w/ projection)

- In general, $\xi \not\simeq \xi^*$ (not equiv. as bundles) — although usually is. But $\xi \simeq (\xi^*)^*$, using the nat'l ident of V & $(V^*)^*$.

If $E = TM$, then $E^* = T^*M$ is called the <u>cotangent bundle</u>.

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Cotangent bundle $T^*M = \bigcup_{p\in M} (T_pM)^*$

* Ex: Check T^*M is a C^∞-mfld.

Identical to what we did for TM – use local trivializations:

(x,U) for M: $(t_x^*)^{-1} : T_U^*M \to U \times (\mathbb{R}^n)^* \xrightarrow{x\times 1} \mathbb{R}^n \times \mathbb{R}^n$ gives coordinates.

$(\mathbb{R}^n)^*$: identify $\mathbb{R}^n$

Then check the transition fcns.

For TM; $(q,a) \longrightarrow (y\circ x^{-1}, [D_i(y_j\circ x^{-1})]\,a)$ (coeff's of vectors) is the transition fcn. For T^*M, get inverse transpose of Jacobian.

→ Do this for any bundle, w/ $(\mathbb{R}^n)^*$ replaced w/ V, a vect sp, which we identify w/ $\mathbb{R}^n$ by choosing a basis.

Def: A section of T^*M is called a 1-form,

$$\omega : M \longrightarrow T^*M.$$

Ex: $f: M \xrightarrow{\text{smooth}} \mathbb{R}$, then we define the 1-form $df(p) \in (T_pM)^*$ as

↳ fcn from tan. vect. (i.e. vect. field) to $\mathbb{R}$.

$df(p)(\overline{X}_p) = \overline{X}_p(f)(p)$ ← a #.

← targ. vector, so a vector field.

Vect. field applied to fcn is differentiation.

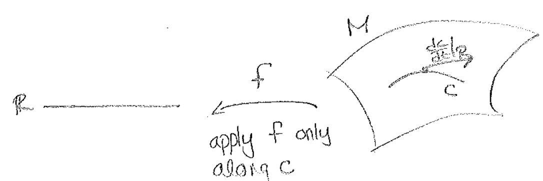
Ex: Let $c: (-\varepsilon,\varepsilon) \to M$ (or $\mathbb{R}$) be a curve w/ $c(0)=p$. Recall

$$\frac{dc}{dt}\Big|_p = c_*\left(\frac{d}{dt}\Big|_0\right).$$

So $df(p)\left(\frac{dc}{dt}\Big|_p\right) = df(p)\left(c_*\left(\frac{d}{dt}\Big|_0\right)\right) = c_*\left(\frac{d}{dt}\Big|_0\right)(f)(p)$

$$= \frac{d}{dt}\Big|_0 (f\circ c)$$

↑ shown before.



$f\circ c: \mathbb{R} \to \mathbb{R}$.

Let (x, U) be coordinates around $p \in M$.

T_pM is gen. by $\{\frac{\partial}{\partial x_i}|_p\}$. Call $\{dx_i(p)\}$ the dual basis to $\{\frac{\partial}{\partial x_i}|_p\}$.

$$dx_i(p)\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \delta_{ij}, \text{ b/c dual basis..}$$

from def from ex above

$$= \frac{\partial x_i}{\partial x_j}\Big|_p = \delta_{ij}$$

* differential applied to vect. is vect. applied to fcn

So $dx_i(p)$ not just a name. They are the same differentials (1-forms) as in the ex's above.

Since $\{dx_i(p)\}$ are a basis, given any section ω, we can write it as

$$\omega(p) = \sum \omega_i(p) dx_i(p).$$

On U we can write

$$\omega = \sum \omega_i dx_i, \text{ where } \omega_i \text{ are fcns } \&$$

ω & dx_i are sections.

<u>Def</u>: We say <u>ω is C^∞</u> is ω_i are C^∞ for an atlas covering M.

<u>Thm</u>: Let (x, U) be coords, f a C^∞ fcn. Then $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$. [Recall, $\frac{\partial f}{\partial x_i} = D_i(f \circ x^{-1})$]

<u>Pf</u>: Let $\overline{X}_p$ be a vector. $\overline{X}_p = \sum \underbrace{\overline{X}_p(x_i)(p)}_{= dx_i(p)(\overline{X}_p)} \frac{\partial}{\partial x_i}\Big|_p$. [recall, for a derivation, we have $l = \sum l(x_i) \frac{\partial}{\partial x_i}$].

$$= \sum dx_i(p)(\overline{X}_p) \frac{\partial}{\partial x_i}\Big|_p, \text{ so}$$

$$df(p)(\overline{X}_p) = \sum_i dx_i(p)(\overline{X}_p) \frac{\partial f}{\partial x_i}\Big|_p = \left(\sum_i \frac{\partial f}{\partial x_i}\Big|_p dx_i(p)\right)(\overline{X}_p)$$

Let $f: M \to N$. We know we have $f_*: TM \to TN$. But we do not have $f^* \dots$:

$f_*: T_pM \to T_{f(p)}N$ is a linear map, so we can dualize

$f^*: (T_{f(p)}N)^* \to (T_pM)^*$. But we cant define $f^*: T^*N \to T^*M$: b/c f may not be inj or surj. Thus f^* does not exist if f is not 1-to-1. [If not surj, can define f^* restricted to $T^*f(M)$].

But we can naturally pullback sections. Let $\omega: N \to T^*N$ be a section. Then we define $f^*\omega: M \to T^*M$, by

$\underbrace{f^*\omega(p)}_{\in T^*M}(\bar{X}_p) = \underbrace{\omega(p)}_{\in T^*N}\underbrace{(f_*(\bar{X}_p))}_{\in TN}$, which is a section on M. If ω & f are C^∞, $f^*\omega$ is C^∞.

Ex: (i) $f^*(dg) = d(g \circ f)$ because:

$$f^*(dg)(\bar{X}_p) = dg(f_*(\bar{X}_p)) = (f_*\bar{X}_p)(g) = \bar{X}_p(g \circ f) = d(g \circ f)(\bar{X}_p)$$

$\boxed{\text{Tensors}}$

Def: A vector field is called a <u>contravariant v.f.</u> & a 1-form is called a <u>covariant v.f.</u>

1-form $\to dy_i = \sum_j \frac{\partial y_i}{\partial x_j} dx_j$ ← same as way to change a basis in $\mathbb{R}^n$, so called covariant

If $\omega = \sum \omega_i dx_i = \sum \hat{\omega}_j dy_j$, then $\omega_i = \sum_j \hat{\omega}_j \frac{dy_j}{dx_i}$

vect. field $\to \frac{\partial}{\partial y_i} = \sum_j \underbrace{\frac{\partial x_j}{\partial y_j}}_{\text{reciprocal than for } dy_i} \frac{\partial}{\partial x_j}$. ← opp as way to change a basis in $\mathbb{R}^n$, so called contravariant

Def: Let $T: \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$ be a multilinear map (i.e. linear in each entry). Define $T^k(V)$ to be the vector sp. of such maps, so $T^1(V) = V^*$. Elts of $T^k(V)$ are called <u>k-tensors</u>.

Given $f: V \to W$ linear, we define $f^*: T^k(W) \to T^k(V)$

$$f^*(T) = T \circ (\underbrace{f \times \cdots \times f}_{k}).$$

Assume $T \in T^k(V)$ & $S \in T^r(V)$. Then $T \otimes S \in T^{k+r}(V)$, &

$$T \otimes S(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+r}) = T(v_1, \ldots, v_k) \cdot S(v_{k+1}, \ldots, v_{k+r}).$$

(This product is a multilinear map, so

$$\otimes: T^k(V) \times T^r(V) \to T^{k+r}(V).$$

- $\otimes$ is non commutative
- $\otimes$ is associative.

Basis for $T^k(V)$: Assume $\{v_1, \ldots, v_n\}$ is a basis for V. Take $\{v_1^*, \ldots, v_n^*\}$, which are 1-tensors, so $v_1^* \otimes \cdots \otimes v_k^*$ is a k-tensor. But $v_2^* \otimes \cdots \otimes v_{k+1}^*$ is also a k-tensor, which is not a multiple. So a basis would be $\{v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*\}$, where $i_j = 1, \ldots, n$, $j = 1, \ldots, k$.

So $\dim T^k(V) = n^k$

Given a mfld M, define the <u>k-covariant tensor bundle</u> to be

$$T^k(M) = \bigcup_{p \in M} T^k(T_pM).$$

$$\downarrow \pi \text{ (takes fiber to } p)$$

$$M$$

Trivializations: Let (x, U) be coords on M

$T^k_U(M) = \bigcup_{p \in U} T^k(T_pM)$. How do we get the following map?

$$T^k_U(M) \longrightarrow U \times \mathbb{R}^{n^k}$$

We have from the tan. bundle:

$t_x : T_U M \to U \times \mathbb{R}^n$

$t_p : T_p M \to \{p\} \times \mathbb{R}^n$, linear

$t_p^* : T^k(\mathbb{R}^n) \to T^k(T_p M)$, the pullback

fix a basis & an isom. $T^k(\mathbb{R}^n) \to \mathbb{R}^{n^k}$, $\}$ use same basis & isom. for all trivializations

$(t_p^*)^{-1} : T^k(T_p M) \to \mathbb{R}^{n^k}$

So bundle these maps to form $T_U^k M \to U \times \mathbb{R}^{n^k}$, so that it's $(t_p^*)^{-1}$ at each fiber.

In coordinates, $dx_{i_1} \otimes \cdots \otimes dx_{i_k}$ generate $T^k(T_p M)$.

A section $A : M \to T^k M$ is called a covariant k-tensor (k-v.f)

$A = \sum \underbrace{a_{i_1, \dots, i_k}}_{= a_I} dx_{i_1} \otimes \cdots \otimes dx_{i_k}$. A is a C^∞-section if a_I are C^∞ for all I & an atlas covering M.

$\leftarrow$ I called a multiindex. (has order)

If $A = \sum_I a_I \, dx_{i_1} \otimes \cdots \otimes dx_{i_k} = \sum_J b_J \, dy_{j_1} \otimes \cdots \otimes dy_{j_k}$, then

$a_I = \sum_J b_J \frac{dy_{j_1}}{dx_{i_1}} \cdots \frac{dy_{j_k}}{dx_{i_k}}$ $\}$ y's on top b/c covariant

Given a covariant tensor, A, we can define a linear map

(1) $\mathcal{A} : V \times \overset{k}{\cdots} \times V \longrightarrow C^\infty(M)$, where V is a vector field

$\leftarrow$ sp. of C^∞ fcns on M

(2) $\mathcal{A}(\overline{X_1}, \dots, \overline{X_k})(p) = A(p)(\overline{X_1}|_p, \dots, \overline{X_k}|_p)$

$\mathcal{A}$ is applied to a vect. field (which exists $\forall p$), while A is applied to vectors.

Thm: If η is a multilinear map as above$^{(1)}$, then there exists a k-tensor A satisfying (2).

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Thm: Let V be the space of C^∞ v.f. on M, & $\mathcal{A}: V \times \overset{(k)}{\cdots} \times V \to C^\infty(M)$ multilinear wrt mult. by $C^\infty(M)$. Then $\exists$ a k, C^∞_r tensor field (covariant) A s.t. $\mathcal{A}(V_1, \dots, V_k)(p) = A(p)(V_1|_p, \dots, V_k|_p)$.

key difference btwn this & identification w/ derivations, which are linear wrt mult. by scalars.

Pf: We need to define $A(p)(v_1, \dots, v_k)$, $v_i \in T_pM$. Given v_i, choose coords at p, (x, U), & let $v_i = \sum_j a^i_j \frac{\partial}{\partial x_j}\big|_p$. Let $f: M \to \mathbb{R}$, $f(p) = 1$, $\operatorname{supp}(f) \subset U$. Define $\overline{X}_i = \begin{cases} \sum_j f a^i_j \frac{\partial}{\partial x_j} & \text{on } U \\ 0 & \text{outside} \end{cases}$ [i.e. bump fcn]

$\overline{X}_i(p) = f v_i(p) = v_i$, & $\overline{X}$ is C^∞. We define

$$A(p)(\underset{\text{vectors}}{v_1}, \dots, v_k) = \mathcal{A}(\underset{\text{vector fields}}{\overline{X}_1}, \dots, \overline{X}_k)(p).$$

This is well-def: i.e. if $\overline{X}_i(p) = Y_i(p)$, $i = 1, \dots, k$, then $\mathcal{A}(\overline{X}_1, \dots, \overline{X}_k)(p) = \mathcal{A}(Y_1, \dots, Y_k)(p)$, i.e. $\mathcal{A}$ depends only on value of $\overline{X}_i$ at pt.

Assume $k = 1$: (This will suffice, since we can apply this case to each entry) By linearity, we'll need show that if $\overline{X}(p) = 0$, then $\mathcal{A}(X)(p) = 0$. (b/c $\mathcal{A}(\overline{X})(p) = \mathcal{A}(Y)(p) \Rightarrow \mathcal{A}(\overline{X} - Y)(p) = 0$)

Assume first that $\overline{X}(p) = 0$ in a nbhd of p, W: Let f be a fcn w/ $\operatorname{supp}(f) \subseteq W$, & $f(p) = 1$.

$$0 \equiv f\overline{X}, \text{ so } \mathcal{A}(f\overline{X})(p) = 0 = f(p)\mathcal{A}(\overline{X})(p) \overset{f(p)=1}{=} \mathcal{A}(\overline{X})(p). \Rightarrow \left(\begin{array}{l}\text{if } \overline{X} = Y \text{ in a} \\ \text{nbhd of } p, \text{ same} \\ \text{value of } \mathcal{A}\end{array}\right)$$

Assume now $\overline{X}(p) = 0$. Let (x, U) be coordinates around p w/ $\overline{X} = \sum a_i \frac{\partial}{\partial x_i}$, $a_i(p) = 0$, on U. [want to take out a_i, by linearity, but can only do that if $\frac{\partial}{\partial x_i}$ are vector fields, not vectors - so we need to extend this restricted $\overline{X}$]

Let $f: M \to \mathbb{R}$, $f|_V = 1$, $\overline{V} \subseteq U$, $\operatorname{supp} f \subseteq U$. Consider $f^2\overline{X}$ on U, $f^2\overline{X} = \sum_i f a_i f\frac{\partial}{\partial x_i}$. So, $\mathcal{A}(f^2\overline{X})(p) \overset{\text{linearity}}{=} \sum_i \underbrace{f a_i(p)}_{=0} \mathcal{A}(f\frac{\partial}{\partial x_i}) = 0$. Since $f^2\overline{X}$ & $\overline{X}$ coincide on a nbhd of p, by prev. case, $\mathcal{A}(f^2\overline{X})(p) = \mathcal{A}(X)(p)$. $\square$

• $f\frac{\partial}{\partial x_i}$: these are global vector fields

One can consider multilinear maps $T: V^* \times \cdots \times V^* \to \mathbb{R}$. Then repeat the construction (b/c V^* a vect. sp). You get <u>contravariant tensors</u>. If $V = T_pM$, $T \in (T_p^*M)^* = T_pM$ ($\&$ $k=1$).

We now take tensor products of these. Locally, a contravariant tensor looks like $\sum a^{i_1,\dots,i_k} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_k}}$.

<u>Note</u>: We can calculate tensor products of sections by doing it ptwise: $(A \otimes B)(p) = A(p) \otimes B(p)$.

One can also construct mixed tensors, in which some are V & others are V^*
apply to forms? apply to vect sps

<u>Contraction</u>: Let $T: \overbrace{V \times \cdots \times V}^{(r)} \times \overbrace{V^* \times \cdots \times V^*}^{(s)} \to \mathbb{R}$ be multilinear, &

R an $(r-1, s-1)$-tensor. Then R is the <u>contraction</u> of T if

$$T = \sum a_{i_1,\dots,i_k}\, dx_{i_1} \otimes \overset{n}{\cdots} \otimes dx_{i_r} \otimes \frac{\partial}{\partial x_{i_{r+1}}} \otimes \overset{m}{\cdots} \otimes \frac{\partial}{\partial x_{i_{r+s}}},$$

$$C_n^m T = \sum_{\substack{i_k \neq i_n, \\ i_m}} \sum_{i_n, i_m} \overset{\delta_{i_n i_m}}{a_{i_1,\dots,i_k}}\, dx_{i_1} \otimes \cdots \otimes \widehat{dx_{i_n}} \otimes \cdots \otimes dx_{i_r} \otimes \frac{\partial}{\partial x_{i_{r+1}}} \otimes \cdots \otimes \widehat{\frac{\partial}{\partial x_{i_m}}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_{r+s}}}.$$

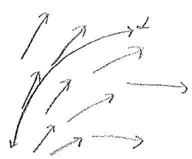
Then $C_n^m T$ is an $(r-1, s-1)$-tensor.

$\boxed{\text{HW}}$ #1, 2, 3 from Ch. 4 (p.127) in Spivak.

Vector Fields & Integration

Goal problem: Given $\{X_1, \dots, X_k\}$, do I locally have a mfld this is tangent to?  Generally no, but we'll look at when this is the case

Assume we have a vector field $\overline{X}$. Can we find a curve $\alpha : (-\varepsilon,\varepsilon) \to M$ s.t. $\overline{X}(\alpha(t))$ is the tangent to α at t. That is, $\alpha_*(\frac{d}{dt}) = \overline{X}(\alpha(t))$ (*)



- In coordinates: $\overline{X}(p) = \sum_i a_i(p) \frac{\partial}{\partial x_i}$,

$$\alpha_*\left(\frac{d}{dt}\right)(f) = \frac{d}{dt}(f \circ \alpha)$$
$$= \sum_i \frac{\partial f}{\partial x_i}(\alpha(t)) \frac{d\alpha_i}{dt}, \text{ where } \alpha_i = x_i \circ \alpha.$$

$$\left[\; (-\varepsilon,\varepsilon) \xrightarrow{\alpha} M \xrightarrow{x} \mathbb{R}^n, \quad \alpha_i = x_i \circ \alpha \;\; (x^i = i^{th} \text{ coord.}) \;\right]$$

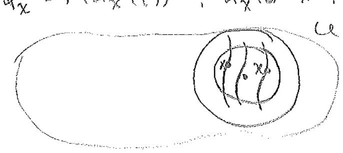
From here, $\alpha_*(\frac{d}{dt}) = \sum_i \frac{d\alpha_i}{dt} \cdot \frac{\partial}{\partial x_i}\Big|_{\alpha(t)}$. If we want (*) to be true, $\sum_i \frac{d\alpha_i}{dt} \cdot \frac{\partial}{\partial x_i}\Big|_{\alpha(t)} = \sum_i a_i(p) \frac{\partial}{\partial x_i}\Big|_{\alpha(t)}$. Thus, locally, the property (*) holds whenever $\frac{d\alpha_i}{dt} = a_i(\alpha(t))$ (a 1^{st} order diff. eq^n ODE), which we can solve, (locally) uniquely if we give an initial condition, i.e. fix $\alpha_i(0) = p_i$, $i = 1, \dots, n$.

<u>Ex</u>: $x' = x^2$

$$\frac{x'}{x^2} = 1 \Rightarrow -\frac{1}{x} = t + \tilde{c} \Rightarrow \frac{1}{x} = t + c \Rightarrow x = \frac{1}{t-c}, \text{ DNE } \forall t. \Rightarrow$$ Solutions will always be local, not global.

<u>Review of ODE</u>: not nec. cts

<u>Thm</u>: Let $f: U \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^n$ (open), $x_0 \in U$, & $r > 0$ s.t. $\overline{B_{2r}(x_0)} \subset U$. Assume f is bdd & Lipschitz on $\overline{B_{2r}(x_0)}$ $\left[\, |f(x) - f(y)| \le k|x-y| \;\; \forall x, y \in \overline{B_{2r}(x_0)} \,\right]$. Then, $\exists\, b > 0$ s.t. for <u>any</u> $x \in \overline{B_r(x_0)}$, $\exists$ a ! sol'n $\alpha_x : (-b,b) \to U$ w/ $\alpha_x' = f(\alpha_x(t))$ & $\alpha_x(0) = x$.



← where all α_x are in $B_r(x_0)$ for the same time, $(-b,b)$!

($\mathcal{C}^1 \Rightarrow$ loc. Lipschitz)

Thm: If $f: U \to \mathbb{R}^n$ is locally Lipschitz, then

$\alpha : (-b, b) \times B_r(x_0) \longrightarrow U$

$\alpha(t, x) = \alpha_x(t)$, as before

is cts (in both t & x). If f is $\mathcal{C}^k$, then α is $\mathcal{C}^k$.

Since $\alpha : \{0\} \times B_r(x_0) \longrightarrow B_r(x_0)$, we can find $\varepsilon > 0$ w/ $\alpha : (-\varepsilon, \varepsilon) \times B_r(x_0) \to B_{2r}(x_0)$, b/c α is cts. Equally, we can find $\hat{\varepsilon} > 0$ w/ $\alpha : (-\hat{\varepsilon}, \hat{\varepsilon}) \times B_{r/2}(x_0) \to B_r(x_0)$, etc.

10/7 - DM

Lie Derivatives

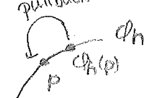
Def: (1) $L_{\bar{X}}(f) = \bar{X}(f)$, the Lie deriv. of a fcn

(2) For a covariant vector field, w/ $M \xrightarrow{\phi_h} \overset{\omega}{M}$, ϕ_h the flow.

$$(L_{\bar{X}}\,\omega)_p = \lim_{h\to 0} \frac{1}{h}\left((\phi_h^*\omega)_p - \omega_p\right).$$

$$(\phi_h^*\omega)_p := \phi_h^*(\omega_{\phi_h(p)})$$

pullback to where you were, b/c can only subtract forms when they're eval. at same pt



$\omega_p \longrightarrow \omega_{\phi_h(p)}$ then pullback so we can compare

Note: $L_{\bar{X}}(f)$ coincides w/ the 2nd def:

$$\lim_{h\to 0} \frac{1}{h}\left((\underset{\substack{\parallel \\ f\circ\phi_h}}{\phi_h^* f})_p - f(p)\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(f(\phi_h(p)) - f(p)\right)$$

$$= \frac{d}{dh}\Big|_{\substack{h=0 \\ \text{(i.e. at } p)}} f(\phi_h) = \bar{X}(f),$$ since $\bar{X}$ is the tangent vector to the curve $h \to \phi_h(p)$ at $h=0$.

