Applications of Differential Equations

For each problem, set up a differential equation modeling the situation. Be sure to define your variables, state the units being used, and determine the initial condition. You do not need to solve.

1. A 100 gallon tank is full of pure water. Let pure water run into the tank at the rate of 2 gals/ min. and a brine solution containing 1/2 lb. of salt per gallon run in at the rate of 2 gals/min. The mixture flows out of the tank through an outlet tube at the rate of 4 gals/min. Assuming perfect mixing, what is the amount of salt in the tank after t minutes?

   Let t be time in minutes and $A(t)$ be the amount of salt in the mixture at time t, measured in pounds. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = (\text{amount of salt in}) - (\text{amount of salt out})$, where there amounts are measured over one minute. First, notice that the volume of solution in the tank is always constant, since every minute 4 gallons flow in and 4 gallons flow out. First let's find the "amount of salt in." In one minute, there is 1 lb. of salt added, because each of the two gallons of brine contains a half pound of salt.. Next, let's find the "amount of salt out." This one is a bit trickier: the mixture leaves the tank at a rate of 4 gals/min, so the amount of salt that flows out of the tank in one minute is $4 \cdot (\text{concentration of salt at time } t)$. The concentration of salt is the amount of salt divided by the volume of solution. At time t, the amount of salt is $A(t)$, and the volume is always 100 gallons, so the concentration is $\frac{A}{100