(3) For a vector field Y,

$$(L_{\bar{X}}(Y))_p = \lim_{h\to 0} \frac{1}{h}\left((\phi_h^* Y)_p - Y_p\right) = \lim_{h\to 0} \frac{1}{h}\left(\left((\phi_{-h})_* Y\right)_p - Y_p\right) =$$

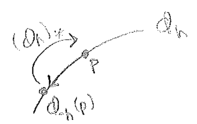
[$M \xrightarrow{\phi_h} M$
diffeo, so can pull back vector fields (ϕ_h^*)

Define $\phi_h^* Y = (\phi_h^{-1})_* Y = (\phi_{-h})_* Y$]

$$= \lim_{h\to 0} \frac{1}{h}\left((\phi_{-h})_* Y_{\phi_h(p)} - Y_p\right)$$

$$\underset{\phi_h(p).}{M} \xrightarrow{\phi_{-h}} \underset{[(\phi_{-h})_* Y]_p}{M}$$

let $h \to -h$: $$= \lim_{h\to 0} \frac{1}{h}\left[Y_p - (\phi_h)_* Y_{\phi_{-h}(p)}\right]$$



* go back in flow & push forward, unlike for forms, where you go forward in flow & pullback

(ω a 1-form)

<u>Prop</u>: Assume $L_{\bar{X}}(Y_i)$ & $L_{\bar{X}}\omega_i$ exist & are C^∞ for $i=1,2$. Then

(1) $L_{\bar{X}}(Y_1+Y_2) = L_{\bar{X}}Y_1 + L_{\bar{X}}Y_2$

(2) $L_{\bar{X}}(\omega_1+\omega_2) = L_{\bar{X}}\omega_1 + L_{\bar{X}}\omega_2$

(3) $L_{\bar{X}}(fY) = \bar{X}(f)Y + fL_{\bar{X}}Y$ ($L_{\bar{X}}$ is a derivation - it has a Leibnitz rule)

(4) $L_{\bar{X}}(f\omega) = \bar{X}(f)\omega + fL_{\bar{X}}\omega$

<u>HW</u> (5) $L_{\bar{X}}(\underbrace{\omega(Y)}_{\text{fcn}}) = (L_{\bar{X}}\underbrace{\omega}_{\text{form}})Y + \omega(L_{\bar{X}}\underbrace{Y}_{\text{vect. field}})$

<u>Pf</u> (of (3)): $\Big[L_{\bar{X}}(\overbrace{fY}^{\text{vect field}})\Big]_p \overset{\text{def}}{=} \lim_{h\to 0}\frac{1}{h}\Big[f(p)Y_p - \underset{\text{linear map}}{(\phi_h)_*}\overbrace{f(\phi_{-h}(p))}^{\#,\text{ so take out}}Y_{\phi_{-h}(p)}\Big]$

$= \lim_{h\to 0}\frac{1}{h}\Big[f(p)Y_p - f(\phi_{-h}(p))(\phi_h)_*Y_{\phi_{-h}(p)}\Big]$

$= \lim_{h\to 0}\frac{1}{h}\Big[f(p)Y_p - f(p)(\phi_h)_*Y_{\phi_{-h}(p)} + f(p)(\phi_h)_*Y_{\phi_{-h}(p)} - f(\phi_{-h}(p))(\phi_h)_*Y_{\phi_{-h}(p)}\Big]$

$= f(p)\lim_{h\to 0}\frac{1}{h}\Big[Y_p - (\phi_h)_*Y_{\phi_{-h}(p)}\Big] + \lim_{h\to 0}\frac{1}{h}\Big[\underset{\xrightarrow{h\to 0}\mathbb{1}}{(\phi_h)_*}\underset{\to Y_p}{Y_{\phi_{-h}(p)}}\Big]\underbrace{(f(p)-f(\phi_{-h}(p)))}_{\xrightarrow{h\to 0}\bar{X}(f)}$ $h\to -h$

$= f(p)(L_{\bar{X}}Y)_p + \bar{X}(f)Y_p$ $\square$

<u>In coordinates</u>:

(1) obvious

(2) $L_{\bar{X}}(dx^i)(p) = \lim_{h\to 0}\frac{1}{h}\Big[(\phi_h^*dx^i)_p - dx^i{}_p\Big]$

$\phi_h^*dx^i = d(x_i\circ\phi_h) = \sum_j \frac{\partial(x_i\circ\phi_h)}{\partial x_j}\circ dx_j$ b/c

$$\begin{tikzcd} M \arrow[d, "x^i"'] & M \arrow[l, "\phi_h"'] \\ \mathbb{R} & \end{tikzcd}$$

dx^i

$= \lim_{h\to 0}\frac{1}{h}\Big[\sum_j\frac{\partial(x_i\circ\phi_h)}{\partial x_j}dx_j - \overbrace{\underset{\frac{\partial x_i}{\partial x_j}}{\delta_{ij}}dx_j}^{\text{same as }dx_i}\Big]$

2/2

$$= \sum_j \frac{\partial}{\partial x_j}\underbrace{\left[\lim_{h\to 0} \frac{1}{h}(x_i \circ \Phi_h - x_i)\right]}_{= L_{\bar{X}}(x_i) = \bar{X}(x_i) = \frac{d}{dh}\big|_{h=0} x_i(\Phi_h)} dx_j$$

$\boxed{\begin{array}{ll} \omega = \sum a_i dx_i & \text{form} \\ Y = \sum b_i \frac{\partial}{\partial x_i} & \text{vect field} \end{array}}$

Assume $\bar{X} = \sum a_i \frac{\partial}{\partial x_i}$, so $\bar{X}(x_i) = a_i$. Thus,

$$L_{\bar{X}}(dx_i) = \sum_j \frac{\partial a_i}{\partial x_j} dx_j$$

So if ω is a form, $\omega = \sum_i \omega_i dx_i$, so apply Leibnitz rule, &

$$L_{\bar{X}}(\omega) = \sum_i \bar{X}(\omega_i) dx_i + \sum_{i,j} \omega_i \frac{\partial a_i}{\partial x_j} dx_j$$

$\boxed{L_{\bar{X}}(\omega) = \sum_i \left[\bar{X}(\omega_i) + \sum_j \omega_j \frac{\partial a_j}{\partial x_i}\right] dx_i}$. If $\bar{X}$ & ω exist & are C^∞, then $L_{\bar{X}}(\omega)$ exists & is C^∞.

(3) $(L_{\bar{X}}(Y))_p = [\bar{X}, Y]_p := \bar{X} Y_p - Y \bar{X}_p$

& $(L_{\bar{X}}(Y))(f) = \bar{X} Y_p(f) - Y \bar{X}_p(f)$:

$L_{\bar{X}} \frac{\partial}{\partial x_i}$: we'll use #5 & the fact that $dx_j(\frac{\partial}{\partial x_i})$ is a constant:

$$\underbrace{dx_j(\tfrac{\partial}{\partial x_i}) = \delta_{ij}}_{\text{constant} \Rightarrow L_{\bar{X}} = 0}, \text{ so } 0 = L_{\bar{X}}(dx_j(\tfrac{\partial}{\partial x_i})) \overset{\text{by (5)}}{=} L_{\bar{X}} dx_j(\tfrac{\partial}{\partial x_i}) + dx_j(L_{\bar{X}} \tfrac{\partial}{\partial x_i})$$

So $dx_j(L_{\bar{X}} \frac{\partial}{\partial x_i}) = -L_{\bar{X}} dx_j(\frac{\partial}{\partial x_i}) = -\frac{\partial a_j}{\partial x_i}$ b/c sub in (2), & terms in sum = 0 if $i \neq j$ from above

Thus, $L_{\bar{X}} \frac{\partial}{\partial x_i} = -\sum_j \frac{\partial a_j}{\partial x_i} \cdot \frac{\partial}{\partial x_j}$ (b/c if dx_j is applied to it we get)

Assume $Y = \sum_i b_i \frac{\partial}{\partial x_i}$. Then,

$$L_{\bar{X}} Y = \sum_i \bar{X}(b_i) \frac{\partial}{\partial x_i} + b_i L_{\bar{X}}(\tfrac{\partial}{\partial x_i}) \text{ by Leibnitz rule } [\bar{X} = \textstyle\sum_j a_j \tfrac{\partial}{\partial x_j})$$

$$= \sum_{i,j} a_j \frac{\partial b_i}{\partial x_j} \frac{\partial}{\partial x_i} - \sum_{i,j} b_i \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j}$$

$\boxed{L_{\bar{X}} Y = \sum_i \left[a_i \frac{\partial b_i}{\partial x_i} - b_i \frac{\partial a_i}{\partial x_i}\right] \frac{\partial}{\partial x_i}}$

Lie bracket:

$$[\bar{X}, Y](f) = \bar{X}\,Y(f) - Y\,\bar{X}(f) = \sum_{j,i} a_j \frac{\partial}{\partial x_j}\left(b_i \frac{\partial f}{\partial x_i}\right) - b_j \frac{\partial}{\partial x_j}\left(a_i \frac{\partial f}{\partial x_i}\right)$$

$$= L_{\bar{X}}(Y)(f) + \sum_{j,i} \underbrace{a_j b_i \frac{\partial^2 f}{\partial x_j \partial x_i} - b_j a_i \frac{\partial^2 f}{\partial x_j \partial x_i}}_{\to 0}$$

* even though the 2 terms in the Lie bracket are not vect. fields, their difference is a vect. field.

Properties:

(1) $L_{\bar{X}}(Y) = -L_Y(\bar{X})$

(2) $L_{\bar{X}}(\bar{X}) = 0$

(3) $[[\bar{X},Y],Z] + [[Y,Z],\bar{X}] + [[Z,\bar{X}],Y] = 0$ (Jacobiian Identity)

$\Rightarrow \underset{\text{cyclic sum}}{\mathfrak{S}} [[\bar{X},Y],Z] = 0$

In terms of Lie derivatives:

$L_Z([\bar{X},Y]) = L_Z(L_{\bar{X}} Y) = -L_{[\bar{X},Y]} Z$

$L_{\bar{X}}[Y,Z] = L_{\bar{X}} L_Y Z$

$L_Y[Z,\bar{X}] = L_Y L_Z \bar{X} = -L_Y L_{\bar{X}} Z$

$\Rightarrow -L_{[\bar{X},Y]} Z + L_{\bar{X}} L_Y Z - L_Y L_{\bar{X}} Z = 0$

$\Rightarrow L_{[\bar{X},Y]} = L_{\bar{X}} L_Y - L_Y L_{\bar{X}} = [L_{\bar{X}}, L_Y]$, so Lie deriv of commutator is commutator of Lie deriv's.

10/9 - DM

Note: • $[\bar{X},Y]$ is not a tensor operation, i.e. cannot be written as a comb. of tensors.

$[\,,\,]$ takes $(\bar{X},Y) \to Z$, input = 2 vect. fields, output = 1 vect. field

In theory, could be (2,1)-mixed tensor

$A = \sum a^i \, dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$. But tensors are linear wrt mult. by fcns, while $[\,,\,]$ is not:

$$[\bar{X}, fY] = [X,Y]f + \bar{X}(f)Y$$
$$\bar{X}(fY) - fY\bar{X} = fXY - fY\bar{X} + \bar{X}(f)Y$$

a derivation, so prod. rule.

• Also, tensors vanish at p when applied to a vector field that vanishes at p, since if $\bar{X} = \sum b^i(p) \frac{\partial}{\partial x_i}\Big|_p$, $b^i(p) = 0$, &

$$A(\bar{X})_p = \sum_{i,j} a^i b^j(p) \ldots \Rightarrow A(\bar{X})_p = 0.$$

But if $\bar{X}_p = 0$, $[\bar{X},Y]_p$ is not always 0. However, if $\bar{X}_p = Y_p = 0$, then $[\bar{X},Y]_p = 0$:

Assume $\bar{X}_p = 0$. Then the flow satisfies $\frac{dc}{dt} = \bar{X}(c(t))$ & $c(0) = p$. Thus $c(t) = p$ is a solution. Hence, $\Phi_t(p) = p \ \forall t$. So,

$$[X,Y]_p = \lim_{h \to 0} \frac{1}{h}\Big(\overset{=0}{Y_p} - (\underbrace{\Phi_{h*}}_{\text{not identity}} Y)_p\Big) = 0$$

$= (\Phi_h)_* \underbrace{Y_{\Phi_{-h}(p)}}_{=0}$, $\Phi_{-h}(p) = p$ b/c flow const at p (use $t=-h$)

More Properties

$$\begin{tikzcd} & TM \arrow[r, "f_*"] & TN \\ M \arrow[ur, "\bar{X}"] \arrow[urr] & & \end{tikzcd}$$

Prop: Let $f: M \to N$ be a diffeomorphism. Let $\bar{X}$ be a vect. field on M & $f_*\bar{X}$ be the v.f. on N. Then the flow of $f_*\bar{X}$ is $\Phi_t^{f_*\bar{X}} = f \circ \Phi_t^{\bar{X}} \circ f^{-1}$.

$q \in N$

Pf: $(f_* \bar{X})_q(g) = (f_* \bar{X}_{f^{-1}(q)})(g) = \underbrace{\bar{X}_{f^{-1}(q)}}_{\text{Lie deriv.}}(g \circ f)$

$= \lim_{h \to 0} \frac{1}{h}\left(g \circ f \circ \phi_h^{\bar{X}}(f^{-1}(q)) - g \circ f(f^{-1}(q))\right)$

$= \lim_{h \to 0} \frac{1}{h}\left(g \circ \underbrace{f \circ \phi_h^{\bar{X}} \circ f^{-1}}_{\Rightarrow\ \phi_h^{f_* \bar{X}}}(q) - g(q)\right)$ □

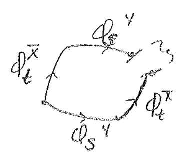
Cor: $f_* \bar{X} = \bar{X} \iff \phi_h^{\bar{X}} \circ f = f \circ \phi_h^{\bar{X}}$

Pf: $\phi_h^{f_* \bar{X}} = f \circ \phi_h^{\bar{X}} \circ f^{-1}$ & $f_* \bar{X} = \bar{X}$. □

So, leaving a vector field invariant is equivalent to commuting w/ the flow.

Prop: $[X, Y] = 0 \iff \phi_t^X \circ \phi_s^Y = \phi_s^Y \circ \phi_t^X$.

Note: This says $[\,,\,]$ measures the lack of commutivity in the flow.



$[\,,\,]$ measures this difference. If $[\,,\,] = 0$, this is closed.

Pf: $(\Leftarrow)$: If $\phi_t^{\bar{X}} \circ \phi_s^Y = \phi_s^Y \circ \phi_t^{\bar{X}}$, then $((\phi_t^{\bar{X}})_* Y)_p = Y_p\ \forall t$, by prev. corollary. Then

$(L_{\bar{X}} Y)_p = \lim_{h \to 0} \frac{1}{h}\left(Y_p - ((\phi_h^{\bar{X}})_* Y)_p\right) = 0$ ✓

$(\Rightarrow)$: If $[\bar{X}, Y]_p = 0$, WTS $(\phi_t^{\bar{X}})_* Y = Y\ \forall t$, and so by cor, the flows commute. Consider the curve $c(t) = ((\phi_t^{\bar{X}})_* Y)_p$. We will show that $c'(t) = 0$, & so $c(t) = c(0) = Y_p\ \forall t$.

$c'(t) = \lim_{h \to 0}\left(\frac{((\phi_{t+h}^{\bar{X}})_* Y)_p - ((\phi_t^{\bar{X}})_* Y)_p}{h}\right)$ [regular derivative]

$\boxed{(\phi_{t+h}^{\bar{X}})_* = (\phi_t^{\bar{X}})_*(\phi_h^{\bar{X}})_*}$

$= (\phi_t^{\bar{X}})_* \cdot \underbrace{\lim_{h \to 0} \frac{1}{h}\left(((\phi_h^{\bar{X}})_* Y)_{\phi_{-t}^{\bar{X}}(p)} - Y_{\phi_{-t}^{\bar{X}}(p)}\right)}_{= -L_{\bar{X}} Y(\phi_{-t}^{\bar{X}}(p)) = 0 \text{ (by assump.}}$ □

<u>Prop</u>: If x is a diffeomorphism, then $x_*[X,Y] = [x_*\bar{X}, x_*Y]$.

<u>Pf</u>: $(x_*\bar{X})_p(f) = \bar{X}_{x^{-1}(p)}(f\circ x) \underset{\uparrow \text{ notation}}{=} \bar{X}(f\circ x)\circ x^{-1}(p)$

So, $[x_*\bar{X}, x_*Y](f) = x_*\bar{X}(x_*Y(f)) - x_*Y(x_*\bar{X}(f))$

$= x_*\bar{X}(Y(f\circ x)\circ x^{-1}) - x_*Y(\bar{X}(f\circ x)\circ x^{-1})$

$= \bar{X}(Y(f\circ x)\circ \cancel{x^{-1}\circ x})\circ x^{-1} - Y(\bar{X}(f\circ x)\circ \cancel{x^{-1}\circ x})\circ x^{-1})$

$= [\bar{X},Y](f\circ x)\circ x^{-1} = x_*[\bar{X},Y]$

D.

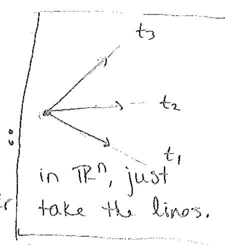
<u>Question</u>: If we have $\bar{X}_1,\dots,\bar{X}_k$ ind. vector fields on $\hat{U}$, can we find coordinates $x: U\subseteq\hat{U}\to\mathbb{R}^n$ s.t. $\bar{X}_i = \frac{\partial}{\partial x_i}$? We did this for one, but can we do it for more?

<u>Ans</u>: No. B/c $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = \frac{\partial^2}{\partial x_i\partial x_j} - \frac{\partial^2}{\partial x_j\partial x_i} = 0$, & so we see that $[\bar{X}_i,\bar{X}_j]=0$ is a necessary condition. It turns out that this is also sufficient. [Essentially, we are straightening the flow]

<u>Thm</u>: If $\{\bar{X}_i\}_{i=1}^k$ are ind. C^∞-v.f.'s on a nbhd of p & $[\bar{X}_i,\bar{X}_j]=0$ $\forall i,j$, then $\exists$ coordinates (x,U) around p s.t. $\bar{X}_i = \frac{\partial}{\partial x_i}$ on U, $i=1,\dots,k$.

<u>Pf</u>: Similarly to $k=1$, assume $M=\mathbb{R}^n$, $p=0$, & $\bar{X}_i|_0 = \frac{\partial}{\partial t_i}\Big|_0$ for some coordinates t in $\mathbb{R}^n$

Define $y(a^1,\dots,a^n) = \Phi^1_{a_1}\circ\Phi^2_{a_2}\circ\cdots\circ\Phi^k_{a_k}(\underbrace{0,\dots,0}_{k},a^{k+1},\dots,a^n)$
where Φ^i is the flow for $\bar{X}_i$. WTS y is a change of var's, i.e. y is a diffeo. If $i=k+1,\dots,n$:



$y_*(\frac{\partial}{\partial t_i}|_0)(f)^{(a)} = \frac{\partial}{\partial t_i}\Big|_0 (f\circ y)(a)$ [can let $a_j=0$ for $i\neq j$ b/c taking ith partial]

$\underset{(i\geq k+1)}{=} \frac{\partial}{\partial t_i}\Big|_0 (f(0,\dots,a_i,\dots,0)) = \frac{\partial f}{\partial t}\Big|_0$.

So $y_*(\frac{\partial}{\partial t_i}|_0) = \frac{\partial}{\partial t_i}\Big|_0$.

If $i=1,\dots,k$, then $\quad$ again, let $a_j = 0$ if $j \neq i$

$$y_*\left(\frac{\partial}{\partial t_i}\Big|_0\right)(f) = \frac{\partial}{\partial t_i}\Big|_0 f\left(\Phi^i_{a_i}(0,\dots,0)\right)$$

$$= \bar{X}_0^i(f)$$

$$= \frac{\partial}{\partial t_i}\Big|_0 (f),$$

by assump.

deriv. of fcn along curve is tan. vector applied to fcn, & $\bar{X}^i_0$ is the tan. vect. to Φ^i_h at $h=0$

Thus $y_*\left(\frac{\partial}{\partial t_i}\Big|_0\right) = \frac{\partial}{\partial t_i}\Big|_0$ for $i=1,\dots,k$. $\quad$ we showed the Jacobian ≠0?

Since $y_*|_0 = I$, we can invert y in a nbhd of 0, so y is a diffeo. in a nbhd of 0.

Choose $x = y^{-1}$. WTS: $x_* \bar{X}^i = \frac{\partial}{\partial t_i}$ on a nbhd of 0 [not just at 0]

$x : U \subseteq M \to \mathbb{R}^n$

Then as $x_*^{-1} \frac{\partial}{\partial t_i} \underset{\uparrow}{=} \frac{\partial}{\partial x_i}$, we have $\bar{X}^i = \partial/\partial x_i$.

by def; see diagram.

$$\begin{array}{c} \partial/\partial x_i \in TM \\ \uparrow \;\; \big\downarrow x_* \\ \frac{\partial}{\partial t_i} \in T\mathbb{R}^n \end{array}$$

Let's look at t_1:

$$y_*\left(\frac{\partial}{\partial t_1}\Big|_a\right)(f) = \frac{\partial}{\partial t_1}\Big|_a (f \circ y) = \frac{\partial}{\partial t_1}\Big|_a \left(f(\Phi^1_{a_1}(g(a))\right),$$ where

$$g^{(a)} = \Phi^2_{a_2} \circ \dots \circ \Phi^k_{a_k}(\overbrace{0,\dots,0}^{k}, a_{k+1},\dots,a_n).$$ Recall $\Phi^1_{a_1}$ is the flow of $\bar{X}^1$, & differentiating f along the flow gives $\bar{X}_1(f)$. So,

$$= \lim_{h\to 0} \frac{1}{h}\left(f(\Phi_{h+a_1}(g(a))) - f(y(a))\right)$$

$$(*) = \lim_{h\to 0} \frac{1}{h}\Big(f(\Phi^1_h \underbrace{\Phi^1_{a_1}(\Phi^2_{a_2} \circ \dots \circ \Phi^k_{a_k}(0,\dots,0,a_{k+1},\dots,a_n))}_{y(a)}) - f(y(a))\Big)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(f(\Phi^1_h(y(a))) - f(y(a))\right) = \bar{X}_1(f) \circ y(a)$$

To do for t_2, need the fact that flows commute in order to pull $\Phi^2_{a_2}$ to the front & repeat. Since $[\bar{X}, Y] = 0$, this is true. $\square$

10/14 - DM

$\boxed{HW}$: Ch. 5 (Vect Fields) # 7, 10, 11, 13, 14b (only for covariant v.f.)

- We have vector fields $\overline{X}_p$ & are trying to find curves $c(t)$ tan. to it. This exists locally, b/c essentially ODE's.
- What if we have k vect. fields, $\overline{X}_p^1, \overline{X}_p^2, \ldots, \overline{X}_p^k$, & want a mfld tan. to them (ideally a k-mfld)
  - When $k=1$, you have a flow; fills everything & don't int.
  - when $k=2$, we want: filling everything

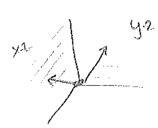
  Doesn't always happen, b/c essentially PDE's.

<u>Def</u>: A <u>k-dim'l distribution</u> on M is a fcn $p \longrightarrow \Delta_p \subseteq T_pM$, where Δ_p is a k-dim'l subsp of T_pM. If for every $p \in M$ $\exists$ coordinates (x, U) s.t. $\overline{X}_q^1, \ldots, \overline{X}_q^k$ are C^∞-v.f.'s generating Δ_q, $q \in U$, then we say Δ is $\underline{C^\infty}$. We say N is an <u>integral k-submfld</u> if $N \subseteq M$ w/ $i: N \to M$ the inclusion, & $i_*(T_pN) = \Delta_p$ for any $p \in N$.

[so at any pt of N, Δ_p is the tangent sp].

<u>Q</u>: Do all k-distributions have integrable k-submflds? No, in general.

<u>Ex</u>: Assume $M = \mathbb{R}^3$, & Δ is gen. by 2 v.f.'s, $\partial/\partial x + f(x,y)\,\partial/\partial z$ $\to \langle 1,0,f\rangle$, & $\partial/\partial y + g(x,y)\,\partial/\partial z$ $\to \langle 0,1,g\rangle$. Suppose we have N a 2-dim'l submfld. Let's



write N as a graph, $z = s(x,y)$. (wlog).

N is given by $(x, y, s(x,y))$. If $p = (x,y,z)$,

$T_pN = \left\langle \begin{pmatrix} 1 \\ 0 \\ \partial s/\partial x \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \partial s/\partial y \end{pmatrix} \right\rangle$. If $i_*(T_pN) = \Delta_p$, then $\partial s/\partial x = f(x,y)$ & $\partial s/\partial y = g(x,y)$. This has a sol'n whenever $\partial^2 s/\partial y \partial x = \partial^2 s/\partial x \partial y \Rightarrow \partial f/\partial y = \partial g/\partial x$. Pick 2 fcns that don't sat. this condition, & there will be no such integral 2-submfld N.

Ex: $\partial/\partial x + b\,\partial/\partial z$, $\partial/\partial y$. If b is not constant wrt y this will not be integrable. If b const. (wrt y) have $\begin{pmatrix}1\\0\\b\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$, & the plane gen. by these 2 vectors at any pt is an integrable distr. if b is constant

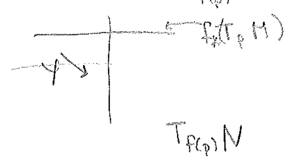
Local Theory of Integration

Def: Let $f: M \to N$ & let $\overline{X}$ be a vf on M, & $\overline{Y}$ on N. We say $\overline{X}$ & $\overline{Y}$ are f-related if $f_*|_p(\overline{X}_p) = Y_{f(p)}$ for any $p \in M$.

I.e., $f_*|_p(\overline{X}_p)(g) = \overline{X}_p(g \circ f)$, so X & Y are f-rel. whenever $Y_{f(p)}(g) = Y(g) \circ f(p)$ $\quad Y(g) \circ f = \overline{X}(g \circ f)$, for any $g: N \to \mathbb{R}$.

Note: If Y is a vf on N, it does not need to be f-rel. to a v.f. in M. It could be that $Y_{f(p)} \notin f_*(T_pM)$, as $f_*(T_pM)$ may have a smaller dim than $T_{f(p)}N$.
$$f_*(T_pM) \subset T_{f(p)}N$$

Could be



Also X may not be unique if f not 1-1, i.e. some $q \in N$ may have 2 preimages.

Prop: Let $f: M \to N$ be a $\mathcal{C}^\infty$ immersion. Then, if Y is a $\mathcal{C}^\infty$ v.f on N w/ $Y_{f(p)} \in f_*|_p(T_pM)$, Then $\exists!$ v.f. $X \in M$ f-rel. to Y.

Pf: Since f is an immersion, $f_*|_p$ is 1-1, & for $Y_{f(p)}$, we can find a unique X_p s.t. $f_*|_p(X_p) = Y_{f(p)}$. To see that X is $\mathcal{C}^\infty$, we choose coordinates: We know there are coords around any p $(x, U), (y, V)$ s.t. $y \circ f \circ x^{-1}(a^1, \dots, a^n) = (a^1, \dots, a^n, 0, \dots, 0)$. In terms of T_pM, this means $f_* \partial/\partial x_i = \partial/\partial y_i$ for $i = 1, \dots, n$, on U. On U, if $X = \sum \alpha_i \partial/\partial x_i$, then $f_* X = \sum \beta_i \partial/\partial y_i$. What are the β_i?

$$\begin{array}{ccc} M & \xrightarrow{f \in C^\infty} & N \\ \alpha_i \downarrow & & \downarrow \beta_i \\ \mathbb{R} & & \mathbb{R} \;\; \in C^\infty \end{array} \quad \text{so} \quad \alpha_i = \beta_i \circ f,$$

If $f_* X = Y \circ f = \sum_i \beta_i \partial/\partial y_i|_p$, then $\alpha_i = \beta_i \circ f$ and if Y is C^∞, then β_i are C^∞, & thus so are α_i. □

<u>Prop</u>: If X_i is f-rel. Y_i, $i=1,2$, then $[X_1, X_2]$ is f-rel. to $[Y_1, Y_2]$.
(i.e. the push-forward preserves the Lie bracket).

<u>Pf</u>: $[Y_1, Y_2](g) \circ f = Y_1(\underbrace{Y_2(g)}_{\text{fcn}}) \circ f - Y_2(\underbrace{Y_1(g)}_{\text{fcn}}) \circ f$

by cond. for X f-rel. to Y: $= X_1(Y_2(g) \circ f) - X_2(Y_1(g) \circ f)$

$= X_1(X_2(g \circ f)) - X_2(X_1(g \circ f))$

$= [X_1, X_2](g \circ f)$. □

<u>Def</u>: We say a v.f. X <u>belongs to a distribution</u> Δ, if $X_p \in \Delta_p \; \forall p$.

<u>Note</u>: Assume N is a k-integrable submfld. Then $i: N \hookrightarrow M$, w/ $i_*|_p T_p N = \Delta_p \; \forall p$. Let $X, Y \in \Delta$ around p. Then $\exists \; \hat{X}, \hat{Y}$, v.f. on N that are i-rel. to X & Y (b/c i an immersion & $X_{i(p)}$, $Y_{i(p)} \in i_* T_p N$. $[\hat{X}, \hat{Y}]_p \in T_p N$, & it is i-rel. to $[X, Y]$, so.

$$i_* [\hat{X}, \hat{Y}]_p = [X, Y]_{i(p)} \in \Delta_p = i_*(T_p N).$$

<u>Def</u>: A distribution Δ is called <u>involutive</u> if for any $\underbrace{X_p, Y_p \in \Delta_p}$, $[X, Y]_p \in \Delta_p$. This is called <u>integrable</u> if for every p, $\exists$ an integrable k-submfld through p.

i.e. 2 vector fields in Δ (local).

<u>Ex</u>: $\partial/\partial x + f \, \partial/\partial z = X$, $\partial/\partial y + g \, \partial/\partial z = Y$, $[X, Y] = \underbrace{(-\partial f/\partial y + \partial g/\partial x)}_{\text{need this incl. of } x \,\&\, y \text{ to be in } \Delta.} \partial/\partial z$

If $\Delta = \langle X, Y \rangle$, then $[X, Y] \in \Delta \Leftrightarrow [X, Y] = 0$

__Frobenius Thm__ (v.f. version): Let Δ be a C^∞ k-distribution, and assume it is involutive. Then for any $p \in M$ $\exists$ coordinates (x, U), $x(U) = (-\varepsilon, \varepsilon)$, s.t. $\{ q \in U \mid x^i(q) = a^i \in \mathbb{R}, |a^i| < \varepsilon, i = k+1, \dots, n \}$ are all integrable submflds.

[i.e., locally looks like ]

Furthermore, any integrable submfld is contained in one of these. (locally)

[i.e. Δ is gen by $\partial/\partial x_i$, $i = 1, \dots, k$, & these x_i are preimages of t_i on $\mathbb{R}^n$]

10/16 - DM

<u>Frobenius Thm</u> (vector field version): let Δ be a C^∞ involutive k-dim'l distribution on M. Then, locally, Δ is integrable; that is, for each $p \in M$, we have (x, U), $x(U) = (-\varepsilon, \varepsilon)^n$ w/

$\{q \in U, x^i(q) = a^i, |a^i| < \varepsilon, i = k+1, \dots, n\}$ an integrable submfld of Δ.

Furthermore, a ctd integrable submfld is locally in one of these sets.

<u>Pf</u>: Assume $M = \mathbb{R}^n$ & $p = 0$. Assume x_i are coordinates s.t. $\frac{\partial}{\partial x_1}\big|_0, \dots, \frac{\partial}{\partial x_k}\big|_0$ generate Δ_0. [Recall, Δ_p is locally gen by X_p, C^∞, & we choose $\{x_i\}$ s.t. $X_0 = \partial/\partial x_i|_0$]. Consider

$\pi: \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the 1^{st} k entries.
$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k)$.

$\pi_* : T\mathbb{R}^n \to T\mathbb{R}^k$ has full rank ($rk = k$) b/c π a proj.)

$\pi_*|_0 : \Delta_0 \to T_0\mathbb{R}^n$ is (the "identity") an isomorphism. requires Δ involutive (*)

For any p close to 0,

$\pi_*|_p : \Delta_p \to T_{\pi(p)}\mathbb{R}^n$ is still an $\cong$, b/c π_* has full rk, it's smooth, & Δ is C^∞

Then let X_p^i be the vector fields w/ $\pi_* X_p^i = \partial/\partial t_i\big|_{\pi(p)}$, p near 0.
X^i & $\frac{\partial}{\partial t_i}$ are π-related, so $[X^i, X^j] = 0$ (b/c $[\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}] = 0$). If $[X^i, X^j] = 0$, then $\exists$ (x, U) w/ $X_p^i = \frac{\partial}{\partial x_i}\big|_p$, $p \in U$, & so $x_i(a) = a^i$ constant for $i = k+1, \dots, n$ are integrable submfld.

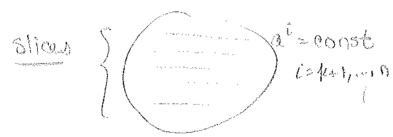
<u>Note</u>: (*) needs 2 things:

(1) need $\Delta_p \to T_{\pi(p)}\mathbb{R}^n$ to be an $\cong$ at each fiber, which comes from continuity

(2). need $\Delta_U \to T_{\pi(U)}\mathbb{R}^n$ to be smooth, & need gen's $\{X_p^i\}$ to map to $\{\partial/\partial t_i|_p\}$, for which one needs involution (b/c $T\mathbb{R}^n$ is closed under $[\cdot,\cdot]$, so Δ_U must be as well to define this map in a smooth way)

If N is an integral submfld, WTS that the ctd components of $N \cap U$ are contained in slices (each in one slice). Consider the inclusion

$i : N \cap U \hookrightarrow U$, & let $X^i \in \Delta_U$.



These are the integrable submflds.

$r = k+1, \dots, n$

$$d(x^r \circ i)(X_q) = X_q(x^r \circ i) =$$
$$= \underbrace{i_* X_q (x^r)}_{} = 0$$
$$\hookrightarrow = 0 \text{ on } U, \text{ since } X_q = \frac{\partial}{\partial x_j}\Big|_q \; j = 1, \dots, k. \quad [\& \; r = k+1, \dots, n]$$

Since $d(x^r \circ i)(X_q) = 0 \;\; \forall X \in \Delta_U$, on any integrable mfld, $x^r \circ i$ is constant. $\square$

<u>Details of step (*)</u>:

$\pi_*|_0 : \Delta_0 \to T_0 \mathbb{R}^k$ is an $\cong$, so for p close to 0,

$\pi_*|_p : \Delta_p \to T_{\pi(p)} \mathbb{R}^k$ is, on each fiber, an isomorphism.

(b/c rk doesn't change in a small nbhd. from continuity).

WTS $\pi_* : \Delta_V \to T_{\pi(V)} \mathbb{R}^k$ is smooth. Choose X^i (C^∞) generating Δ_V s.t. $\pi_*(X^i) = \frac{\partial}{\partial t_i}$. [Why can we choose these?]

Then, if Δ is involutive, π_* is smooth.

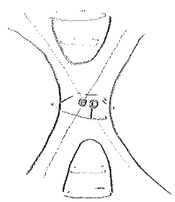
<u>What is the global picture?</u>

<u>Def</u>: Let M be a mfld. A <u>k-dim'l smooth & regular foliation</u> of M is a set $\{N_\alpha\}_{\alpha \in I}$ of ctd k-dim'l immersed submflds N_α s.t. $M = \bigcup_{\alpha \in I} N_\alpha$, & s.t. for any $p \in M$, $\exists$ (x, U) with $N_\alpha \cap U$ being contained in slices of the form

$$S_a(p) = \{ q \in M \mid x^i(q) = a^i, |a^i| < \varepsilon, \; i = k+1, \dots, n \}$$

<u>Ex</u>: The leaves of foliation are level sets of $x^2+y^2-z^2$:

- this is a 2-dim'l foliation w/ singularities.



- not regular, b/c looks like  at every pt except at p, where you have a singularity & the dimension drops.

[note: 2 open cones are each a leaf, & the pt p is a leaf]

- The N_α are called <u>leaves</u> of the foliation.

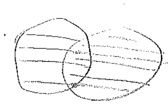
<u>Ex</u>: Foliation of $\mathbb{R}^3$ by action of $SO(3)$ is also singular: leaves are spheres except 0, which is also a leaf.

<u>Note</u>: $N_\alpha \cap U$ could be contained in multiple slices, not just 1.

<u>Thm</u>: Let Δ be a C^∞ involutive distribution on M. Then M is foliated by integral immersed submflds. Each ctd component is called a maximal integrable submfld.

<u>Ex</u>: Torus  This flow is a leaf of a foliation, but is not embedded in any way.

<u>Pf</u>: Let $\{U_i\}_{i=1}^{\infty}$ be a cover of M, sliced as in Frobenius thm. Let's say $p \sim q$ if $\exists\ U_{\alpha_1}, \dots, U_{\alpha_r}$ and slices $S_{\alpha_1}, \dots, S_{\alpha_r}$ w/ $S_{\alpha_i} \subseteq U_{\alpha_i}$ and $S_{\alpha_i} \cap S_{\alpha_{i+1}} \neq \emptyset$. Let $N_p = \{q \in M \mid q \sim p\}$.



We can split M as the disjoint union of these classes $M = \bigsqcup_{\alpha \in I} N_\alpha$, $N_\alpha = N_{p_\alpha}$. Note: $N_p = N_q$ or $N_p \cap N_q = \emptyset$.

On N_α, define the top. generated by that of S_i as an immersed submfld of U_i. We need to show this is 2^{nd} countable.

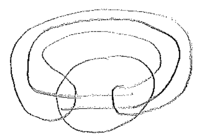
Rk] $N_\alpha \cap U_i$ could have countably many ctd components.

Think:  dense, so nbhd has countably many ctd comp's of ONE leaf.

Can we have more than countably many ctd comp's of an N_α in one nbhd U_i? If not, then have countably many slices (covered by countably many open sets from rat'l balls) in countably many nbhds, so done.

First, if S_{α_i} is a slice in U_i & $U_i \cap U_j \neq \emptyset$, how many slices of U_j's slices can S_{α_i} intersect?

- More than 1:



Note: If $p \in S_{\alpha_i} \cap S_{\alpha_j}$, by continuity, an entire nbhd of p is in the intersection. But we can cover an intersection w/ at most countably many such nbhds. Thus, S_{α_i} can intersect at most countably many other slices.

Next, done. A pt on a leaf lies on a slice, & there are only countably many choices of where to go, in each of countably many open sets covering it. Thus the leaf containing p is covered by countably many slices.

Thus, N_α is an immersed submfld.

$\square$

10/21 - DM

From last time: $\pi: \mathbb{R}^n \to \mathbb{R}^k$ proj.

$\pi_*|_0 : \Delta_0 \to T_0\mathbb{R}^k$ an isomorphism b/c Δ_0 gen, by assump, by $\partial/\partial x_1, \dots, \partial/\partial x_k$

$\pi_*|_p : \Delta_p \to T_p\mathbb{R}^k$ an isomorphism for p close to 0 from continuity of $\pi_*|_p$ (which comes from $\begin{array}{ccc} \Delta_p & & T_p\mathbb{R}^k \\ \text{cts}\downarrow & \nearrow \text{cts} & \\ T_p\mathbb{R}^n & & \end{array}$) which implies rk const. in nbhd of 0.

$\left.\frac{\partial}{\partial t_i}\right|_p \in T_p\mathbb{R}^k$

$\pi_*|_p X^i = \left.\frac{\partial}{\partial t_i}\right|_p$, so X^i & $\partial/\partial t_i$ are π-related, i.e.

$\pi_*[X^i, X^j] = [\pi_* X^i, \pi_* X^j] = [\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}] = 0$. But π_* an $\cong$ on Δ_p so $[X^i, X^j] = 0$, which only occurs when $[X^i, X^j] \in \Delta_p$, which it is, since Δ is involutive.

[i.e. if Δ_p involutive, then Δ is integrable]

Def: If $T \in \mathcal{T}^k(V)$ is a k-tensor on a vect. sp V, we say T is alternating if $T(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = 0$ whenever $v_i = v_j$, $i \neq j$.

This is equivalent to saying T is skew-symm. in any 2 entries, i.e. exchanging any 2 elts v_i & v_j, $i \neq j$, changes the sign of T:

$T(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots v_n) = 0$ (b/c alternating)

$\Rightarrow T(v_1, \dots, v_i, \dots, v_j, \dots, v_n) + T(v_1, \dots, v_j, \dots, v_i, \dots, v_n) = 0$ [b/c 2 terms w/ v_i, v_i etc. then $T = 0$]

&, of course, skew-sym $\Rightarrow$ alt b/c exchanging v_i & v_j.

Denote by $\Omega^k(V)$ the set of alternating tensors. $\Omega^k(V)$ is a vector subsp. of $\mathcal{T}^k(V)$. If $f: V \to W$ is linear, $f^*: \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ w/ $f^*T(v_1, \dots, v_k) = T(f(v_1), \dots, f(v_k))$. If $T \in \Omega^k(W)$, then $f^*T \in \Omega^k(V)$, since if $v_i = v_j$, $f(v_i) = f(v_j)$ & T alt so $T^{(f(v))} = 0$ & then $f^*T(v) = 0$. So we have $f^*: \Omega^k(W) \to \Omega^k(V)$.

Recall, $\dim \mathcal{T}^k(V) = n^k$ if $\dim V = n$. What is $\dim \Omega^k(V)$?

Def: Let S_k be the gp of permutations of $\{1, \dots, k\}$. For $\sigma \in S_k$, $\sigma: \{1, \dots, k\} \to \{\sigma(1), \dots, \sigma(k)\}$. $\operatorname{sign} \sigma = (-1)^{|\sigma|}$ ($|\sigma|$ = # of involutions to take $\{\sigma(1), \dots, \sigma(k)\}$ to $\{1, \dots, k\}$.

S_k acts on k-vectors as $\sigma \cdot (v_1, \dots, v_k) = (v_{\sigma(1)}, \dots, v_{\sigma(k)})$.

If we compose 2 permutations, ρ & σ:

$$(\rho \circ \sigma)(v_1, \dots, v_k) = \rho(\underset{w_1}{v_{\sigma(1)}}, \dots, \underset{w_k}{v_{\sigma(k)}}) = (w_{\rho(1)}, \dots, w_{\rho(k)}) = (v_{\sigma(\rho(1))}, \dots, v_{\sigma(\rho(k))})$$

b/c $w_i = v_{\sigma(i)} \Rightarrow w_{\rho(i)} = v_{\sigma(\rho(i))}$.

Thus, $(\rho \circ \sigma)(v_1, \dots, v_k) = (v_{\sigma(\rho(1))}, \dots, v_{\sigma(\rho(k))})$ is a right action on k-vect's.

Def: Given a k-tensor T, we define its <u>alternation</u>, $\operatorname{Alt}(T)$, as the tensor

$$\operatorname{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S} (\operatorname{sign} \sigma) \cdot T \circ \sigma$$

i.e., $\operatorname{Alt}(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S} (\operatorname{sign} \sigma) \cdot T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$.

ex: $k = 2$: $\operatorname{Alt}(T)(v_1, v_2) = \frac{1}{2}[T(v_1, v_2) - T(v_2, v_1)]$

Note: If T is any tensor, $\operatorname{Alt}(T)$ is skew-symm in every pair of entries.

<u>Properties of Alt</u>:

① If $T \in \mathcal{T}^k(V)$, $\operatorname{Alt}(T) \in \Omega^k(V)$.

② If $\omega \in \Omega^k(V)$, $\operatorname{Alt}(\omega) = \omega$. [this is why $1/k!$ is necessary]

③ $\operatorname{Alt}^2(T) = \operatorname{Alt}(\operatorname{Alt}(T)) = \operatorname{Alt}(T)$. [follows from 1 & 2]

Pf: ① $\operatorname{Alt}(T)(v_1, \dots, v_k) = 1/k! \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$. Assume $v_i = v_j$, $i \neq j$. For each $\sigma \in S_k$, consider $\sigma \circ s_{ij}$, where $s_{ij}(l) = l$, $l \neq i, j$, & $s_{ij}(i) = j$ & $s_{ij}(j) = i$. $\operatorname{sign}(\sigma) = -\operatorname{sign}(\sigma \circ s_{ij})$, but since $v_i = v_j$, $T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = T(v_{\sigma \circ s_{ij}(1)}, \dots, v_{\sigma \circ s_{ij}(k)})$. Thus those 2 terms will vanish in the sum. But all permutations can be covered w/ this type of pairing, so $\operatorname{Alt}(T)(v_1, \dots, v_k) = 0$, so $\operatorname{Alt}(T) \in \Omega^k(V)$.

② Assume $\omega \in \Omega^k(V)$.

$$\mathrm{Alt}(\omega)(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sign}\sigma\, \omega(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

use involutions to rearrange back to $v_1,\dots,v_k$, but ω is skew-symm, so each switch gives a minus.

$$= \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sign}\sigma \operatorname{sign}\sigma\, \omega(v_1,\dots,v_k)$$

$$= \frac{1}{k!}\cdot k!\, \omega(v_1,\dots,v_k)$$

$$= \omega(v_1,\dots,v_k).$$

③ Obvious. □

Note: The tensor product of 2 alt. tensors is no longer alt.:

$$(\eta\otimes\omega)(\underbrace{v_1,\dots,v_k}_{\text{if 2 = here, will vanish}},\overbrace{v_{k+1},\dots,v_{k+\ell}}^{\text{if 2 = here, will vanish}})$$

But if $v_i = v_j$, $1\le i\le k$, $k+1\le j\le k+\ell$, $\eta\otimes\omega$ will not vanish.

Def: Let $\eta\in\Omega^k(V)$, $\omega\in\Omega^\ell(V)$. Define the wedge product of η & ω to be

$$\eta\wedge\omega = \frac{(k+\ell)!}{k!\,\ell!}\mathrm{Alt}(\eta\otimes\omega).$$

Properties:

(1) $\wedge$ is bilinear. [b/c alternation is linear & $\otimes$ is bilinear]

(2) $f^*(\eta\wedge\omega) = (f^*\eta)\wedge(f^*\omega)$. [b/c f^* changes v_i to $f(v_i)$ but doesn't change the order]

(3) $\omega\wedge\eta = (-1)^{k\ell}\eta\wedge\omega$.

Pf: (3): $(\omega\wedge\eta)(v_1,\dots,v_k,v_{k+1},\dots,v_{k+\ell})$ ← rearrange these to be $(v_{k+1},\dots,\dots,v_k)$

$= (-1)^k(\omega\wedge\eta)(v_{k+1},v_1,\dots,v_k,v_{k+2},\dots,v_{k+\ell})$

(*) $= (-1)^{k\ell}(\omega\wedge\eta)(v_{k+1},\dots,v_{k+\ell},v_1,\dots,v_k)$

$= (-1)^{k\ell}(\eta\wedge\omega)(v_1,\dots,v_k,v_{k+1},\dots,v_{k+\ell})$. □

If k is odd, then k^2 is odd, so $(-1)^{k^2} = -1$, & thus $\omega\wedge\omega = -\omega\wedge\omega \Rightarrow$ $\omega\wedge\omega = 0$.

Details of $(*)$: Think of $\omega \wedge \eta$ as $\mathrm{Alt}(\omega \otimes \eta)$ [up to constants]

$\mathrm{Alt}(\omega \otimes \eta)(v_{k+1}, \dots, v_{k+\ell}, v_1, \dots, v_k)$ vs. $\mathrm{Alt}(\eta \otimes \omega)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell})$

a sum of $\omega \otimes \eta$ applied to all permutations of inputs } so first apply $\sigma \in S_{k+\ell}$ that takes 1^{st} input to 2^{nd} input, then do sum over all permutations. Thus they differ only by $\mathrm{sign}(\sigma)$.

Thm (Associativity):

(1) If $S \in T^k(V), T \in T^\ell(V)$, with $\mathrm{Alt}(S) = 0$, then $\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$.

(2) $\mathrm{Alt}(\mathrm{Alt}(S \otimes T) \otimes R) = \mathrm{Alt}(S \otimes \mathrm{Alt}(T \otimes R)) \underset{\text{def}}{:=} \mathrm{Alt}(S \otimes T \otimes R)$

(3) $\underset{\Omega^k}{\omega} \wedge (\underset{\Omega^\ell}{\eta} \wedge \underset{\Omega^r}{\nu}) = (\omega \wedge \eta) \wedge \nu \underset{\text{def}}{:=} \omega \wedge \eta \wedge \nu \underset{\text{def}}{=} \dfrac{(k+\ell+r)!}{k!\,\ell!\,r!} \cdot \mathrm{Alt}(\omega \otimes \eta \otimes \nu)$

Pf:

(1) [For $\mathrm{Alt}(T \otimes S)$] Let $H_0 \subseteq S_{\ell+k}$ be all perms that fix $1, \dots, \ell$.

$\mathrm{Alt}(T \otimes S)(v_1, \dots, v_\ell, v_{\ell+1}, \dots, v_{\ell+k}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} (-1)^{|\sigma|} (T \otimes S)(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}, \dots, v_{\sigma(\ell+k)})$

So, $\sum_{\sigma \in H_0} (-1)^{|\sigma|} (T \otimes S)(v_1, \dots, v_\ell, v_{\sigma(\ell+1)}, \dots, v_{\sigma(\ell+k)})$

$= \sum_{\sigma \in H_0} (-1)^{|\sigma|} T(v_1, \dots, v_\ell) S(v_{\sigma(\ell+1)}, \dots, v_{\sigma(\ell+k)}) = T(v_1, \dots, v_\ell) \sum_{\sigma \in H_0} (-1)^{|\sigma|} S(\cdot \ \cdots) = 0$

Let $\sigma_1 \notin H_0$, & let $H_1 := \sigma_1 H_0$. (Note, $H_1 \cap H_0 = \emptyset$)

Then, similarly, we'll have $T(v_{\sigma_1(1)}, \dots, v_{\sigma_1(\ell)})$, constant, so can take out of Σ

$\sum_{\sigma \in H_1} = \sum_{\sigma \in H_0} (-1)^{|\sigma|} \underbrace{(-1)^{|\sigma_1|} T(v_{\sigma_1(1)}, \dots, v_{\sigma_1(\ell)})}_{\text{const}} S(v_{\sigma(\sigma_1(\ell+1))}, \dots, v_{\sigma(\sigma_1(\ell+k))}) = 0$

Continue: let $\sigma_2 \notin H_0 \cup H_1$, then $\sigma_2 H_0 = H_2$, & $H_2 \cap (H_0 \cup H_1) = \emptyset$, & sum will vanish.... Go on until $\bigcup_i \sigma_i H_0$ covers $S_{k+\ell}$.

<u>Thm</u>: (Associativity)

(1) If $S \in T^k(V)$, $T \in T^l(V)$, & $\mathrm{Alt}(S) = 0$, then $\mathrm{Alt}(T \otimes S) = \mathrm{Alt}(S \otimes T) = 0$

(2) $\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \nu) = \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \nu)) = \mathrm{Alt}(\omega \otimes \eta \otimes \nu)$

(3) If $\omega \in \Omega^k(V)$, $\eta \in \Omega^l(V)$, $\nu \in \Omega^r(V)$, then

$$\omega \wedge (\eta \wedge \nu) = (\omega \wedge \eta) \wedge \nu := \omega \wedge \eta \wedge \nu = \frac{(k+l+r)!}{k!\,l!\,r!} \mathrm{Alt}(\omega \otimes \eta \otimes \nu)$$

<u>Pf</u>: <u>Correction to (1)</u>: $\sigma_0 \notin H$, $\sigma_1 \notin \sigma_0 H \cup H$, etc.

Goal: Write $S_{k+l} = H \cup \sigma_0 H \cup \cdots$,

Should have been $\underline{H\sigma_0}$, $H\sigma_1, \ldots,$ etc

<u>(2)</u>: Let $\mathrm{Alt}(\omega \otimes \eta) - \omega \otimes \eta = S$. Then $\mathrm{Alt}(S) = \underbrace{\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta))}_{= \mathrm{Alt}(\omega \otimes \eta)} - \mathrm{Alt}(\omega \otimes \eta)$

$= 0.$

From (1), we have $\mathrm{Alt}(S \otimes \nu) = 0$. But then

$\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \nu - (\omega \otimes \eta) \otimes \nu) = 0$, so

$\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \nu) = \mathrm{Alt}(\omega \otimes \eta \otimes \nu)$.

To get other equality tensor on the left.

<u>(3)</u>: $\overset{k}{\omega} \wedge (\overset{l}{\eta} \wedge \overset{r}{\nu}) = \dfrac{(l+r+k)!}{(l+r)!\,k!} \mathrm{Alt}(\omega \otimes \underbrace{(\eta \wedge \nu)}_{= \frac{(l+r)!}{l!\,r!} \mathrm{Alt}(\eta \otimes \nu)})$

$$= \frac{(l+r+k)!}{l!\,r!\,k!} \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \nu)) \quad \text{(since Alt is linear)}$$

$$= \frac{(l+r+k)!}{l!\,r!\,k!} \mathrm{Alt}(\omega \otimes \eta \otimes \nu) \quad \text{by (2).}$$

Similarly for $(\omega \wedge \eta) \wedge \nu$.

$\square$

Assume $\{v_1, \dots, v_n\}$ is a basis for V. Then v_i^* gen $\Omega^1(V) = T^1(V) = V^*$.

Then consider $(v_1^* \wedge \dots \wedge v_n^*)(v_1, \dots, v_n) = \frac{n!}{1!1!1!\cdots 1!} \mathrm{Alt}(v_1^* \otimes \cdots \otimes v_n^*)(v_1, \dots, v_n)$

$$= n! \cdot \frac{1}{n!} \sum_{\sigma \in S_n} \underbrace{(v_1^* \otimes \cdots v_n^*)(v_{\sigma(1)}, \dots, v_{\sigma(n)})}_{=0 \text{ unless } \sigma = \mathbb{1}} = \frac{n!}{n!}(v_1^* \otimes \cdots \otimes v_n^*)(v_1, \dots, v_n) = 1.$$

Thus, $v_1^* \wedge \dots \wedge v_n^*$ is dual to $(v_1, \dots, v_n)$.

What is the dimension of $\Omega^k(V)$?

If $\dim V = n$,

Thm: Let $\{v_i\}$ be a basis for V & $\{v_i^*\}$ dual. Then $\{v_{i_1}^* \wedge \dots \wedge v_{i_k}^*\}$ w/ $1 \le i_1 < i_2 < \dots < i_k \le n$ form a basis for $\Omega^k(V)$.

Hence $\dim \Omega^k(V) = \binom{n}{k}$.

Note: • $<$ b/c if any 2 are $=$, the wedge $= 0$.

• ordered b/c if switch 2, just change sign, so this just gets rid of repeats.

Pf: If $\omega \in \Omega^k(V) \subseteq T^k(V)$, then

$$\omega = \sum_{\substack{i_r = 1 \\ r = 1, \dots, k}}^{n} a_{i_1, \dots, i_k} v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*$$

$$\omega \underset{\substack{\uparrow \\ \text{b/c } \omega \in \Omega}}{=} \mathrm{Alt}(\omega) = \sum a_{i_1, \dots, i_k} \mathrm{Alt}(v_{i_1}^* \otimes \cdots \otimes v_{i_k}^*)$$

$$= \sum_{\substack{i_r = 1 \\ r = 1, \dots, k}} a_{i_1, \dots, i_k} \frac{1!1!\cdots 1!}{k!} v_{i_1}^* \wedge \dots \wedge v_{i_k}^*$$

reorganize, i.e. ignore 0's & add negatives to get i_r in increasing order

$$= \sum_{1 \le i_1 < \dots < i_k \le n} \hat{a}_{i_1, \dots, i_k} v_{i_1}^* \wedge \dots \wedge v_{i_k}^*$$

$\square$

<u>Notation</u>: $(i_1, \dots, i_k) = I$ (multiindex)

$$\left.\begin{aligned} V_I &= v_{i_1} \otimes \cdots \otimes v_{i_k} \\ &= v_{i_1} \wedge \cdots \wedge v_{i_k} \end{aligned}\right\} \text{ambiguous.}$$

In this class, we'll use I for wedge products only, i.e.
above $= \sum_I a_I V_I^*$ (<u>note</u>: understood for wedge have $1 \le i_1 < i_2 < \dots < i_k \le n$)

<u>Comment</u>: $\Omega^n(\mathbb{R}^n) = n$-forms $\underset{\text{only}}{\hat{=}} v_1^* \wedge \cdots \wedge v_n^*$ (if you have a basis)
$= \{c \det(v_1, \dots, v_n) \mid c \in \mathbb{R}\}$ / if v_i written in terms of standard basis.

<u>note</u>: . $\dim \Omega^n(\mathbb{R}^n) = \binom{n}{n} = 1$
- All properties of Ω^n are the props. of determinant;
If $e_1^* \wedge \cdots \wedge e_n^*(e_1, \dots, e_n) = 1$, then this is exactly the determ.

<u>Thm</u>: Let $\omega_1, \dots, \omega_k \in \Omega^1(V)$. Then $\omega_1, \dots, \omega_k$ are linearly independent iff $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$.

[In this sense, it's generalizing the determinant]

<u>Pf</u>: $(\Leftarrow)$: If $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$, then $\{\omega_i\}$ are independent, since otherwise $\omega_i = \sum_{j \neq i} \omega_j \alpha_i$, & $\omega_1 \wedge \cdots \wedge \omega_j \wedge \cdots \omega_k = 0$ (b/c will always have repeated index)

$(\Rightarrow)$: If $\omega_1, \dots, \omega_k$ are lin. independent, consider their duals, $\omega_i^* \in V$, which are also independent. Then we can extend them to a basis $\{v_i\}_{i=1}^n$, $v_i = \omega_i^*$ $i = 1, \dots, k$. Then
$(\omega_1 \wedge \cdots \omega_k)(v_1, \dots, v_k) = 1 \neq 0$, so $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$. $\square$

<u>Note</u>: To check $\{v_1, \dots, v_k\}$ are independent, calculate $v_1^* \wedge \cdots \wedge v_k^*$.

<u>Thm</u>: (Change of basis) Let $\{v_i\}_{i=1}^n$ be a basis for V, $\omega \in \Omega^n(V)$, & let $\{u_i\}$ be a different basis, $u_i = \sum_{j=1}^n \alpha_{ij} v_j$. Then
$\omega(u_1, \dots, u_n) = \det(\alpha_{ij})\, \omega(v_1, \dots, v_n)$.

Pf: Define $\eta \in \Omega^n(\mathbb{R}^n)$ as $\eta(\vec{a}_1, \dots, \vec{a}_n) = \eta\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1}\end{pmatrix}\begin{pmatrix} a_{12} \\ \vdots \\ a_{n2}\end{pmatrix}\cdots\begin{pmatrix} a_{1n} \\ \\ a_{nn}\end{pmatrix}\right)$

$= \omega(a_{11}v_1 + a_{21}v_1 + \cdots + a_{n1}v_n, \dots, a_{1n}v_1 + \cdots + a_{nn}v_n)$

$\eta \in \Omega^n(\mathbb{R}^n) \Rightarrow \eta = c\det$, so $\eta(\vec{a}_1, \dots, \vec{a}_n) = c \cdot \det(\vec{a}_1, \dots, \vec{a}_n)$. To find c,

$\eta(\vec{e}_1, \dots, \vec{e}_n) = c \cdot 1$

$\overset{\shortparallel}{\omega(v_1, \dots, v_n)}$

$\Rightarrow \omega(u_1, \dots, u_n) = c \cdot \det(\vec{a}_1, \dots, \vec{a}_n) = \omega(v_1, \dots, v_n) \cdot \det(a_{ij})$. □

Cor: If $\omega \in \Omega^n(V)$, $\omega \neq 0$, then we can orient V by stating $\{v_1, \dots, v_n\}$ is positively or. $\Leftrightarrow$ $\omega(v_1, \dots, v_n) > 0$.

Pf: ω preserves orientation, since 2 bases have same or. $\Leftrightarrow$ $\det(a_{ij}) > 0$, so by formula above, ω keeps same sign on both bases. □

$\boxed{\text{Homework}}$ Ch. 6 # 1, 3; Ch. 5 # 7, 10, 11, 13, 14(b) [covar. vect. field only]; 2 more Frobenius problems via email.

Due: Thurs. Nov. 6

Let $\overset{(E,B,\pi)}{\overset{\shortparallel}{\xi}}$ be a bundle. Define $\Omega^k(\xi)$ to be the bundle for which the fibers are $\Omega^k_p(\pi^{-1}(p))$. This construction is identical to that of $T^k(\xi)$ [the tensor bundle]. [where $\bigcup_p \Omega^k_p(\pi^{-1}(p)) = \Omega^k(E)$]

A section of $\Omega^k(\xi)$ is a map $B \xrightarrow{s} \Omega^k(E)$. If we have 2 sections $\omega: B \to \Omega^k(E)$ & $\eta: B \to \Omega^l(E)$, we can define the wedge product pointwise:

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p),$$

since $\omega(p) \in \Omega^k(\pi^{-1}(p))$, $\eta(p) \in \Omega^l(\pi^{-1}(p))$, so

$$\omega \wedge \eta: B \to \Omega^{k+l}(E).$$

If $E = TM$, a section ω is called a <u>k-form</u>.

10/28 - DM

$\dim V = n$, $v_i \in V$, $v_1^* \wedge \cdots \wedge v_k^*$ $\underset{\text{"}}{0} \Leftrightarrow v_i$ dep. } are these the minors?

Fix a basis $\{e_i\}$ in $\mathbb{R}^n$. Then $\{e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*\}_{i_1 < \cdots < i_k}$ a basis.

These match exactly the minors

(take the $i_1, \dots, i_k$ rows)

So $v_1^* \wedge \cdots \wedge v_k^* = \sum \underbrace{c_{i_1 \dots i_k}}_{\text{minor wrt } i_1, \dots, i_k \text{ of } [v_1 \cdots v_k] \text{ in basis } \{e_i\}} e_{i_1}^* \wedge \cdots \wedge e_{i_k}$.

Let $\hat{V} = \operatorname{span}\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$, a k-dim'l subsp, so $\hat{V} \in Gr(k,n)$

$\hat{V} \longrightarrow \underset{\text{(projectivization)}}{[v_1^* \wedge \cdots \wedge v_k^*]}$, called Plücker coords.

Last Time. We defined $\Omega^k(TM)$.

Thm (Pullback locally): Let $f: M^n \to N^n$, (x,U), $p \in U$, (y,V), $f(p) \in V$.
Then $f^*(g\, dy_1 \wedge \cdots \wedge dy_n) = g \circ f \det\left(\frac{\partial y_i \circ f}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n$.

Pf: $f^*(g\, dy_1 \wedge \cdots \wedge dy_n)\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) = g \circ f\, dy_1 \wedge \cdots \wedge dy_n \left(f_* \frac{\partial}{\partial x_1}, \dots, f_* \frac{\partial}{\partial x_n}\right)$

$= g \circ f\, dy_1 \wedge \cdots \wedge dy_n \left(\sum_i \frac{\partial y_i \circ f}{\partial x_1} \frac{\partial}{\partial y_i}, \dots, \sum_i \frac{\partial y_i \circ f}{\partial x_n} \frac{\partial}{\partial y_i}\right)$

$= g \circ f \det\left(\frac{\partial y_i \circ f}{\partial x_j}\right) \underbrace{dy_1 \wedge \cdots \wedge dy_n \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right)}_{=1}$ by change of basis formula.

So $f^*(g\, dy_1 \wedge \cdots \wedge dy_n) = g \circ f \det\left(\frac{\partial y_i \circ f}{\partial x_j}\right) \underbrace{dx_1 \wedge \cdots \wedge dx_n}$. b/c this applied to $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) = 1$.

(Recall, $\dim \Omega^n(M) = \binom{n}{n} = 1$) ☐

Cor (Change of variable - in n-forms): If (x,U), (y,V) are 2 coordinate systems around p [i.e. $f = \mathbb{1}$], then if

$\omega = g\, dy_1 \wedge \cdots dy_n = h\, dx_1 \wedge \cdots \wedge dx_n$, then

$h = g \det\left(\frac{\partial y_i}{\partial x_j}\right)$.

Pf: Prove $f = \mathbb{1}$ in thm. ☐

Let ω be an n-form. It can be applied to n-vect's in tangent:
Say $\omega(p)(v_1,\ldots,v_n) \neq 0$. It behaves like det (if switch v_i & v_j, sign switches)
$v_i \in T_pM$
We usually use det to orient vectors, so lets use $\omega(p)$ to orient vects — need $\omega(p) \neq 0$ everywhere — & make signs to match everywhere, so can patch it up.
↳ b/c $\omega(p)$ cts & changes ctsly.

<u>Thm</u>: Let M be a C^∞-mfld. M is orientable $\iff \exists$ a nonvanishing C^∞ n-form (i.e. $\omega(p) \neq 0$ $\forall p$).
↳ i.e. $\neq 0$ when apply to basis. Will, of course, $= 0$ when applied to dependent vectors.

<u>Pf</u>: ($\Leftarrow$): HW.
($\Rightarrow$): Locally (in nbhd u), we know that we have ω_u w/ $\omega_u(p) \neq 0$. For ex, choose $\omega_u = dx_1 \wedge \cdots \wedge dx_n$. ($\neq 0$ b/c applied to $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ get 1).
(The only way $f dx_1 \wedge \cdots \wedge dx_n$ can vanish is if f vanishes smwhere.)
Cover M w/ an <u>oriented</u> atlas. (i.e. the det of change-of-coord is +)
Then all ω_u are going to give the same orientation, as $\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$. Now we need to patch this together:
Let $\{\psi_u\}$ be a partition of unity assoc. to the atlas. $\{(x,u)\}$.
Define $\omega = \sum \psi_u \omega_u$. Then

$\omega(p) = \sum_{p \in u} \psi_u(p)\omega_u(p) > 0$ b/c $\psi_u(p) > 0$ if $p \in u$. $p \in u$ for at least one u, so there neither are 0, so $\neq 0$. Also $\psi_u(p) \geq 0$ (whether or not $p \in u$). Thus, one term is > 0 & rest are $\geq 0 \Rightarrow \omega(p) > 0$.

Alternatively, if $\{v_1,\ldots,v_n\}$ is a basis for T_pM, & $p \in u$, then $\omega_u(p)(v_1,\ldots,v_n) \neq 0$ since $\omega_u(p) \neq 0$. But $\psi_u(p) \geq 0$ & $\psi_u(p) \neq 0$ if $p \in u$, so $\omega(p)(v_1,\ldots,v_n) \neq 0$.
↳ & either > 0 or < 0 for all bases.
□

Def of Differential

By def, a 0-form is a fcn $f: M \to \mathbb{R}$, i.e. $\Omega^0(M) = C^\infty(M)$.

$\boxed{\text{Locally}}$

Given a 0-form f, we define a 1-form $df = \sum \frac{\partial f}{\partial x_i} \cdot dx_i$, a local section of $\Omega^1(TM) = T^*M$.

Q: Can we define $d: \Omega^k(M) \to \Omega^{k+1}(M)$ $\forall k$?

Assume $\omega \overset{(p)}{=} \sum a_I dx_I$ is a C^∞ k-form, $p \in U$. Define

$$d\omega(p) = \sum_I \sum_{i=1} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I.$$

$$= \sum_I da_I \wedge dx_I.$$

Properties:

(1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

(2) If ω_1 is a k-form, then

$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$. b/c wedge at beginning, & need to move it all the way back

(3) $d^2 = 0$

(4) df = classical df (as above)

$= \sum \frac{\partial f}{\partial x_i} dx_i$

Pf: (1) Obvious.

(2) If $\omega_1 = \sum a_I dx_I$, $\omega_2 = \sum b_J dx_J$, then

$$\omega_1 \wedge \omega_2 = \sum_{I,J} a_I b_J dx_I \wedge dx_J$$

multilinear in fcns, so can move a_I, b_J around

So $d(\omega_1 \wedge \omega_2) = \underbrace{\sum_{I} \sum_i \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I}_{d\omega_1} \wedge \underbrace{\sum_J b_J dx_J}_{\omega_2} + \sum_{I,J} \sum_i a_I \, dx_i \wedge dx_I \wedge \left(\frac{\partial b}{\partial x_i} dx_J\right)$

need to move dx_i to get $d\omega_2$

$+ (-1)^k \underbrace{\sum_I a_I dx_I}_{\omega_1} \wedge \underbrace{\sum_J \sum_i \frac{\partial b}{\partial x_i} dx_i \wedge dx_J}_{d\omega_2}$

$= d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$

(3) $d^2\omega = d(d\omega) = d\left(\sum_I \sum_i \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I\right)$

$$= \sum_I \sum_{i=1} \sum_{j=1} \frac{\partial^2 a_I}{\partial x_j \partial x_i} \underbrace{dx_j \wedge dx_i}_{\text{not ordered}} \wedge \underbrace{dx_I}_{\text{ordered}}$$

These come in pairs given by

$$\frac{\partial^2 a_I}{\partial x_j \partial x_i} dx_j \wedge dx_i + \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_i \wedge dx_j = 0 \quad \& \quad dx_i \wedge dx_i = 0$$

So sum = 0.

□

Prop: Assume there is $d' : \Omega^k(U) \to \Omega^{k+1}(U)$ that satisfies properties (1) → (4). Then $d' = d$ on $U \subseteq M$.

Pf: From (1) (linearity), we can assume $\omega = a_I dx_I$.

From (2) (product rule), $\omega = \underset{\substack{\uparrow \\ \text{0-form}}}{a_I} \wedge dx_I$ & so

$$d'\omega = d'a_I \wedge dx_I + (-1)^0 a_I \wedge d'(dx_I).$$

From (4) $d'a_I = da_I$ (they are both the classical one), so we only need to check that $d'(dx_I) = 0$, & then $d'\omega = da_I \wedge dx_I = d\omega$.

$$d'(\underset{\substack{\| \\ d'x_{i_1}}}{dx_{i_1}} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) \overset{\text{prod.}}{\underset{\text{rule}}{=}} (d')^2 x_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} - dx_1 \wedge d'(\underbrace{dx_{i_2} \wedge \cdots \wedge dx_{i_k}}_{(k-1)\text{-form}})$$

(b/c coincide on fcns)

$$= 0 - dx_i \wedge d'((k-1)\text{-form}).$$

By induction: $d'(df) = d'^2 f = 0$.

So if $\underset{\text{for any }(k-1)\text{-form}}{d'(d\omega^{k-1}) = 0} \Rightarrow \underset{\text{for any }k\text{-form}}{d'(d\omega^k) = 0}$, from induction step above.

Thus $d'\omega = d\omega$

□

Cor: $\exists$ a ! operator $d: \Omega^k(M) \to \Omega^{k+1}(M)$ w/ properties (1) - (2) - (3) - (4).

Pf: Choose an atlas $\{(x, U)\}$, construct the unique local d_U on U, & define $(d\omega)(p) = d_U(\omega|_U)(p)$. Well-def. b/c of uniqueness on $U \cap V$.

10/30 - DM

Coordinate free def of d: Given ω a k-form we define $d\omega$ as the $(k+1)$-form

$$d\omega(X_1,\dots,X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1}))$$
$$+ \sum_{1\le i<j\le n} (-1)^{i+j}\,\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}).$$

* forms, globally, are multi-linear wrt to mult. by fcns maps on vect. fields, i.e. a fcn on k vect. fields so this def makes sense, since vect fields, X_i take fcns as input & gives a fcn

Recall $a\,dx(\partial/\partial x) = a$, a fcn.

Thm: Both (local & global) definitions of d coincide.

Pf: We need to prove:

(1) $d\omega$ is a form

(a) $d\omega$ is multilinear wrt $C^\infty(M)$ multiplication (wrt to sum is obvious)

(b) $d\omega$ is alternating (i.e. if $X_i = X_j$, $d\omega(X_1,\dots,X_{k+1}) = 0$)

(2) Both def's coincide locally.

Pf of (1):

(a) Consider fX_r in the def of $d\omega$.

$$d\omega(X_1,\dots,fX_r,\dots,X_{k+1}) = \underbrace{f[(-1)^{r+1}X_r(X_1,\dots,\hat{X}_r,\dots,X_{k+1})}_{i=r,\text{ ok, b/c the } f \text{ came out}} + \dots$$

Look at $i\ne r$ in first sum: can pull f out of ω, so have $f\omega$, so use product rule.

$$1^{st}\text{ sum} = \sum_{i\ne r} X_i(f)\,\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1}) + \sum_{i=1}^{n} (-1)^{i+1} f X_i(\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1}))$$

$$2^{nd}\text{ sum} = \sum_{\substack{1\le i<j\le n\\ i\ne r\\ j\ne r}} (-1)^{i+j} f\,\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}) \quad [\text{ok, } f \text{ comes out of } \Sigma]$$

$$+ \sum_{\substack{i<r\\ (j=r)}} (-1)^{i+r}\,\omega([X_i,fX_r],X_1,\dots,\hat{X}_i,\dots,\hat{X}_r,\dots,X_{k+1}) \quad (*)$$

$$+ \sum_{\substack{r<j\\ (i=r)}} (-1)^{r+j}\,\omega([fX_r,X_j],X_1,\dots,\hat{X}_r,\dots,\hat{X}_j,\dots,X_{k+1}) \quad (**)$$

1st term of 2nd sum – from pulling out the commutators from rule below

$$= \sum_{1\le i<j\le r} (-1)^{i+j} f\omega([X_i,X_j],\dots) + \sum_{i<r} (-1)^{i+r} \omega\left(X_i(f) X_r, X_1, \dots, \hat{X}_i, \dots, \hat{X}_r, \dots, X_{k+1}\right) +$$

$$\sum_{r<j} (-1)^{r+j} \omega(-X_j(f) X_r, X_1, \dots, \hat{X}_r, \dots, \hat{X}_j, \dots, X_{k+1})$$

* using $[gX, Y] = gXY - Y(gX) = g[X,Y] - Y(g)X$ & $[X, gY] = X(gY) - gYX$

$Y(gX) = Y(g)X + gYX$ (chain rule) $\qquad = X(g)Y + g[X,Y]$

Now just rearrange:

$$= f \sum_{1\le i<j\le r} (-1)^{i+j} \omega([X_i,X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}) \Big\} (+)$$

$$+ \sum_{i<r} (-1)^{i+r} X_i(f) (-1)^{r-1} \omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1})$$

$$+ \sum_{r<j} (-1)^{r+j+1} X_j(f) (-1)^{r-1} \omega(X_1, \dots, \hat{X}_j, \dots, X_{k+1})$$

$$= (+) + \sum_{i<r} (-1)^i X_i(f) \omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) + \sum_{r<j} (-1)^j X_j(f) \omega(X_1, \dots, \hat{X}_j, \dots, X_{k+1})$$

$$= (+) + \sum_{i\ne r} (-1)^i X_i(f) \omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1}).$$

exactly the opposite of first half of 1st sum.

Then

$d\omega(X_1, \dots, fX_r, \dots, X_{k+1}) = 1^{st}$ sum $+ 2^{nd}$ sum

$= 2^{nd}$ ½ of 1st sum $+ (+) = f d\omega(X_1, \dots, X_{k+1})$ ✓

(b) Assume $X_r = X_s$, $r \ne s$, $r < s$.

If $i \ne r, s \ne j$, then the sum in def. will vanish, since ω is alternating.

We need to check in the first sum:

$$(-1)^{r+1} X_r(\omega(X_1, \dots, \hat{X}_r, \dots, X_s, \dots, X_{k+1})) + (-1)^{s+1} \underset{X_r}{\overset{}{X_s}} (\omega(X_1, \dots, X_r, \dots, \hat{X}_s, \dots, X_{k+1}))$$

so these are same, but $X_r = X_s$ is in diff. positions, so rearrange the 1st – have to move X_s to place r, move $s-r-1$ spots. So have $(-1)^{r+1}(-1)^{s-r-1} = (-1)^s$ & thus the 2 have opp signs, so $= 0$

So 1st sum $= 0$.

In 2^{nd} sum:

- if take out both X_r & X_s, i.e. $i=r$ & $j=s$, then $[X_i, X_j]=0$, so that term would be 0.
- if $i \neq r, s \neq j$, then those terms $=0$ b/c ω alternating.
- if one is s or r:

$$i=r \quad \sum_{1\le r<j\le n} (-1)^{r+j}\,\omega([X_r,X_j], X_1,\dots,\hat{X}_r,\dots,\hat{X}_j,\dots X_{k+1})$$

has X_s

has X_r

$$i=s \quad + \sum_{1\le s<j\le n} (-1)^{s+j}\,\omega([X_s,X_j], X_1,\dots,\hat{X}_s,\dots,\hat{X}_j,\dots X_{k+1})$$

Only terms that remain are those w/ $r<j<s$, since for $j>s$ same as above – have same X's, but not in same place, so Σ's cancel.

$$j=r \quad + \sum_{1\le i<r\le n} (-1)^{i+r}\,\omega([X_i,X_r], X_1,\dots,\hat{X}_i,\dots,\hat{X}_r,\dots, X_{k+1})$$

$$j=s \quad + \sum_{1\le i<s\le n} (-1)^{s+i}\,\omega([X_i,X_s], X_1,\dots,\hat{X}_i,\dots,\hat{X}_s,\dots, X_{k+1})$$

$i<r$, 2 sums = w/ opp signs, so cancel as above

so only terms left are for $r<i<s$.

[$\underline{\underline{n=k+1}}$]

$$= \sum_{1\le r<j<s\le n} (-1)^{r+j}\,\omega([X_r,X_j], X_1,\dots,\hat{X}_r,\dots,\hat{X}_j,\dots, X_{k+1})$$

$$+ \sum_{1\le r<i<s\le n} (-1)^{s+i}\,\omega(\underbrace{[X_i,X_s]}_{}, X_1,\dots,\hat{X}_i,\dots,\hat{X}_s,\dots, X_{k+1})$$

when we move X_s to r position, we have one move less than before b/c need to go over X_i, which is missing.

$$[X_i, X_s] = -[X_s, X_i]$$

Thus, after reorganizing, we have opp. signs, so the whole sum $=0$.

(2) Since d is linear, we can assume $\omega = a_I dx_I$. For simplicity, assume $dx_I = dx_1 \wedge \cdots \wedge dx_k$, so $\omega = a_I dx_1 \wedge \cdots \wedge dx_k$.

$$(d\omega)\left(\frac{\partial}{\partial x_{i_1}}, \dots, \partial/\partial x_{i_{k+1}}\right) = \sum_{1\le r\le k+1} (-1)^{r+1} \frac{\partial}{\partial x_{i_r}}\left(\omega\left(\frac{\partial}{\partial x_{i_1}}, \dots, \widehat{\frac{\partial}{\partial x_{i_r}}}, \dots \frac{\partial}{\partial x_{i_{k+1}}}\right)\right)$$

Note: $[\partial/\partial x_{i_l}, \partial/\partial x_{i_m}] = 0$, so bracket part of sum $=0$

These $\omega(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_r}}, \dots, \frac{\partial}{\partial x_{i_{k+1}}})$ are all 0 unless $(i_1, \dots, i_{k+1}) = (1, \dots, k, s)$ $s > k$.
$[i_1 < \dots < i_{k+1}]$

In order to not vanish, need to take out the s.

If $s > k$ $= (-1)^{k+2} \frac{\partial}{\partial x_s} \big(\underbrace{\omega(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}, \hat{\frac{\partial}{\partial x_s}})}_{= a_I} \big)$

the position of the one you remove +1

$= (-1)^k \frac{\partial a_I}{\partial x_s}$

so $d\omega(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_s}) = (-1)^k \frac{\partial a_I}{\partial x_s}$

so $d\omega = \sum_{s>k} (-1)^k \frac{\partial a_I}{\partial x_s} dx_1 \wedge \dots \wedge dx_k \wedge dx_s$

$= \sum_{s>k} (-1)^k (-1)^k \frac{\partial a_I}{\partial x_s} dx_s \wedge dx_1 \wedge \dots \wedge dx_k$ [but, for all other values of s this will vanish

$= \underbrace{\sum_{s=1}^{n} \frac{\partial a_I}{\partial x_s} dx_s}_{da_I} \wedge dx_1 \wedge \dots \wedge dx_k$

$= da_I \wedge dx_I$ ✓

□

For $k=1$,
$d\omega(X_1, X_2) = X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2])$

<u>Def</u>: Let $\Omega(M)$ be the ring of all forms on M (wrt $\wedge$).
Let $\Delta^\circ \subseteq \Omega(M)$ be $\Delta^\circ = \{\omega \in \Omega(M) \mid$ if ω is an r-form, $\omega(X_1, \dots, X_r) = 0$ whenever $X_i \in \Delta$ $\forall i\}$.

Δ° is called the <u>annihilator</u> of Δ.

Δ° is an ideal, since if $\omega \in \Delta^\circ$, then $\eta \wedge \omega \in \Delta^\circ$ $\forall \eta \in \Omega(M)$

<u>Thm</u> (Frobenius, 2nd version) Δ is integrable $\iff$ $d\Delta^\circ \subset \Delta^\circ$.

11/4 - DM (Arnold - Ch. 8 - Math. Methods of Classical Mechanics - Symplectic Geom)

Frobenius Thm (2nd version). Let Δ be a C^∞ k-distribution. Then Δ is integrable $\Leftrightarrow d\Delta^\circ \subseteq \Delta^\circ$

Pf: Let $\{X_i\}_{i=1}^k$ be generators of Δ_u and extend them to a basis for T_uM, $(i=k+1,\dots,n)$ & let $\{\omega_i\}$ be the basis dual to $\{X_i\}$, so $\omega_{k+1},\dots,\omega_n$ generate Δ°.

In coords: Choose coords so that $\frac{\partial}{\partial x_i}\big|_p$ $i=1,\dots,k$ gen. Δ_p (can always do this at a pt - in a nbhd $\Leftrightarrow \Delta$ integrable). Then $(dx_1\wedge\cdots\wedge dx_k)(p)\neq 0$ since $(dx_1\wedge\cdots\wedge dx_k)\big|_p(\partial/\partial x_1,\dots,\partial/\partial x_k)\big|_p = 1$. By continuity, $(dx_1\wedge\cdots\wedge dx_k)(q)\neq 0$ on Δ_q $\forall q\in U\ni p$. Thus, $\{dx_i(q)\}^k$ are independent on Δ_q. Therefore, $dx_j = \sum_{i=1}^{k} a_{ij}\,dx_i$ (dual has dim k), $j=k+1,\dots,n$, on Δ_u.

$\Rightarrow dx_j - \sum a_{ij}dx_i = \omega_j$ generate Δ_u° [the annihilator of Δ], $j=k+1,\dots,n$.

Note: $\{\underbrace{dx_1,\dots,dx_k}_{\text{distribution}}, \underbrace{\omega_{k+1},\dots,\omega_n}_{\text{annihilator}}\}$ basis for T_u^*M.

Recall. $d\omega_i(X_r,X_s) = X_r(\omega_i(X_s)) - X_s(\omega_i(X_r)) - \omega_i([X_s,X_r])$, $i=k+1,\dots,n$

If $r,s<k$, so $\{X_i\}_{i=1}^k$ gen Δ_u, then

$d\omega_i(X_r,X_s) = \underbrace{X_r(0) - X_s(0)}_{\text{b/c } \omega_i \text{ gen the annihilator}} - \omega_i([X_s,X_r]) = -\omega_i([X_s,X_r])$.

Then $d\omega_i(X_r,X_s) = 0 \ \forall r,s \Leftrightarrow \omega_i([X_s,X_r]) = 0 \ \forall r,s \Leftrightarrow [X_s,X_r]\in\Delta \ \forall r,s$, since ω_i gen Δ°. □

In coords, how can we tell if $d\Delta^\circ\subset\Delta^\circ$? Note: Δ° a ring, not a v.s.

Assume $\{\underbrace{\omega_1,\dots,\omega_k}_{\text{dual to }\{X_1,\dots,X_k\}\text{ which gen }\Delta}, \overbrace{\omega_{k+1},\dots,\omega_n}^{\text{gen }\Delta^\circ}\}$ is chosen as before. Then, for $r=k+1,\dots,n$,

$d\omega_r = \sum_{\substack{i,j=1\\ i<j}}^{n} a^r_{ij}\,\omega_i\wedge\omega_j = \sum_{j=1}^{n}\theta_j^r\wedge\omega_j$, $\theta_j = \sum_{i<j} a^r_{ij}\,\omega_i$.

If $d\Delta^\circ\subseteq\Delta^\circ$, then $d\omega_r(X_i,X_j)=0$ when $i<j\le k$. So,

$d\omega_r(X_i,X_j) = \theta_j^r\wedge\omega_j(X_i,X_j) - \omega_i\wedge\theta_i^r(X_i,X_j)$

But $\theta_i^r(X_j)=0$ b/c $\theta_i^r(X_j) = \sum_{s<i} a^r_{si}\,\omega_s$, but $s<i<j$, so $\omega_s(X_j)=0$. $\forall s$

& then $d\omega_r(X_i,X_j) = \theta_j^r\wedge\omega_j(X_i,X_j) = \theta_j^r(X^i) = 0 \ \forall i<j$. (b/c $d\Delta^\circ\subseteq\Delta^\circ$)

$[\omega_j(X_j)=1]$

But by def, $\theta_j^r = \sum_{i<j} a_{ij}^r \omega_i$, so if $\theta_j^r(X_i) = 0$ for all $i<j$,
then $a_{ij}^r = 0\ \forall i<j$ (b/c $\omega_i(X_i)=1$). Thus, $\theta_j^r = 0,\ \forall j \le k$.

Therefore, $\boxed{d\omega_r = \sum_{j=k+1}^{n} \theta_j^r \wedge \omega_j}$ ← this is what it means for $d\Delta^\circ \subseteq \Delta^\circ$.

<u>Prop</u>: $d(f^*\omega) = f^* d\omega$.

<u>Pf</u>: For 0-forms, $d(f^*g) = f^*dg$:
(0-forms = fcn)

$d(f^*g) = d(g\circ f)$.

$f^*dg(X) = dg(f_* X) = (f_* X)(g) = X(g\circ f) = d(g\circ f)(X)$. ✓

It suffices to check $\omega = a_I dx_I$.

$d\omega = da_I \wedge dx_I$

$f^* d\omega = f^*(da_I) \wedge f^*(dx_I)$

$= d(f^* a_I) \wedge f^*(dx_I)$

$= d(f^*(a_I \wedge dx_I)) = df^*\omega$.

<u>Def</u>: We say a form ω is <u>closed</u> if $d\omega = 0$. ω is <u>exact</u> if $\omega = d\theta$, θ a $(k-1)$-form.

<u>Note</u>: • exact ⇒ closed (b/c $d^2=0$)

• in general, closed ⇏ exact:

let $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ (the differential of the ∡ on the plane).

$$d\omega = \frac{\partial \frac{-y}{x^2+y^2}}{\partial y} dy \wedge dx + \frac{\partial \frac{x}{x^2+y^2}}{\partial x} dx \wedge dy$$

$$= \underbrace{\left[-\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right]}_{=\frac{-x^2+y^2}{(x^2+y^2)^2}} dy\wedge dx + \underbrace{\left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right]}_{=\frac{y^2-x^2}{(x^2+y^2)^2}} dx \wedge dy$$

$= 0.$ ⇒ ω closed.

If $\omega = d\theta = \underbrace{\frac{\partial\theta}{\partial x}}_{=\frac{-y}{x^2+y^2}} dx + \underbrace{\frac{\partial\theta}{\partial y}}_{\frac{x}{x^2+y^2}} dy$ ⇒ $\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2}$ ⇒ $\theta(x,y) = \tan^{-1}(\frac{y}{x}) + c(y)$

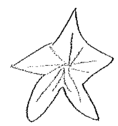
But $\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} + c'(y)$ ⇒ $c(y) = C$.

but θ is not C^∞ for any const. C.

The problem is that the domain of ω does not include $(0,0)$.

<u>Def</u>: M is <u>smoothly contractible</u> to p_0 if $\exists$ C^∞ map $H: M\times[0,1] \to M$, $(p,t) \mapsto H(p,t)$
with $H(p,0)=p_0$ & $H(p,1)=p$.

<u>Ex</u>: Star-shaped region: A region M in $\mathbb{R}^n$ is called <u>star-shaped</u> if $\exists$ $p_0 \in M$ s.t. $\overline{pp_0} \subseteq M$ $\forall p \in M$:



- Clearly, a star-shaped region is contractible since we can define
$H: M\times[0,1] \to M$
$(p,t) \longmapsto p_0 + t(p-p_0)$

Define: $i_s: M \to M\times[0,1]$,
$p \longmapsto (p,s)$

<u>Lemma</u>: If ω is a closed form on $M\times[0,1]$, then $i_1^*\omega - i_0^*\omega$ is exact.

<u>Pf</u>: <u>Idea</u>: find a transformation $I(\omega)$ s.t.
$i_1^*\omega - i_0^*\omega = d(I(\omega)) - I(d\omega)$. Then if ω closed, get
$i_1^*\omega - i_0^*\omega = d(I(\omega))$, so exact.

Assume ω is a 1-form. In coordinates:

$$\begin{array}{ccc} M & \longrightarrow & M\times[0,1] \\ \swarrow \hat{x} & & \downarrow (x,t) \\ \mathbb{R}^n & & \mathbb{R}^{n+1} \end{array}$$

$x = \hat{x} \circ \pi_M$
$t = \hat{t} \circ \pi_{[0,1]}$

It suffices to assume $\omega = \sum \omega_i dx_i + f dt$.

$\underset{M}{\overset{\wedge}{i_s^*}}\omega = \sum i_s^*(\omega_i)\, i_s^* dx_i + \underbrace{i_s^* f dt}_{=0}$ b/c time is constant when pull back: $i_s^*(fdt) = fdt(\underbrace{i_{s*}(X)}_{\text{depends only on } X}) = 0$

$\Rightarrow i_s^*\omega(p) = \sum i_s^*\omega_i\, i_s^* dx_i = \sum \omega_i(p,s)\, i_s^* dx_i$

$i_s^* dx_i|_p\left(\frac{\partial}{\partial \hat{x}_i}\Big|_p\right) = dx_i\left((i_s)_*\left(\frac{\partial}{\partial \hat{x}_i}\Big|_p\right)\right)$

$\left(i_{s*}\left(\frac{\partial}{\partial \hat{x}_i}\Big|_p\right)\right)(f(q,t)) = \frac{\partial}{\partial \hat{x}_i}\Big|_p (f(q,s)) = \frac{\partial}{\partial x_i}\Big|_{p,s} (f(q,s))$, so

$i_{s*}\left(\frac{\partial}{\partial \hat{x}_i}\Big|_p\right) = \frac{\partial}{\partial x_i}\Big|_{(p,s)}$. $\Rightarrow (i_s^* dx_i)\left(\frac{\partial}{\partial \hat{x}_i}\right) = dx_i\left(\frac{\partial}{\partial x_i}\right) = 1$, so $i_s^* dx_i = d\hat{x}_i$.

Thus, $(i_s^* \omega)(p) = \sum_i \omega_i(p,s)\, d\hat{x}_i$.

From here,

$i_1^*\omega - i_0^*\omega = \sum_i [\omega_i(p,1) - \omega_i(p,0)]\, d\hat{x}_i$.

If ω is closed, $d\omega = 0$, and then:

$d\omega = \sum_{i=1}^{n} \omega_i \wedge dx_i + df \wedge dt$

$= \sum_1^n \frac{\partial \omega_i}{\partial t} dt \wedge x_i + \sum_1^n \frac{\partial f}{\partial x_i} dx_i \wedge dt$ + terms w/ no dt.

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Lemma: If ω is a closed 1-form on $M\times[0,1]$, then $i_1^*\omega - i_0^*\omega$ is exact, where $i_s : M \to M\times[0,1]$, $p \mapsto (p,s)$.

Pf: Coordinates

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\hat{x}"] & M\times[0,1] \arrow[d, "(x{,}t)"] \\ \mathbb{R}^n & \mathbb{R}^{n+1} \end{tikzcd} \qquad \begin{array}{l} x = \hat{x}\circ\pi_M \\ t = t\circ\pi_{[0,1]} \end{array}$$

Assume $\omega = \sum_{i=1}^n \omega_i dx_i + f dt$, then $i_{s*}\frac{\partial}{\partial \hat{x}_i}\Big|_p = \frac{\partial}{\partial x_i}\Big|_{(p,s)}$ &

$i_s^*\omega = \sum_i \omega_i(p,s)\,d\hat{x}_i$. (recall, dt disappears b/c i_s fixes t, so $dt=0$)

So $i_1^*\omega - i_0^*\omega = \sum_i (\omega_i(p,1) - \omega_i(p,0))\,d\hat{x}_i$.

ω is closed iff $d\omega = 0$, &

$$d\omega = \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x_j} dx_j \wedge dx_i + \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dt + \sum_{i=1}^n \frac{\partial \omega_i}{\partial t} dt \wedge dx_i$$

If $d\omega = 0$, then $\frac{\partial \omega_i}{\partial t} = \frac{\partial f}{\partial x_i}$, from the last 2 terms above, $\forall i$.

So, since

$$i_1^*\omega - i_0^*\omega = \sum_{i=1}^n \left[\int_0^1 \frac{\partial \omega_i}{\partial t}(p,s)\,ds\right] d\hat{x}_i \overset{\text{we get}}{=} \sum_{i=1}^n \left[\int_0^1 \frac{\partial f}{\partial x_i}(p,s)\,ds\right] d\hat{x}_i.$$

if $= \sum_{i=1}^n \frac{\partial g}{\partial \hat{x}_i} d\hat{x}_i$, then $i_1^*\omega - i_0^*\omega = dg$. To show this, let

$g = \int_0^1 f(p,s)\,ds$. we WTS $\frac{\partial g}{\partial \hat{x}_i} = \int_0^1 \frac{\partial f}{\partial x_i}(p,s)\,ds$. If so, then done.

$$\frac{\partial g}{\partial \hat{x}_i}(p) = D_i(g\circ\hat{x}_i^{-1}) = D_i\left(\int_0^1 f(\hat{x}^{-1}(p),s)\,ds\right) = \int D_i f((x,t)^{-1}(p,s))\,ds$$
$$= \int_0^1 \frac{\partial f}{\partial x_i}(p,s)\,ds$$

since x just projection then $\hat{x}$.
So $(x,t)(a,s) = (\hat{x}(a), s)$

The fcn f, & hence g, does not depend on the coordinate, since $T(M\times[0,1]) = \ker(\pi_M)_* \oplus \ker(\pi_{[0,1]})_*$, so if $\omega : M\times[0,1] \to T(M\times[0,1])$, $\omega = \omega_M \oplus \omega_{[0,1]}$, as $\omega(v) = \omega(v_M + v_{[0,1]})$ & $\underbrace{\omega_M(v) = \omega(v_M),\ \omega_{[0,1]}(v) = \omega(v_{[0,1]})}_{\text{not forms on } M \text{ or } [0,1], \text{ as } \omega_M \text{ depends on } [0,1] \text{ \& vice versa}}$

$\omega_{[0,1]} = f dt$ for some global f, since $[0,1] \simeq \mathbb{R}$ is trivial. $\square$

If we have a k-form in $M\times[0,1]$, then $\omega=\omega_M+dt\wedge\eta$, w/ η a $(k-1)$-form. [Homework]. Now, η will play the role of f, as for 1-forms, we integrated a 0-form, so now for k-forms, we need to integrate a $(k-1)$-form.

Define $I\omega(p)(v_1,\dots,v_{k-1})=\int_0^1 \eta(p,s)(i_{s*}v_1,\dots,i_{s*}v_{k-1})\,ds$, a $(k-1)$-form on M.

or, shortly, $I\omega=\int_0^1 i_s^*\eta\,ds$. Note! I is a linear operator.

<u>Thm</u>: For any k-form ω on $M\times[0,1]$, $i_1^*\omega-i_0^*\omega=d(I\omega)+I(d\omega)$.

* In particular, if ω closed, LHS is exact b/c $=d(I\omega)$.

<u>Pf</u>: It suffices to prove this in coordinates, since the split $\omega=\omega_M+dt\wedge\eta$ is independent of coords.

(a) Assume $\omega=a_I\,dx_I$ (no dt). Then can assume, b/c I,d are linear wrt sum.

$d\omega=\frac{\partial a_I}{\partial t}dt\wedge dx_I+(\text{terms w/o } dt)\Rightarrow \eta=\frac{\partial a_I}{\partial t}dx_I$

$\Rightarrow\quad I(d\omega)^{(p)}=\int_0^1 i_s^*\left(\frac{\partial a_I}{\partial t}dx_I\right)(p)\,ds.$

Since $i_{s*}\frac{\partial}{\partial \hat{x}_i}\Big|_p=\frac{\partial}{\partial x_i}\Big|_{(p,s)}$, $i_s^*dx_i(p,s)=d\hat{x}_i(p)$. (as $\langle f^*\omega,v\rangle:=\langle\omega,f_*v\rangle$)

So $I(d\omega)^{(p)}=\left(\int_0^1\frac{\partial a_I}{\partial t}(p,s)ds\right)d\hat{x}_I=(a_I(p,1)-a_I(p,0))d\hat{x}_I$

Also, $I\omega=0$, as $\eta=0$ in this case $[\omega=a_Idx_I]$.

$I(d\omega)=(a_I(p,1)-a_I(p,0))d\hat{x}_I=i_1^*(a_Idx_I)-i_0^*(a_Idx_I)=i_1^*\omega-i_0^*\omega$ ✓.

(b) Assume $\omega=f\,dt\wedge dx_I$ (w/ this I one smaller than prev. I)

$\eta=f\,dx_I$. i_s^*dt can only applied to $\partial/\partial\hat{x}_i$ (as no t in M), & then $i_s^*dt(\frac{\partial}{\partial\hat{x}_i})=dt(\frac{\partial}{\partial x_i})=0$ OR $i_s^*dt=d(t\circ i_s)=0$, as $t\circ i_s=s$, const.

$d\omega=\sum_{j=1}^n\frac{\partial f}{\partial x_j}dx_j\wedge dt\wedge dx_I$ (note, don't need $\partial f/\partial t\cdot dt\wedge dt\wedge dx_I$ b/c $=0$)

$I(d\omega)=-\int_0^1 i_s^*\left(\sum_{j=1}^n\frac{\partial f}{\partial x_j}dx_j\wedge dx_I\right)ds=-\sum_j^n\left(\int_0^1\frac{\partial f}{\partial x_j}(p,s)\,ds\right)d\hat{x}_j\wedge d\hat{x}_I.$

$I\omega=\int_0^1 i_s^*(f\,dx_I)ds=\int_0^1 f(p,s)ds\,d\hat{x}_I$ (on M, so $\partial/\partial\hat{x}_j$)

$d(I\omega)=\sum_{j=1}^n\frac{\partial}{\partial\hat{x}_j}\int_0^1 f(p,s)ds\;d\hat{x}_j\wedge d\hat{x}_I$

We NTS: $\int_0^1 \frac{\partial f}{\partial x_j}(p,s)\,ds = \frac{\partial}{\partial x_j}\int_0^1 f(p,s)\,ds$, then done.

$$\frac{\partial}{\partial x_j}\int_0^1 f(p,s)\,ds = \frac{\partial}{\partial x_j}\int_0^1 \iota_s^* f\,ds = \int_0^1 \frac{\partial}{\partial x_j}(\iota_s^* f)\,ds = \int_0^1 d(\iota_s^* f)\frac{\partial}{\partial x_j}\,ds$$
$$= \int_0^1 \iota_s^*(df \cdot \tfrac{\partial}{\partial x_j})\,ds = \int_0^1 df\, \iota_{s*}\tfrac{\partial}{\partial x_j}\,ds = \int_0^1 df\,\tfrac{\partial}{\partial x_j}\,ds = \int_0^1 \frac{\partial f}{\partial x_j}\,ds$$

From here, $d(I\omega) = \sum_{j=1}^{n}\int_0^1 \frac{\partial f}{\partial x_j}\,ds\, d\hat{x}_j \wedge dx_I$, so $I(d\omega) + d(I\omega) = 0$. □

<u>Poincare's Lemma</u>: If M is C^∞-contractible to $p_0 \in M$, then $d\omega = 0 \iff \omega = d\eta$ for some η.

<u>Pf</u>: Since ω is contractible, we have a C^∞ map $H: M \times [0,1] \to M$ w/ $H(p,0) = p_0$, $H(p,1) = p$.

Let $H^*\omega$ be a form on $M \times [0,1]$. By the prev. thm,

$$i_1^*(H^*\omega) - \underbrace{i_0^*(H^*\omega)}_{=(H\circ i_0)^*} = I(d(H^*\omega)) + d(I(H^*\omega)).$$

$H \circ i_0(p) = p_0$, so $(H\circ i_0)^*\omega = 0$

$H \circ i_1(p) = p \Rightarrow H\circ i_1 = \mathbb{1}_M$, so $(H \circ i_1)^*\omega = \mathbb{1}_M^*\omega = \omega$.

If $d\omega = 0$, then $d(H^*\omega) = H^*(d\omega) = 0$.

Then $\omega = I(d(H^*\omega)) + d(I(H^*\omega))$, so $\forall$

ω is closed, $\omega = d(I(H^*\omega))$. □

<u>Homework</u>: Do 2 Frobenius problems for forms; Ch. 7/6, 17, 21, 27; earlier problem from today $\omega = \omega_H + dt \wedge \eta$.

Symplectic Structures on Manifolds

Let M^{2n} be a $2n$-dim'l mfld. A <u>symplectic structure</u> on M is a nondegenerate closed 2-form ω on M^{2n}. i.e, $d\omega = 0$ and for any $p \in M$ & $v \in T_pM$, $\exists\, u \in T_pM$ w/ $\omega(p)(v,u) \neq 0$.

<u>Ex</u>: Cotangent bundle T^*M has a nat'l symp. structure (which is exact) defined as:

Let θ be a 1-form in T^*M, $\xi \in T_\alpha(T_q^*M)$, α a form on M. (i.e. a pt in T^*M)

We have $\pi : T^*M \to M$.

$$\xi \longrightarrow \pi_* \xi$$

So let $\theta(\xi) = \alpha(\pi_* \xi)$.

q coords in M, (p,q) coords in T^*M.

Assume $\alpha = \sum_i p_i dq_i$ in coords.

$$\xi = \sum_i a_i \frac{\partial}{\partial q_i} + \sum_i b_i \frac{\partial}{\partial p_i}$$

$$\pi_* \xi = \sum_i a_i \frac{\partial}{\partial q_i}$$

$$\theta\left(\frac{\partial}{\partial q_i}\right) = \alpha\left(\pi_* \frac{\partial}{\partial q_i}\right) = p_i \text{ if } \alpha = \sum p_i dq_i .$$

Then $\theta = \sum p_i dq_i$.

Then $\omega = d\theta = \sum_i dp_i \wedge dq_i$.

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Symplectic Manifold : (M^{2n}, ω), $d\omega = 0$, ω a 2-form, nondegenerate. → symplectic structure

Ex : T^*M is a $(2n)$-dim'l. locally, $\omega = \sum_i dq_i \wedge dp_i = dq \wedge dp$.

Hamiltonian vector fields :

Given a vector field $X \in \mathfrak{X}(M)$ = set of all vect. fields, i.e. maps to tangent, we can define a 1-form as

$$\omega_X(Y) = \omega(Y, X).$$

This relation defines a map.

$$\mathfrak{X}(M) \to \Omega^1(M)$$

The map is linear & since ω is nondegenerate, there is no v.f. in kernel

Recall: For any $p \in M$ & any $v \in T_pM$, $\exists\, u$ s.t. $\omega(p)(v,u) \neq 0$.

This map is invertible: $\Omega^1(M) \xrightarrow{I} \mathfrak{X}(M)$.

In particular, given a fcn $H: M \to \mathbb{R}$, dH is a 1-form, and so $\exists$ v.f. $I(dH) = \xi_H$ with the property that

$$dH(\eta) = \omega(\eta, \xi_H) \text{ for any v.f. } \eta.$$

ξ_H is called the Hamiltonian vector field associated to H.

Ex: on T^*M w/ symplectic form $\omega = dp \wedge dq = \sum_i dp_i \wedge dq_i$.

If $X = \sum_i a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial p_i}$ is a vector field, then

$$\omega(\cdot, X) = -\omega(X, \cdot) = -\sum_i (b_i dq_i - a_i dp_i) \quad [\text{a 1-form}]$$

locally consider the basis $\{\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}\}$ of $\mathfrak{X}(M)$ & $\{dq_i, dp_i\}$ of $\Omega^1(M)$

X in the basis is $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ & the image is $\begin{pmatrix} -b_i \\ a_i \end{pmatrix}$. Thus, the map

$\mathfrak{X}(M) \to \Omega^1(M)$ is $\begin{pmatrix} a_i \\ b_i \end{pmatrix} \mapsto \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix}$.

Then the inverse $I: \Omega^1(M) \to \mathfrak{X}(M)$ locally looks like

$$\begin{pmatrix} c \\ d \end{pmatrix} \longmapsto \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}.$$

Locally, $dH = \sum_i \underbrace{\frac{\partial H}{\partial q_i}}_{c} dq_i + \underbrace{\frac{\partial H}{\partial p_i}}_{d} dp_i$, so $\xi_H = \sum_i \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}$

The eqns for the flow are

$$\frac{dq_i}{dt} = \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \& \quad \dot{p}_i = \frac{-\partial H}{\partial q_i}.$$

<u>Property</u>: $H(\phi^t_{\xi_H}(x)) = H(x)$ if $\phi^t_{\xi_H}$ is the flow of ξ_H, i.e. the flow preserves H. [energy is preserved]

<u>Pf</u>: $\xi_H(H) = \frac{d}{dt}(H(\phi^t_{\xi_H}(x))) =$

$$= dH(\xi_H) \quad \text{(differential applied to vect. field is v.f. applied to fcn)}$$
$$= \omega(\xi_H, \xi_H)$$
$$= 0$$

□

<u>Def</u>: Let $F, H: M \to \mathbb{R}$ be 2 Hamiltonians. Define $\{F,H\}: M \to \mathbb{R}$ as the fcn given by
$\{F, H\} = \omega(\xi_H, \xi_F)$. $\{F,H\}$ is called the <u>Poisson bracket</u> associated to ω. Note: $\omega(\xi_H, \xi_F) = dF(\xi_H) = \xi_H(F)$

<u>Properties</u>:

(1) $\{F, H\} = -\{H, F\}$, i.e. $\xi_H(F) = -\xi_F(H)$

(2) $\{,\}$ is bilinear over $\mathbb{R}$.

(3) $\{FG, H\} = F\{G,H\} + G\{F,H\}$ (Leibniz property):
$d(FG)(\xi_H) = FdG(\xi_H) + GdF(\xi_H)$ ✓

(4) $\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0$ (Jacobi identity):

$$0 = d\omega(\xi_H, \xi_F, \xi_G) \overset{\text{by def}}{=} \underbrace{\xi_G(\omega(\xi_H,\xi_F))}_{\xi_G(\{F,H\})} - \underbrace{\xi_F(\omega(\xi_H,\xi_G))}_{\xi_F(\{G,H\})} + \underbrace{\xi_H(\omega(\xi_F,\xi_G))}_{\xi_H(\{G,F\})}$$
$$- \omega([\xi_H,\xi_F],\xi_H) + \omega([\xi_H,\xi_G],\xi_F) - \omega([\xi_F,\xi_G],\xi_H) \quad (*)$$
$$= \{\{F,H\},G\} - \{\{G,H\},F\} + \{\{G,F\},H\} + (*)$$
$$= \{\{F,H\},G\} + \{\{H,G\},F\} + \{\{G,F\},H\} + (*)$$

$$\omega([\xi_H,\xi_F],\xi_G) = [\xi_H,\xi_F](G)$$
$$= \xi_H(\underbrace{\xi_F(G)}_{\{G,F\}}) - \xi_F(\underbrace{\xi_H(G)}_{\{G,H\}})$$
$$= \{\{G,F\},H\} - \{\{G,H\},F\}$$

In $(*)$, this term is negative, so get: $\{\{G,H\},F\} - \{\{G,F\},H\}$.

When we add all 3 terms in $(*)$, we get:

$$2\{\{G,H\},F\} - 2\{\{G,F\},H\} + 2\{\{H,F\},G\}$$

Thus the whole sum yields $0 = \{\{F,H\},G\} + \{\{H,G\},F\} + \{\{G,F\},H\}$. ✓

Closer Look at Jacobi identity

$$0 = -\{\{F,H\},G\} - \{\{H,G\},F\} - \{\{G,F\},H\}$$
$$= \xi_{\{F,H\}}(G) + \xi_F(\xi_H G) - \xi_H(\xi_F G)$$
$$\Rightarrow \xi_{\{F,H\}} = [\xi_H, \xi_F]$$

Cor: The Hamiltonian vector fields form a subalgebra of the vector fields, i.e. Lie bracket of 2 H. v.f.'s is an H. v.f.

Cor: If G & F are preserved by ξ_H, then $\{G,F\}$ is also preserved. Such quantities are called first integrals.

Pf: Notice that if G is preserved by ξ_H, then $\xi_H(G) = 0 = \{G,H\}$. (i.e. if flow keeps G constant, then the deriv. along flow $= 0$). If $\{G,H\} = \{F,H\} = 0$, then by Jacobi identity, $\{\{G,F\},H\} = 0$.

Cor: The Hamiltonian flows for ξ_H & ξ_G commute $\Leftrightarrow$ $\{H,G\}$ is constant.

Pf: Flows commute $\Leftrightarrow$ Lie bracket $= 0$, but $[\xi_H,\xi_G] = \xi_{\{H,G\}}$.
But $\xi_F = 0$ only if F is locally constant, b/c $0 = \xi_F(R) = -dF(\xi_R)$
$\xi_F = 0 \Leftrightarrow \omega(\ ,\xi_F) = 0$, but $\exists$ 1-1 corresp. w/ forms, so $dF = 0$, & F is locally constant.

Problems:

① Find ξ, ζ locally on T^*M.

② Show that on $\mathbb{R}^{2n}$ w/ form $\overset{\omega}{\overset{\shortparallel}{\sum_i dp_i \wedge dq_i}}$, every 1-parameter gp of diffeomorphisms preserving ω $((\varphi^t)^*\omega = \omega)$ is a Hamiltonian flow.

<u>Hint</u>: In $\mathbb{R}^{2n}$, any closed 1-form is exact.

<u>Thm</u> (Darboux's thm) Let ω be a closed, nondegenerate 2-form in a nbhd of $p \in \mathbb{R}^{2n}$. Then, $\exists$ coordinates (p_i, q_i) s.t. $\omega = \sum_i dp_i \wedge dq_i$

<u>Thm</u>: (Liouville's Thm): Assume M^{2n} is a symplectic mfld & we have n independent fcns $F_1, \dots, F_n$ in involution (i.e. $[F_i, F_j] = 0$). Let $M_a = \{x \mid F_i(x) = a_i, i = 1, \dots, n\}$. Then (a) M_a is a smooth mfld invariant under the flow ξ_{F_i}.

(b) If M_a is cpt & ctd, then it is diffeomorphic to the torus T^n.

(c) The flow of ξ_{F_1} can be written in coordinates (ω, φ) w/ $\frac{d\omega}{dt} = 0$ & $\frac{d\varphi}{dt} = \omega$.

Called <u>action-angle variables</u>

flow.

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Integration

Recall: Line integrals: $\gamma : [a,b] \to \mathbb{R}^2$, $\gamma(t) = (x(t), y(t))$

$$\underbrace{\int_\gamma}_{\text{a singular 1-cube}} f\,dx + g\,dy = \int_a^b \underbrace{[f(\gamma(t))x'(t) + g(\gamma(t))y'(t)]\,dt}_{\text{pullback of the form } f\,dx + g\,dy}$$

Def: We call $I^k : [0,1]^k \to \mathbb{R}^k$ given by the inclusion the <u>standard k-cube</u>. We say $c : [0,1]^k \to M$, (C^∞) is a <u>singular k-cube</u>. We can write $c \circ I^k$ instead of c. By definition, $[0,1]^0 = \{0\}$, & a 0-cube is $c : \{0\} \to M$, a pt in M.

Ex: $c : [0,\pi] \times [0, 2\pi] \to \mathbb{R}^3$ (or S^2) (can reparametrize to be $[0,1]$)

$(\varphi, \theta) \longrightarrow$ spherical coords

Note, $\operatorname{Im}(c) = S^2$, so S^2 is a singular cube. The torus is also a cube.

Def: If ω is a k-form on $[0,1]^k$, $\omega = f\,dx_1 \wedge \cdots dx_k$ w/ $\{x_i\}$ coords on $[0,1]^k$ (or $\mathbb{R}^k$), then

$$\int_{[0,1]^k} \omega := \underbrace{\int_{[0,1]^k} f\,dx_1 \cdots dx_k}_{\text{Riemann integral.}}$$

If c is a singular cube & ω is a k-form on M $(c : [0,1]^k \to M)$, then

$$\int_c \omega := \int_{[0,1]^k} c^*\omega.$$

If $k = 0$, & ω is a zero-form, $\omega = f$, then

$$\int_{[0,1]^0} \omega := f(0)$$

Note: We can only integrate a k-form over a k-cube.

Lemma: Let c be a 1-1 singular n-cube, $c : [0,1]^n \to \mathbb{R}^n$ w/ $\det c' \geq 0$ on $[0,1]^n$. Let $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$. Then $\int_c \omega = \int_{c([0,1]^n)} f\,dx_1 \cdots dx_n$

Pf: $\int_c \omega := \int_{[0,1]^n} c^*\omega \overset{\text{def of pullback}}{:=} \int_{[0,1]^n} f \circ c \det c'\,dx_1 \wedge \cdots \wedge dx_n$. Since $\det c' \geq 0$,

$$= \int_{[0,1]^n} f \circ c\,|\det c'|\,dx_1 \cdots dx_n \underset{\text{Riemann change of var.}}{=} \int_{c([0,1]^n)} f\,dx_1 \cdots dx_n.$$

$\square$

Prop (Independence from reparametrization): Let $p : [0,1]^k \to [0,1]^k$ be 1-1 & onto. Assume $\det p' \geq 0$. Let c be a k-cube & ω a k-form in M. Then,

$$\int_c \omega = \int_{c \circ p} \omega .$$

Pf: $\int_{c\circ p} \omega = \int_{[0,1]^k} (c \circ p)^* \omega = \int_{[0,1]^k} p^*(c^*\omega) \underset{\text{lemma}}{=} \int_{p([0,1]^k)} c^*\omega$

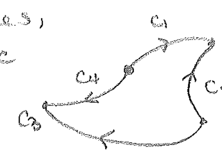
$\underset{p \text{ onto.}}{=} \int_{[0,1]^k} c^*\omega = \int_c \omega$ $\square$

Def: If $\det p' \neq 0$ everywhere, then p is <u>orientation preserving</u> if $\det p' > 0$, <u>reversing</u> if $\det p' < 0$.

Note: If p preserves orientation, $\int_c \omega = \int_{c\circ p} \omega$. If p reverses orientation, then $\int_c \omega = -\int_{c \circ p} \omega$.

Def: A formal sum of singular k-cubes w/ integer coefficients is called a <u>k-chain</u>. $c = \sum a_i c_i$

ex: If c_i are singular 1-cubes, there <u>may</u> be a geometric meaning:

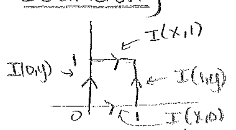


$c_1 - c_2 + c_3 - c_4$ means, <u>geometrically</u>, follow in this order w/ this orientation.

Def: If ω is a k-form, $\int_c \omega := \sum a_i \int_{c_i} \omega$.

Note: $-c_i$ represents c_i w/ opp. orientation, since $\int_{-c_i} \omega = -\int_{c_i} \omega$. kc_i represents c_i traveled k times in same direction (for a curve).

Boundary



think of this as the image of 1-cubes.

$\partial I^2 = I(x,0) + I(1,y) - I(x,1) - I(0,y)$, oriented boundary

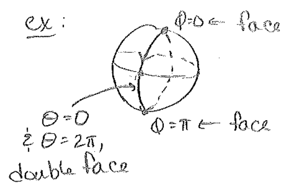
$\underbrace{2 \qquad 2}_{\text{even}}$ $\underbrace{3 \qquad 1}_{\text{odd}}$

add the number you plug in (0 or 1) to the position where you put it. (1 or 2)

For ⬡, plug in 0 or 1, but the positions are 1, 2, or 3.

Def: Given a k-cube, we define the <u>(i,α)-face</u> to be a $(k-1)$-cube $I^k_{(i,\alpha)} : [0,1]^{k-1} \longrightarrow M$, $I^k_{(i,\alpha)}(x_1, \dots, x_{k-1}) = I^k(x_1, \dots, \underset{i\text{th position}}{\alpha}, \dots, x_{k-1})$. If c is a singular k-cube, we define the (i,α)-face to be $c_{i,\alpha} = c \circ I^k_{(i,\alpha)}$.

ex:



So 4 faces, 2 are pts, & the other 2 are the same arc, at $\theta = 0, \theta = 2\pi$.

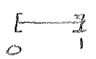
Def: The <u>boundary</u> of I^k is $\partial I^k = \sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^k_{(i,\alpha)}$. Similarly, $\partial c = \sum_{i=1}^{k} \sum_{\alpha=1,0} (-1)^{i+\alpha} c_{i,\alpha}$.

ex: • $\partial S^2 = -(0,\theta) + (\phi,0) + (\pi,\theta) - (\phi,2\pi)$

if (ϕ,θ) $= -N + f + S - f$

when you integrate, $f - f$ disappears, so just integral over 2 pts, so the integral of a 1-form is just 0 (b/c one var. always constant), i.e. $\int_{\partial S^2} \omega = 0$.

• $\partial[0,1] = -\{0\} + \{1\}$, i.e. $\int_0^1 f\,dx = F(1) - F(0)$.



• $\partial(\partial I^2) = I^2(1,0) - I^2(0,0) + I(1,1) - I(1,0) - I(1,1) + I(0,1) - I(0,1) + I(0,0)$
$= 0$.

Thm: $\partial^2 c = 0$, for c a k-chain.

Pf: Assume c a singular k-cube.

$\partial(\partial c) = \partial\left[\sum_{i=1}^{k} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{i,\alpha}\right]$

$= \sum_{j=1}^{k-1} \sum_{i=1}^{k} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} (c_{i,\alpha})_{j,\beta}$

$(I_{i,\alpha})_{j,\beta}(x_1,\dots,x_{k-2}) = I_{i,\alpha}(x_1,\dots,x_{j-1},\beta,x_j,\dots,x_{k-2})$. Assume $i<j$.

$= I(x_1,\dots,x_{i-1},\alpha,x_{i+1},\dots,x_{j-1},\beta,x_j,\dots,x_{k-2})$

$= (I_{j+1,\beta})_{i,\alpha}$

But in the sum, these will have opp. signs b/c first is $(-1)^{i+j+\alpha+\beta}$, & 2nd is $(-1)^{i+j+1+\alpha+\beta}$, so these cancel.

$$\left.\begin{array}{l} i: 1,2,3,\dots,k \\ j: 1,2,\dots,k-1 \end{array}\right\} \to \begin{array}{l} (i,j): \\ \left.\begin{array}{l} \boxed{(1,1)}, \cancel{(1,2)}, \dots, (1,\cancel{k-1}) \\ \boxed{(2,1)}\ \boxed{(2,2)}\ \cancel{(2,3)}, (2,k-1) \\ \cancel{(3,1)}\ (3,2) \end{array}\right\} i<j \\ \left.\begin{array}{l} \qquad\qquad\qquad (k-1,k-1) \\ (\cancel{k},1), (\cancel{k},2) \dots, \boxed{(k,k-1)} \end{array}\right\} \text{remaining is diagonal \& subdiag.} \end{array} \qquad (i,j) \leftrightarrow (j+1,i)$$

same w/ opp signs just by counting.

So, $\partial^2 c = 0$.

$\square$

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Thm: $\partial^2 c = 0$.

Pf: Done on I^k.

$\partial c = c \circ \partial I^k = \sum_{i=1}^{k} \sum_{\alpha=1,0} (-1)^{i+\alpha} c_{i,\alpha}$ ← faces of the boundary

Then the proof of $\partial^2 c = 0$ is identical to $\partial^2 I^k = 0$. (substitute $c_{i\alpha}$ for $I^k_{i,\alpha}$). And if c is a chain, $c = \sum a_i c_i$, then linearity ensures $\partial^2 c = 0$.

$\square$

Note: • $\partial^2 c = 0$ parallel to $d^2\omega = 0$

• c is closed if $\partial c = 0$ parallel to ω closed if $d\omega = 0$

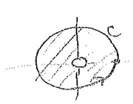
↑ chain is 0 when $\int$ over chain $= 0$.

Recall: sphere is a closed chain.

• $d\omega = 0 \not\Rightarrow \omega = df$. we have the example $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $\mathbb{R}^2 \setminus \{(0,0)\}$.

parallel to

In general, $\partial c = 0 \not\Rightarrow c = \partial \hat{c}$

ex: 

$\partial c = 0$, but $c \neq \partial \hat{c}$, b/c there's a hole, & $\hat{c}$ would have to be the image of $[0,1]^2$.

Stokes Thm (for chains): Let ω be a $(k-1)$-form on M & c a k-chain. Then

$$\int_{\partial c} \omega = \int_c d\omega. \qquad \left[\begin{array}{l} d\omega \text{ a } k\text{-form,} \\ \partial c \text{ a } (k-1)\text{-chain} \end{array}\right]$$

Pf: Case I: $c = I^k$, & $M = \mathbb{R}^k$. & $\omega = f\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ ← one missing. For simplicity, assume $\omega = f\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$

Let's pull back ω to one of the faces: (recall, $\partial c = \partial I^k = \sum_{i,\alpha} (-1)^{i+\alpha} I^k_{i,\alpha}$)

$(I^k_{j,\alpha})^* \omega = (I^k_{j,\alpha})^* (f\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k)$

& $I_{j,\alpha}(x_1, \dots, x_k) = I^k(x_1, \dots, \underset{j}{\alpha}, \dots, x_k) = (x_1, \dots, \alpha, \dots, x_k)$ so the $\mathbb{1}$ on all but j^{th} spot

(if $j \neq i$, then $I^{k\,*}_{j,\alpha} dx_j = 0 = d\alpha$ ↑ b/c α a const ($=0$ or 1) so $\uparrow \neq 0$ only when $j = i$)

$= \delta^j_i\, f(x_1, \dots, \underset{i}{\alpha}, \dots, x_k)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$

So, $\displaystyle\int_{\partial I^k} \omega = \sum_{j=1}^{n} \sum_{\alpha=1,0} (-1)^{i+\alpha} \int_{I^k_{j,\alpha}} \omega$

$\displaystyle \overset{\text{by def}}{\int_{I^k_{j,\alpha}}} = \int_{[0,1]^{k-1}} \underbrace{(I^k_{j,\alpha})^* \omega}_{=0 \text{ if } j \neq i}$

$$= (-1)^{i+1} \Big[\int_{[0,1]^{k-1}} f(x_1, \dots, \overset{i}{1}, \dots, x_k)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \Big\} \alpha = 1$$

$$- \int_{[0,1]^{k-1}} f(x_1, \dots, \overset{i}{0}, \dots, x_k)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \Big] \Big\} \alpha = 0$$

$\displaystyle\int_{I^k} d\omega$:

move

$$d\omega = \sum_j D_j f\, dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k \quad (\text{all} = 0 \text{ if } i \neq j)$$

$$= D_i f\, dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx^k \; (-1)^{i-1}$$

Riemann integral → b/c this is the def of $\int d\omega$, but need the dx_i in correct order.

So $\displaystyle\int_{[0,1]^k} d\omega = \int_{[0,1]^k} (-1)^{i-1} D_i f\, dx_1 \cdots dx_k$

Fubini

$$= (-1)^{i-1} \int_{[0,1]^{k-1}} \left[f(x_1, \dots, \overset{i}{1}, \dots, x_k) - f(x_1, \dots, \overset{i}{0}, \dots, x_k) \right] dx_1 \cdots \widehat{dx_i} \cdots dx_k$$

integrate in i^{th} comp. (dx_i)

$$= \int_{\partial I^k} \omega \quad ✓$$

<u>Case II</u>: c is a singular cube on $[0,1]^k$, then

$$\int_{\partial c} \omega \overset{\partial c = c \circ \partial I^k}{=} \int_{\partial I^k} c^* \omega \overset{\text{case I}}{=} \int_{I^k} d(c^* \omega) \overset{d \,\&\, * \text{ commute.}}{=} \int_{I^k} c^* d\omega = \int_c d\omega \quad ✓$$

(b/c $c^* \omega$ is on $\mathbb{R}^k$)

<u>Case III</u>: $c = \sum a_i c_i$.

Linearity proves this case. ✓

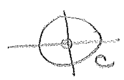
□

Applications of Stoke's Thm

① Assume a form ω, & $0 \neq \int_c \omega$ for some closed c. Then ω is not exact: [usually use this for closed forms which we suspect are exact]

If $\omega = d\eta$, then $\int_c \omega = \int_c d\eta = \int_{\partial c} \eta = 0$ ↯ as $\partial c = 0$ b/c closed.

* Thus can check non-exactness of forms by integrating.

Ex: $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$  , $\partial c = (1,0)-(1,0) = 0$.

$c: [0,1] \to \mathbb{R}^2$

$t \mapsto (\cos^2 2\pi t, \sin^2 2\pi t)$

$$\int_c \omega = \int_0^1 \underbrace{(\sin^2 2\pi t \cdot 2\pi + \cos^2 2\pi t \cdot 2\pi)}_{c^*\omega} dt$$ ← subst. image of c in ω.

$= 2\pi \neq 0.$

Thus $\omega \neq d\theta$.

② If $0 \neq \int_c \omega$ & ω is closed, then $c \neq \partial \hat{c}$:

$$\int_c \omega = \int_{\partial \hat{c}} \omega = \int_{\hat{c}} d\omega \overset{d\omega = 0}{=} 0 \text{ ↯}$$

Goal: Stoke's Thm on Mfld. We want to parametrize the mfld.

We need to know: (1) What does $\int_M \omega$ mean? (2) Is $\int_{C_1} \omega$ $\int_{C_2} \omega$ on the intersection? (3) How do you orient ∂M? (4) What is T_pM, $p \in \partial M$?

Thm: Let M be an orientable n-mfld, & let c_1, c_2 be 2 chains $c_i: [0,1]^n \to M$ s.t. c_i are diffeomorphisms on a nbhd of $[0,1]^n$ & they both preserve (or reverse) orientation. Let $C = c_1([0,1]^n) \cap c_2([0,1]^n) \subseteq M$ and assume ω is an n-form w/ supp $\omega \subseteq C$. Then

$$\int_{c_1} \omega = \int_{c_2} \omega.$$

Pf: $$\int_{c_1} \omega \overset{\text{supp } \omega \subseteq C}{=} \int_{\underbrace{c_2 \circ c_2^{-1} \circ c_1}_{\text{b/c diffeo. def. on } C \text{ only}}} \omega = \int_{c_2^{-1} \circ c_1} c_2^* \omega$$, call $c_2^{-1} \circ c_1 = c$.

$c_2^* \omega$ an n-form in $[0,1]^n$, so $c_2^* \omega = f\, dx_1 \wedge \cdots \wedge dx_n$, so

$c^* c_2^* \omega = f \circ c\, (\det c')\, dx_1 \wedge \cdots \wedge dx_n = f \circ c\, |\det c'|\, dx_1 \wedge \cdots \wedge dx_n$

↑ both pres or rev or.

Riemann int $\qquad c: c_1^{-1}(C) \to c_2^{-1}(C)$

$$\int_c c_2^* \omega = \int_{c_1^{-1}(C)} c^* c_1^* \omega = \int_{c_1^{-1}(C)} f \circ c \, |\det c'| \, dx_1 \cdots dx_n$$

$$\overset{\text{change of vars.}}{=} \int_{c_2^{-1}(C)} f \, dx_1 \cdots dx_n$$

$$= \int_{c_2^{-1}(C)} f \, dx_1 \wedge \cdots \wedge dx_n$$

$$= \int_{c_2^{-1}(C)} c_2^* \omega$$

supp $\omega \subseteq C$ & supp $c_2^*\omega \subseteq c_2^{-1}(C) \subseteq [0,1]^n$

$$= \int_{[0,1]^n} c_2^* \omega$$

$$= \int_{c_2} \omega$$

$\square$

Def: Assume c is an n-chain, & let ω be an n-form in M w/ supp $\omega \subseteq c([0,1]^n)$. Then $\int_M \omega := \int_c \omega$.

Note: By the previous thm, this is well-def.

If supp $\omega \not\subseteq c([0,1]^n)$ for any c, we cover M with open sets $\{\Theta\}$ & a partition of unity associated to $\{\Theta\}$, $\{\psi_\Theta\}$ w/ each $\Theta \subseteq c_\Theta([0,1]^n)$ for some chain c_Θ. We can always do this by covering M w/ coord charts $(\chi(p) = 0)$; then pick a box around 0 in $\mathbb{R}^n$, $[-\varepsilon,\varepsilon]^n$ & an $\cong$ $[-\varepsilon,\varepsilon]^n \xrightarrow{f} [0,1]^n$. Then $c = \chi^{-1} \circ f^{-1}$. Then, by def, if ω has cpt supp, $\int_M \omega := \sum_\Theta \int_{c_\Theta} \psi_\Theta \cdot \omega$ [cpt supp ensures sum is finite].

But does this def. depend on the partition of unity? If both have same cover

$$\sum_\Theta \int_{c_\Theta} \psi_\Theta \cdot \omega = \sum_\Theta \int_{c_\Theta} \underbrace{\left(\sum_{\Theta'} \varphi_{\Theta'}\right)}_{=1} \psi_\Theta \, \omega = \sum_{\Theta'} \int_{c_\Theta} \underbrace{\sum_{\Theta} \varphi_{\Theta'} \psi_\Theta}_{\sum_\Theta \psi_\Theta = 1} = \sum_{\Theta} \int_{c_\Theta} \varphi_\Theta \, \omega$$

If both don't have same cover – next time.

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* If $\mathrm{supp}\,\omega \subseteq c([0,1]^k)$, ω a k-form, $\int_M \omega := \int_c \omega$.

HW: Prove $\int_M$ is a linear map.

If this is true, then $\int_M \omega = \sum_\varphi \int_M \varphi\omega$, $\{\varphi\}$ a partition of unity does not depend on $\{\varphi\}$, since

$$\sum_\varphi \int_M \varphi\omega \underset{\uparrow}{=} \sum_\varphi \int_M \sum_\psi \varphi\psi\omega = \sum_\psi \int_M \sum_\varphi \varphi\psi\omega = \sum_\psi \int_M \psi\omega.$$

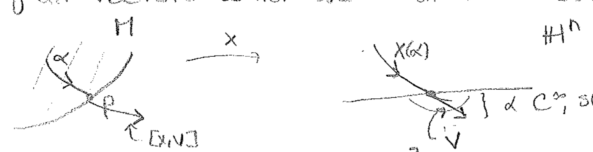
b/c $\sum_\psi \psi = 1$.

Manifolds w/ Boundary

A mfld M, & for any $p \in M$ $\exists$ $x: U \to \mathbb{R}^n$ or $U \to \mathbb{H}^n = \{(x_1, \dots, x_n) \mid x_n \geq 0\}$,
If $\mathbb{H}^n$, we say $p \in \partial M$.

To define T_pM, $p \in \partial M$, we have to allow curves that start / end at p, giving outward-pting / inward-pting vectors.

Define $T_p^{out} M = \{\alpha'(0) \mid \alpha: (-\varepsilon, 0], \alpha(0) = p\}$, where $\alpha' = [x, v]$ is the class of all vectors which are $=$ in diff. coords.



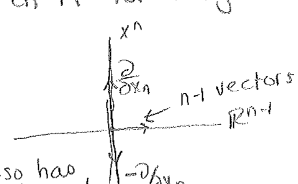
$\} \alpha\ C^\infty$, so can extend a bit

$T_p^{in} M = \{\alpha'(0) \mid \alpha: [0, \varepsilon) \to M, \alpha(0) = p\}$

Then, $T_p\partial M = T_p^{out} M \cap T_p^{in} M$, so vectors you can get from both curves that end & that start at p. So the tan. plane, for ex, is the whole plane, not just half.

Def: Assume M is an oriented mfld. ∂M is positively oriented with the induced orientation if $\{v_1, \dots, v_{n-1}\}$ are positively or. whenever $\{v, v_1, \dots, v_{n-1}\}$ is positively oriented in M for any v pointing outwards.

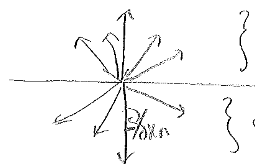
Ex: How is $\partial \mathbb{H}^n$ or. w/ this or?
If $(x_1, \dots, x_n)$ are usual coords, then $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}$ gen ∂M. What is sign of $\det(-e_n, e_1, \dots, e_n) = (-1)^n$



* so has induced or $\Leftrightarrow$ n even.

So if the or. in $\partial \mathbb{H}^{n-1} = \mathbb{R}^{n-1}$ is pos. if $\{e_1, \dots, e_{n-1}\}$ is pos. (the standard or. of $\mathbb{R}^{n-1}$), then it is the induced or. if n is even, & the opp. one if n is odd.

Note:



} any vector here will give opp. sign as $\partial/\partial x_n$.

$\mathbb{R}^{n-1} = \{x_n = 0\}$

} any vector here will give same sign as $-\partial/\partial x_n$

Note: ∂M as a submfld of M has an atlas given by intersections of the atlas of M w/ the boundary. The image is the $\partial \mathbb{H}^n \simeq \mathbb{R}^{n-1}$, thus $\dim \partial M = n-1$. If the atlas is or. in M, then the atlas in ∂M will be oriented.

What is the integral of a mfld w/ ∂?

Case I: Assume $M \subseteq \mathbb{R}^m$, & $\partial M \neq \emptyset$. If f has cpt supp, we define

$$\int_M \underbrace{f dx_1 \wedge \cdots \wedge dx_n}_{\text{an } n\text{-form}} = \int_M f dx_1 \cdots dx_n, \text{ the Riemann integral}$$

($m(\partial M) = 0$, so doesn't change the R. integral)

Case II: $\operatorname{supp} \omega \subseteq c([0,1]^n)$, & $\int_M \omega = \int_c \omega = \int_{[0,1]^n} c^* \omega$, & back to case I.

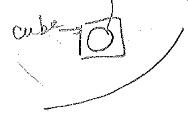
Case III: use partition of unity to patch up $\int_M \omega$ as before.

Prop: If $f: M^n \to N^n$ is a diffeomorphism and ω is an n-form in N w/ cpt supp, then
$$\int_M f^* \omega = \begin{cases} \int_N \omega & \text{if } f \text{ preserves or.} \\ -\int_N \omega & \text{if } f \text{ reverses or.} \end{cases}$$

Pf: Follow the proof of the change of variables. □

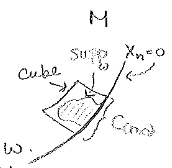
Thm (Stoke's): Let M be oriented, n-dim'l mfld w/ $\partial M \neq \emptyset$, & assume ∂M has the induced orientation. Let ω be an $(n-1)$-form w/ cpt supp. Then $\int_M d\omega = \int_{\partial M} \omega$.

Pf: __Case I__: Assume $\text{supp}\,\omega \subseteq \overset{\circ}{\overline{c([0,1]^n)}} = \text{int}(c([0,1]^n))$, and $c([0,1]^n) \subseteq M \setminus \partial M$.



Then $\int_{\partial M} \omega = \int_{\partial c} \omega \underset{\substack{\uparrow \\ \text{Stoke's for} \\ \text{chains}}}{=} \int_c d\omega = \int_M d\omega \quad [=0]$

__Case II__: Assume $\partial M \cap c([0,1]^n) = c_{(n,0)}([0,1]^{n-1})$ & $\text{supp}\,\omega \subseteq \overset{\circ}{\overline{c([0,1]^n)}}$, which includes $c_{(n,0)}([0,1]^{n-1})$.



Then $\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega \underset{\uparrow}{=} (-1)^n \int_{c_{(n,0)}} \omega = (-1)^n(-1)^n \int_{\partial M} \omega.$

b/c $\text{supp}\,\omega \subseteq \overset{\circ}{\overline{c([0,1]^{n-1})}}$, so on ∂c, $\omega \neq 0$ __only__ on $c_{(n,0)}$, & the sign of $c_{(n,0)}$ (in the sum for ∂c) is $(-1)^n$.

But the or. of $[0,1]^{n-1}$ as $\partial \mathbb{H}^n$ is $(-1)^n$ the induced one that M has

__Case III__: Cover M w/ coordinate systems $\{\theta\}$ & a partition of unity $\{\phi\}$ s.t. $d\omega$ is of type I or II. Then

$$\int_M d\omega = \sum_\phi \int_M \phi\, d\omega = \sum_\phi \int_M \underbrace{d\phi \wedge \omega}_{\downarrow} + \phi\, d\omega = \sum_\phi \int_M d(\phi\omega) \overset{\text{by case I/II}}{=} \sum_\phi \int_{\partial M} \phi\omega$$

$$= \int_{\partial M} \omega$$

b/c $d\phi \wedge \omega = 0$, since

$$\sum_\phi \int_M d\phi \wedge \omega = \int_M \underbrace{\sum_\phi d\phi}_{\substack{= d(1) = 0 \\ \text{b/c } d \text{ linear}}} \wedge\, \omega = 0$$

$\square$

__Recall__: $c_{(n,0)}(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, 0)$, so $c_{(n,0)}$ identifies $\mathbb{R}^{n-1}$ w/ $\partial \mathbb{H}^n$.

Applications

① Assume (M,μ) [i.e. M w/ an or.] is cpt & $\partial M = 0$. Then

$$\int_M d\omega = \int_{\partial M} \omega = 0 \text{ for any form } \omega.$$ (not can integrate any form b/c M cpt)

Hence, if $\int_M \omega \neq 0$, then ω is not exact, even if it's closed.

Cor: A cpt or mfld w/ $\partial M = 0$ cannot be contracted to a pt.

Pf: If M is or, there is always an n-form ω s.t. $\int_M \omega \neq 0$: Let ω be the form giving an orientation, so that $\omega(p) \neq 0$ & $\omega(p)(v_1, \ldots, v_n) > 0$ if $(v_1, \ldots, v_n)$ is positively or. So, locally, in or. charts, $\omega = f dx_1 \wedge \cdots \wedge dx_n$ w/ $f > 0$, & for $\int_M \omega$ you just patch up w/ partition of unity,

$$\int_M \omega = \sum_\phi \int_M \phi\omega > 0 \quad (\text{b/c } \phi > 0 \text{ around a pt}).$$

But $d\omega = 0$, b/c ω is already top dim'l. □

11/25 - DH

de Rham Cohomology

(1) In $\mathbb{R}^2 \setminus \{0\}$, we say the closed, non-exact form $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$

(2) On S^{n-1} we have ω, an $(n-1)$-form, giving the orientation, w/ $\int_{S^{n-1}} \omega > 0$. If $\omega = d\eta$, $\int_{S^{n-1}} \omega = \int_{\partial S^{n-1}} \eta = \int_{S^{n-1}} \underset{0}{\underset{\shortparallel}{d\omega}} \Rightarrow \omega \neq d\eta$ for any smooth η. $d\omega = 0$, since $\dim S^{n-1} = n-1 =$ order of ω, so $d\omega$ an n-form on an $(n-1)$dim'l mfld, so $=0$.

- $\omega + d\eta$ also closed but not exact also.

$0 \to \Omega^0(M) \to \cdots$
$\to \Omega^{k-1}(M) \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \to \cdots$, $d^2 = 0$

Let $Z^k(M) =$ the vect. sp. over $\mathbb{R}$ gen by closed k-forms $\leq \Omega^k(M)$
$= \ker d_k$

$B^k(M) =$ the vect sp. gen by exact forms $\leq \Omega^k(M)$
$= \operatorname{Im} d_{k-1}$

Since $d^2 = 0$, $B^k(M) \leq Z^k(M)$.

Define $H^k(M) := Z^k(M)/B^k(M)$, called the <u>k-de Rham cohomology</u> vector sp.

- The zero class is formed by closed, exact forms.
- $[\omega] = [\tilde{\omega}] \Leftrightarrow \omega = \tilde{\omega} + d\eta$, where $d\eta \in B^k(M)$.

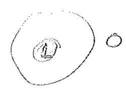
<u>Note</u>. de Rham proved this coincides w/ the simplicial cohom. of a mfld.

① Poincaré's Lemma says that if M is contractible, then $H^k(M) = 0$, $k > 0$.

② $H^0(M)$: $Z^0(M) =$ zero forms (fcns) w/ $df = 0$, ie. const. on ctd comp's.
$B^0(M) = 0$

$\Rightarrow \dim H^0(M) = \#$ of ctd components, ℓ, of M. (take $f = 1$ in ctd comp & $g = 1$ in diff ctd comp & 0 else. So all fcns will be lin. comb. of these, so have same # of fcns as ctd comps.

$\Rightarrow H^0(M) = \mathbb{R}^\ell$

③ If M is or & $\partial M = 0$, then $\exists \omega$ an n-form s.t. $\int_M \omega > 0$:
Let ω be the orienting form, & let $f =$  . Then
$f\omega = g\,dx_1\wedge\cdots\wedge dx_n$ w/ $g \geq 0$, & $g > 0$ in a nbhd.
Then $\int_M f\omega = \int_U g > 0$.

$d\omega = 0$ b/c ω an n-form on an n-dim'l mfld. By Stoke's, if $\omega = d\eta$
$0 \neq \int_M \omega = \int_{\partial M} \eta = 0$, so $\omega \neq d\eta \Rightarrow H^n(M) \neq 0$. (& so $H^{n-1}(S^{n-1}) \neq \{0\}$)

④ $H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$, b/c of our 1-form $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, closed but not exact.

<u>Def</u>: Let $A \subseteq X$ be a subset. We say $r: X \to A$ is a <u>retraction</u> if $i: A \hookrightarrow X$ inclusion & $r \circ i = \mathbb{1}_A$. A is a retraction of X.

<u>Note</u>: S^{n-1} is a retraction of $\mathbb{R}^n \setminus \{0\}$.

$$r: \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$$
$$p \longmapsto \frac{p}{|p|}$$

Let ω be the orienting $(n-1)$-form on S^{n-1} & consider $r^*\omega$.
Then $d(r^*\omega) = r^*(d\omega) = 0$, so $r^*\omega$ is exact. Assume $r^*\omega = d\eta$.
Then $\underbrace{i^*r^*}_{=\mathbb{1}^*}\omega = i^*d\eta = di^*\eta \Rightarrow \omega = d(i^*\eta)$. But ω is not exact, so neither is $r^*\omega$.

<u>Note</u>: $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is $r^*\omega$.

Define ω on S^{n-1} as $\omega(p)(v_1,\dots,v_{n-1}) = \det\begin{pmatrix} p \\ v_1 \\ \vdots \\ v_{n-1}\end{pmatrix}$. <u>Note</u>: ω is the restriction to S^{n-1} of $\sigma = \sum_{i=1}^{n} (-1)^{n-1} x_i\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n$; an $(n-1)$-form in $\mathbb{R}^n$:

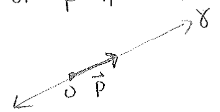
if $(v_1,\dots,v_{n-1})$ are tangent to S^{n-1} at $p \in S^{n-1}$, & we calculate $\sigma(p)(v_1,\dots,v_{n-1})$, then if $v_s = \sum a_j^s\, dx_j$, then $\sigma(p)(v_1,\dots,v_{n-1})$ is the expansion by minors of $\det\begin{pmatrix} p \\ v_1 \\ \vdots \\ v_{n-1}\end{pmatrix}$, b/c the x_i will give the i^{th} comp. of p, & the wedge is defined as a det, but never have the i^{th} comp's of v_s, so det of a minor.

Lemma: Let σ & ω be as before. $(r^*\omega)(p) = \frac{\sigma(p)}{|p|^n}$, that is,

$$(r^*\omega) = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{\nu^n} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \text{ where } \nu(p) = |p|$$

Note: This is our usual form for $n = 2$.

Pf: For $\vec{p} \in T_p(\mathbb{R}^n \setminus \{0\})$, $r_* \vec{p} = \frac{d}{dt}(\gamma(t)) = \frac{d}{dt}\left(\frac{\gamma(t)}{|\gamma(t)|}\right) = \frac{d}{dt}\left(\frac{\vec{p}}{|\vec{p}|}\right) = 0$



For $\vec{v} \in T_p(\underbrace{S^{n-1}(|p|)}_{(n-1)\text{-sphere w/ radius } |p|})$, $r_* \vec{v} = \frac{d}{dt}\Big|_0 (r \circ \gamma(t)) = \frac{d}{dt}\Big|_0 \left(\frac{\gamma(t)}{|\gamma(t)|}\right) = \frac{\gamma'(0)}{|\gamma(0)|} - \frac{\gamma(0)}{|\gamma(0)|^2} |\gamma(t)|'\Big|_{t=0}$

for $\gamma(0) = p$, $\gamma'(0) = \vec{v}$ $\quad = \frac{\vec{v}}{|p|}$ $\quad\quad |\gamma(t)|' = \frac{\gamma(t)\cdot\gamma'(t)}{|\gamma(t)|} = \frac{\vec{p}\cdot\vec{v}}{|p|} = 0$

Notice $T_p(\mathbb{R}^n \setminus \{0\}) = \underbrace{\langle \vec{p} \rangle}_{\text{radius}} \oplus T_p(S^{n-1}(|p|))$ ← called the polar splitting.

So $r_* p = 0$ & $r_* \vec{v} = \frac{\vec{v}}{|p|}$.

ω is an $(n-1)$-form on $\mathbb{R}^n \setminus \{0\}$.

Case I: $r^*\omega(p)(\vec{p}, v_1, \ldots, v_{n-2}) = \omega(r(p))(\underbrace{r_* \vec{p}}_{=0}, r_* v_1, \ldots, r_* v_{n-2}) = 0$.

$\sigma(p)(\vec{p}, v_1, \ldots, v_{n-2}) = \det\begin{pmatrix} \vec{p} \\ v_1 \\ \vdots \\ v_{n-2} \end{pmatrix} = 0$.

Case II: $r^*\omega(p)(v_1, \ldots, v_{n-1}) = \omega(r(p))(r_* v_1, \ldots, r_* v_{n-1}) = \omega\left(\frac{p}{|p|}\right)\left(\frac{v_1}{|p|}, \ldots, \frac{v_{n-1}}{|p|}\right)$

$v_i \in T_p S^{n-1}(|p|)$ $\quad = \frac{1}{|p|^n} \det\begin{pmatrix} p \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$

$\sigma(p)(v_1, \ldots, v_{n-1}) = \det\begin{pmatrix} p \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$.

So $r^*\omega = \frac{\sigma(p)}{|p|^n}$ $\square$

Lemma: Let $f: B_1(0) = B \to \mathbb{R}$, & define $g = \int_0^1 u^{n-1} f(up)\, du$. Then

$\underbrace{\int_B f}_{\text{Riemann int}} = \int_B f\, dx_1 \wedge \cdots \wedge dx_n = \int_{S^{n-1}} g\omega$. [convert from int whole ball to just sphere]

Pf: Strategy, Move $\int_{S^{n-1}} \to \int_{S^{n-1} \times \underbrace{[0,1]}_{\text{radius}}} \to \int_B$.

Let
$$\begin{array}{ccc} & S^{n-1}\times[0,1] & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ S^{n-1} & & [0,1] \\ \omega & & dt \end{array}$$

& define $\omega\wedge dt$ on $S^{n-1}\times[0,1]$ as
$\omega\wedge dt = (\pi_1^*\omega)\wedge(\pi_2^* dt)$, by abuse of notation.

Let $h(t,p) := t^{n-1} f(tp)$, $h: S^{n-1}\times[0,1]\to\mathbb{R}$, $g = \int_0^1 h(u,p)\,du$

$$\int_{S^{n-1}} g\,\omega \underset{\substack{\uparrow \\ HW}}{=} \int_{S^{n-1}\times[0,1]} h(t,p)\,\omega\wedge dt$$

Now let $\Phi: B\setminus\{0\}\to S^{n-1}\times[0,1]$
$$p \longmapsto (\underbrace{r(p)}_{\frac{p}{|p|}}, \underbrace{\nu(p)}_{=|p|})$$ (polar decomp, $\measuredangle$, radius)

$$\Phi^*(\omega\wedge dt) = \Phi^*(\pi_1^*\omega\wedge\pi_2^* dt) = (\pi_1\circ\Phi)^*\omega\wedge(\pi_2\circ\Phi)^* dt = r^*\omega\wedge\underbrace{\nu^* dt}_{=d|p|}$$
$$= r^*\omega\wedge\sum_i \frac{x_i}{|p|}dx_i \underset{\substack{\uparrow \\ \text{from def } r^*\omega}}{=} \sum_i \frac{x_i^2}{|p|^{n+1}} dx_1\wedge\cdots\wedge dx_n \underset{\substack{\uparrow \\ \sum_i x_i^2 = |p|^2}}{=} \frac{1}{|p|^{n-1}} dx_1\wedge\cdots\wedge dx_n$$

$$\Phi^*(h\,\omega\wedge dt) = (h\circ\Phi)(\Phi^*\omega\wedge dt) = h\left(|p|, \frac{p}{|p|}\right)\frac{1}{|p|^{n-1}} dx_1\wedge\cdots\wedge dx_n$$
$$= \cancel{|p|^{n-1}} f\left(|p|\cdot\frac{p}{|p|}\right)\frac{1}{\cancel{|p|^{n-1}}} dx_1\wedge\cdots\wedge dx_n$$
$$= f(p)\,dx_1\wedge\cdots\wedge dx_n.$$

So, $\int_B f\,dx_1\wedge\cdots\wedge dx_n = \int_{B\setminus\{0\}} f\,dx_1\wedge\cdots\wedge dx_n = \int_{B\setminus\{0\}} \Phi^*(h\,\omega\wedge dt) = \int_{S^{n-1}\times[0,1]} h\,\omega\wedge dt$

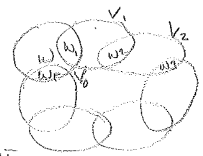
$= \int_{S^{n-1}\times[0,1]} h\,\omega\wedge dt = \int_{S^{n-1}} g\,\omega.$

12/4 - DM

Thm: Let M be a non-or, ctd n-mfld. Then $H^n_c(M) = 0$.

Pf: Let ω be an n-form w/ cpt supp. Let $\{\phi_i\}$ be a partition of unity: $\omega = \sum_{i=1}^{k} \phi_i \omega$ (finite b/c supp cpt, $\mathrm{Supp}(\phi_i \omega) \subseteq V_i \cong \mathbb{R}^n$). wlog, we can assume supp $\omega \subseteq V \cong \mathbb{R}^n$.

Assume (V, χ_0) is a coord. system. We claim that we can construct a sequence (V_i, χ_i) w/ $V_0 = V$, $V_r = V$, & s.t. $V_i \cap V_{i+1} \neq \emptyset$ & $\chi_{i+1} \circ \chi_i^{-1}$ is local. or-preserving, & $\chi_r \circ \chi_{r-1}^{-1}$ is local or-reversing. If such a chain DNE, then M is or ($\frac{1}{2}$). Choose one such chain.



Assume $\int_{V=V_0} \omega > 0$. Note that since $V \cong \mathbb{R}^n$, if $\int_V \omega = 0$, then $\omega = d\eta$ for some η w/ cpt. support (b/c $\mathbb{R}^n$ or, so we proved this already). $\mathbb{R}^n$ contractible, so $\omega = d\eta$ for some η, but not nec. w/ cpt supp.

We define ω_i on $V_i \cap V_{i+1}$ w/ $\int_{V_i} \omega_i > 0$, for $i = 1, \dots, r$, supp $\omega_i \subseteq V_i$.

Since $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$, $\omega_{i+1} = c_i \omega_i + d\eta_i$, & recall $c_i = \dfrac{\int_{V_i} \omega_{i+1}}{\int_{V_i} \omega_i} > 0$.

On the other hand, ω, as related to ω_r, will have opposite orientation, so if $\omega = c_r \omega_r + d\eta_r$, $c_r < 0$

(In other words, if we want to compare ω_r w/ a form $V = V_r$ s.t. c_r is positive, it will need to be $-\omega$).

$$\left.\begin{array}{l} \omega_1 = c_0 \omega + d\eta_0 \\ \omega_2 = c_1 \omega_1 + d\eta_1 \\ \vdots \\ \omega = c_r \omega_r + d\eta_r \end{array}\right\} \Rightarrow \omega = c\omega + d\eta, \quad c = c_0 c_1 \cdots c_r < 0$$

So $\underbrace{(1-c)}_{\neq 0} \omega = d\eta \Rightarrow \omega = \frac{1}{1-c} \cdot d\eta$ $\square$

Let $f: M \to N$. If $\omega \in \Omega^k(N)$, $f^* \omega \in \Omega^k(M)$. Can we define a map

$f^*: H^k(N) \to H^k(M)$? Is it well-def.? Yes: If $\omega = \hat{\omega} + d\eta$,

$[\omega] \longmapsto [f^* \omega]$ $\quad$ $f^* \omega = f^* \hat{\omega} + d(f^* \eta)$ [pullback & differentials commute].

What about $f^*: H_c^k(N) \to H_c^k(M)$? We would need to know $f^*\omega$ has cpt supp. whenever ω does.
$[\omega] \longmapsto [f^*\omega]$

$\text{supp } f^*\omega \subseteq f^{-1}(\text{supp }\omega)$, so we need $f^{-1}(K)$ cpt whenever K is,
i.e. that f is proper.

Assume f is proper. If M,N are or, ctd n-mflds, then
$H_c^n(M) \cong \mathbb{R} \cong H_c^n(N)$, so $f^*: \mathbb{R} \to \mathbb{R}$, & is linear $\Rightarrow$ $\exists$ a number α s.t.
$\hookrightarrow \Rightarrow$ is mult. by #

$$\underbrace{\int_M f^*\omega}_{f^*([\omega])} = \alpha \underbrace{\int_N \omega}_{\alpha[\omega]}, \quad \alpha \text{ independent of } \omega.$$

Def: α is the degree of f.

Def: Let $p \in M$. We define $\text{sign}_p f = \begin{cases} 1 & \text{if } f_*: T_pM \to T_pN \text{ preserves or.} \\ -1 & \text{if } \quad \text{---''---} \quad \text{reverses} \text{ ---} \end{cases}$
(Let M,N be ctd, or n-mflds)

Thm: Let $f: M \to N$ be a proper map of ctd oriented n-mflds. Let $q \in N$ be a regular value. (i.e. Jacobean has full rk at all the preimages)
Then $\deg f = \sum_{p \in f^{-1}(q)} \text{sign}_p f$, or $=0$ if $f^{-1}(q) = \emptyset$.

Note: This implies $\deg f \in \mathbb{Z}$.

Pf: Since q is a regular value, $f^{-1}(q)$ must be discrete, as an accumulation pt would give a vanishing directional derivative, contradicting that they are regular pts. $\{q\}$ is cpt, so $f^{-1}(q)$ is cpt as well, since f proper, so $f^{-1}(q) = \{p_1, \dots, p_k\}$ is finite.
If we have $\{U_i\}$ w/ $p_i \in V_i$ & $f: U_i \to V$ an iso. $\forall i$, $q \in V \cong \mathbb{R}^n$, all open, then construct ω w/ $\text{supp}\,\omega \subseteq V$ w/ cpt supp & $\int_V \omega > 0$.

If $\int_M f^*\omega \underset{\uparrow}{=} \sum_i \int_{U_i} f^*\omega = \sum_i \text{sign}_{p_i} f \int_V \omega$, then $\sum_i \text{sign}_{p_i} f = \deg f$.
$\text{supp } f^*\omega \subseteq f^{-1}(\text{supp }\omega)$

So it remains only to show we can construct such V_i:
Let U_i be s.t. $p_i \in U_i$ & $U_i \cap U_j = \emptyset$, & each U_i having only regular pts & locally a diffeo. (by inv. fcn thm)
Let W be a cpt nbhd of q. Let $\hat{\omega} = f^{-1}(W) \cap [\bigcup_{i=1}^k U_i]^c$. $\hat{\omega}$ is cpt, & so $f(\hat{\omega}) \subseteq W$ is cpt. Therefore $W \setminus f(\hat{\omega})$ is a nbhd of q. Let $q \in V \subseteq (W \setminus f(\hat{\omega}))$ be an open set. Define $\hat{U}_i = U_i \cap f^{-1}(V)$.

Def: Two maps $f, g: M \to N$ are C^∞ homotopic if $\exists\ H: [0,1] \times M \to N$, C^∞ s.t. $H(0,p) = f(p)$ & $H(1,p) = g(p)$.

Thm: If $f \sim g$, then $f^*: H^k(N) \to H^k(M)$, $f^* = g^*$.

Pf: $f \sim g$ if $i_t: M \to [0,1] \times M$, $p \mapsto (t,p)$ is such that $H \circ i_0 = f$ & $H \circ i_1 = g$

Then $g^*\omega - f^*\omega = i_1^*(H^*\omega) - i_0^*(H^*\omega) \overset{\text{Poincaré's lemma}}{=} I(\underbrace{d(H^*\omega)}_{=0 \text{ if } \omega \text{ closed}}) + d(I(H^*\omega))$

So $g^*\omega = f^*\omega + d\eta \Rightarrow [g^*\omega] = [f^*\omega]$.

Cor: (1) If M, N are cpt & oriented, & $f \sim g$, $\deg f = \deg g$.

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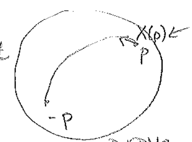
Recall: • $f \sim g \Rightarrow f^* = g^* \Rightarrow \deg f = \deg g$

Cor: If n is even, there is no nonvanishing vector field on S^n.

Pf: Assume there is. Let $A: S^n \to S^n$, $A(p) = -p$. Construct a htpy from $A(p)$ to $\mathbb{1}(p)$:

Let $\gamma_X^p(t)$ be the large circle s.t. $\gamma_X^p(0) = p$ & $\gamma_X^p(1) = -p$, & $(\gamma_X^p)'(0) = \overline{X}(p)$. Since X is C^∞, one can prove that



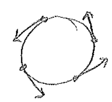
← choose a direction at p.

$$H: [0,1] \times S^n \to S^n \quad \text{is } C^\infty.$$
$$(t,p) \mapsto \gamma_X^p(t)$$

But $H(0,p) = \mathbb{1}(p)$ & $H(1,p) = A(p)$, so $A \sim \mathbb{1}_{S^n}$, so $\deg A = \deg \mathbb{1}_{S^n} = 1$.

But $\deg A = \sum_{p \in A^{-1}(q)} \operatorname{sign}_p(A) = -1$, since $A^{-1}(q) = \{-q\}$ & $\operatorname{sig}_p A = -1$ since if n is even, A reverses the orientation. □

Note: If n is odd, such a v.f. always exists: If $p = (a_1, a_2, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$, define $X(p) = (a_2, -a_1, \dots, a_{n+1}, -a_n) \in TS^n$.



Cor: $H^k(S^{n-1}) \simeq H^k(\mathbb{R}^n \setminus \{0\})$ $\forall k$.

Pf: Recall the retraction $r: \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, $p \mapsto p/\|p\|$. Then $\mathbb{1}_{S^n}: r \circ i: S^{n-1} \to S^{n-1}$,

So $(r \circ i)^* = \mathbb{1}^*_{S^{n-1}} = \mathbb{1}_{H^k(S^{n-1})}$. WTS: $(i \circ r)^* = \mathbb{1}_{H^k(\mathbb{R}^n \setminus \{0\})}$ so that r^* an $\simeq$, & we're done.

We'll show $i \circ r \sim \mathbb{1}_{\mathbb{R}^n \setminus \{0\}}$:

$$H(t,p) = tp + (1-t)\, p/\|p\| \quad \text{(straight-line htpy)}$$

□

Lie Groups

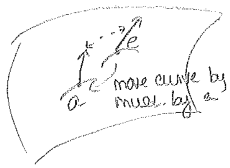
Def: G is a Lie gp if G is a gp and a manifold s.t. the maps $(a,b) \to ab$ & $a \to a^{-1}$ are smooth. This is equivalent to $(a,b) \to a^{-1}b$ C^∞.

Ex: • $(\mathbb{R} \setminus \{0\}, \cdot)$

• $O(n)$, $U(n)$, $SL(n)$, $Sp(n)$, ... : all matrix gps.

* If G is a Lie gp, define homeos $L_a : G \to G$, $b \to ab$ and $R_a : G \to G$, $b \to ba$: homeos since $L_a^{-1} = L_{a^{-1}}$ & $R_a^{-1} = R_{a^{-1}}$.

Def: A v.f. X on G is <u>left-invariant</u> if $L_{a*}X = X$. Similarly for rt-invariant.



• If $\gamma(t) \in G$ w/ $\gamma(0) = e$, then $a\gamma(t) \in G$ w/ $a\gamma(0) = a$. Since $\gamma'(0) \in T_eG$, $L_{a*}\gamma'(0) \in T_aG$, so this def. makes sense.

<u>Note</u>: Saying $L_{a*}X = X$ is equivalent to saying X is L_a-related to X, so the space of l-inv. v.f.'s is closed under the Lie bracket

Def: $T_eG = \mathfrak{g}$, the <u>Lie algebra</u> (a v. sp.) with Lie bracket $[v,w] = \underbrace{[X_v, X_w]}_{\text{extend the } l\text{-inv. v.f.'s from } v \text{ \& } w}(e)$, where X_v is the left v.f. corresp. to v. (we are assuming X_v is C^∞, proven below)

Thm: The tangent bundle of G is trivial. (i.e. X_v is C^∞)

Pf: We only need to find n $(= \dim G)$ global independent v.f.'s. If we have them, then the map $TG \to G \times \mathbb{R}^n$, $v \in T_pG \mapsto (p, c_1, \dots, c_n)$, $v = \sum c_i X_i(p)$ is a triv. of the bundle.

Let $X(a) = L_{a*}v$, $v \in T_eG$. Consider coords (x, U) $(e \in U)$ around a & $V \subseteq U$ s.t. if $a, b \in V$, $ab \in U$. WTS $X(a)(x_j)$ is C^∞ (since $X = \sum X(x_j)\,\partial/\partial x_j$)

$X(a)(x_j) = (L_{a*} v)(x_j) = v(L_a(x_j))$.

$L_a(x_j)(b) = x_j(ab)$, & ab is C^∞, so $x_j(ab)$ is C^∞, so $v(L_a(x_j))$ is C^∞. (b/c v takes C^∞ to C^∞). $\square$

Recall: $\exists$ conjugation map $\begin{array}{ccc} G \times G & \xrightarrow{A} & G \\ (g,h) & \longrightarrow & ghg^{-1} \end{array}$. Fixing g, this maps $e \to e$.

Fixing g, then $A_*(g): \mathfrak{g} \to \mathfrak{g}$. This defines an action $\begin{array}{c} G \times \mathfrak{g} \to \mathfrak{g}, \\ (g,v) \longrightarrow A_*(g)(v) \end{array}$ called the adjoint action on the Lie algebra, usually written $Ad(g)(v) = A_*(g)(v)$.

→ you're conjugating the derivative of a curve.

We can differentiate this map (take curve in gp thru e & diff) to get an action of $\mathfrak{g}$ on itself, the adjoint action of $\mathfrak{g}$

$\begin{array}{c} \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\ (\omega, v) \longrightarrow Ad(\omega)(v) \end{array}$, which defines the Lie bracket.

Problem: Think about $ad(\omega)(v) = [\omega, v]$.

Ex: Assume G a matrix gp.

$Ad(g)h = ghg^{-1}$.

If $B = h'(0)$, then $Ad(g)(B) = gBg^{-1}$ (i.e. skew-symm conj by rotation is still skew-symm)

If $C = g'(0)$, $g(0) = e$, then $g'(0) g^{-1}(0) \underset{e}{} \quad (g^{-1})' = g^{-1} g' g^{-1}$

$Ad(C)(B) = \frac{d}{dt}\Big|_0 (gBg^{-1}) = CB - BC = [C,B]$

Recall: We had a problem: $O(n) = \{A \in GL(n), AA^T = I\}$ is a mfld. We do this by showing $A \to AA^T$ has full rk around those for which $AA^T = I$, for ex, around I. So I is a regular value, & its preimage is a submfld of $GL(n)$. The rk is same everywhere $AA^T = I$, b/c it's $L_A R_{A^T}(e)$, so rk at A = rk at e. At e, we ~~~~ed to diff. $AA^T = I$, which gives representation of a Lie alg.

$A \to AA^T$ at $AA^T = I$.

$A(0) = e$

$A'(0) = B \in \mathfrak{g} = T_I(U(n))$

$AA^T = I \overset{\text{dif.}}{\rightsquigarrow} A'(0) + A^{T\prime}(0) = 0$, so $B = -B^T$

So showing dim Lie alg = dim Lie gp shows the Lie gp a mfld.

Def: $H \subseteq G$ is a <u>Lie subgp</u> if H is a submfld and a subgp of G.

Let $\mathfrak{h} = T_eH \subseteq T_eG = \mathfrak{g}$. Since

$$L_a|_H \to \hat{L}_a \quad \begin{tikzcd} H \arrow[r, hook, "i"] \arrow[d] & G \arrow[d, "L_a"] \\ H \arrow[r, hook, "i"] & G \end{tikzcd}$$

commutes, $\mathfrak{h}$ is a <u>Lie subalgebra</u> of $\mathfrak{g}$. Note, if $a \in H$, you can extend a vector by L_a: $v \in \mathfrak{h}$, $X(a) = L_{a*}v$, then take brackets. So if take Lie bracket of 2, you'll end up in $\mathfrak{h}$:

$[v,w] = [X_v, X_w](e)$, but $[X_v(a), X_w(a)] = [L_{a*}v, L_{a*}w] = L_{a*}\underbrace{[v,w]}_{\in \mathfrak{h}}$

So Lie subgp $\Rightarrow$ Lie subalg. But we'll show next time that Lie subalg $\Rightarrow$ Lie subgp, as well, which is like being able to integrate.

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Thm: $\mathfrak{h} \leq \mathfrak{g}$ a subalg, then $\exists!$ $H \leq G$, Lie Subgp, ctd & $\mathfrak{h}$ a Lie alg. of H.

↳ subgp that is also a submfld
ex: $O(n) \leq_{Lie} GL(n)$

Pf: Define $\Delta_a = \{v \in T_aG \mid v = X(a)$, where X is left-inv. & $X(e) \in \mathfrak{h}\}$ (the portion of T_aG in $\mathfrak{h}$ direction) This is integrable, b/c $v = L_{a*}X(e)$, & since L_{a*} preserves Lie brackets & $\mathfrak{h}$ is a subalg, then Δ is involutive &, by Frobenius, integrable. So we have many leaves, & we let H be the max'l integrable submfld w/ $e \in H$, i.e. leaf through e.

H is a subgp: $L_{b*}(\Delta_a) = \Delta_{ba}$. L_b takes integral submflds to integral submflds. (In fact, the int. submfld through a to the one through ab). Thus, if $a \in H$, $L_{a^{-1}}(H) = H$ ← the leaf containing $aa^{-1} = e$.
So H is closed under $b \to a^{-1}b$, & hence it's a subgp. It's a submfld by def, & the maps $(a,b) \to ab$ & $a \to a^{-1}$ are C^∞ since they are in G & H has the induced structure, so H a Lie subgp. □

Def: The $\mathfrak{g}$-valued form ω is, for $v_a \in T_aG$, $\omega(v_a) = L_{a^{-1}*}v_a$, i.e. pull v_a back to the identity, & ω is called the Maurer-Cartan form.

$\omega = \sum \alpha_i dx_i$, where $\alpha_i : G \to \mathfrak{g}$

- ω is C^∞
- ω is left-invariant, i.e. $(L_b^* \omega)(v_a) = \omega(\underbrace{L_{b*}v_a}_{\in T_{ba}G}) = L_{(ba)^{-1}*}L_{b*}v_a$

$= L_{a^{-1}*}L_{b^{-1}*}L_{b*}v_a = L_{a^{-1}*}v_a = \omega(v_a)$

So ω depends only on its value at e, so constant on a on $\mathfrak{g}$

Ex: $\underset{G}{GL_n(\mathbb{R})}$: $\omega(v_a) = L_{a^{-1}*}(\overset{TG}{a,v}) = (e, a^{-1}v_a)$

v_a: matrix ∈ Lie alg moved to a.
$a^{-1}v_a$: matrix ∈ Lie alg

Classically, one writes $\omega = g^{-1}dg$, representing by g the identity map on G.

Def:

$$d\omega(X,Y) = \underbrace{X(\overbrace{\omega(Y)}^{\text{constant} = \omega(Y_e)})}_{\text{zero}} - \overbrace{Y(\omega(X))}^{\text{zero}} - \omega([X,Y])$$

So $d\omega(X,Y) + \omega([X,Y]) = 0$

$\boxed{d\omega + \omega\wedge\omega = 0}$ called the <u>structure equation</u>.

Def: Let F be a top. sp. We say (E,B,π,F) is a <u>fiber bundle</u> if $E \xrightarrow{\pi} B$ is cts & for every pt $p\in B$ $\exists$ open set $p\in U\subseteq B$ & homeomorphisms Φ s.t. (diagram) & $\Phi: \pi^{-1}(U)\to F$ a homeo.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U\times F \\ & \pi\searrow \quad \circ \quad \swarrow \mathrm{proj}_U & \\ & U & \end{array}$$

Def: An <u>action</u> of a Lie gp on a mfld M is a smooth map

$\mu: G\times M \to M$ s.t. (1) $\mu(e,p) = p$

$(g,p)\longmapsto \mu(g,p)$ (2) $\mu(h,\mu(g,p)) = \mu(hg,p)$

Def: μ is <u>free</u> if $(\mu(g,p) = \mu(h,p)$ for some $p) \Rightarrow g = h$. μ is <u>transitive</u> if given any $p,q\in M$, $\exists\, h$ s.t. $\mu(h,p) = q$.

$G_p = \{\mu(g,p)\mid g\in G\}$ is the <u>orbit</u> of p, & μ is transitive $\Leftrightarrow G_p = G$.

Def: Assume we have a fiber bundle & an action of G on F, $G\times F\to F$. A <u>G-atlas</u> is an atlas s.t. the transition fcns $xy^{-1}: U\cap V\times F \to U\cap V\times F$ for some smooth map $h: U\cap V\to G$,

$xy^{-1}(p,f) = (p,\mu(h(u),f))$

A <u>G-bundle</u> is a fiber bundle w/ a max'l G-atlas.

Ex: Let $H\leq G$ be closed. Then one can prove that G/H is a mfld

- closed needed b/c $\mathbb{Q}\leq(\mathbb{R},+)$ & $\mathbb{R}/\mathbb{Q}$ = irrationals, not a mfld.

Def: A principle G-bundle is a G-bundle with a transitive & free action of G on E preserving the fibers $G \times E \to E$

ex $G \to G/H$ is a H-principle bundle w/ fiber H.

This principle bundle w/ the Maurer-Cartan form is the Klein geometry.

Ex: $E(n)$, Euclidean sp.

$E(n) = SO(n) \ltimes \mathbb{R}^n$

$$(\theta, v)(\hat{\theta}, \hat{v}) = (\theta\hat{\theta}, \theta\hat{v} + v)$$

Let $H = SO(n)$, $G = E(n)$. Then $G/H \cong \mathbb{R}^n$

$$[(\theta, v)] = [(e,v), (\theta, p)] = [(e,v)]$$

There is a natural action $G \times G/H \to G/H$. If we identify

$$(g, [p]) \to [gp]$$

$E(n)/SO(n)$ w/ the section $\mathbb{R}^n \to E(n)$, $p \mapsto (e,p)$, then

$$(\theta, v) \cdot [(e,p)] = [(e, \theta p + v)]$$, which is the Euclidean action.

Note: Almost all classical geometries can be thought of this way.

Def: A pair (G,H), $H \leq_{cl} G$, together w/ $G \to G/H$ & a Maurer-Cartan form, is called a Klein geometry.

Properties of M-C form, ω:

(1) ω is a linear iso. on the fibers of the tangent bundle

(2) $R_h^* \omega = Ad(h^{-1})\omega \quad \forall h \in H$

(3) $\omega(X_v) = v \quad \forall v \in \mathfrak{g} \quad X_v|_F = \mu_{*(p,e)}(0,v)$.

Cartan considered other forms w/ these 3 properties, & other principle bundles. Then the structure eqn for that form $\neq 0$, but a different form, which measures things like curvature (which you usually need a distance to measure (from Riemannian geom)).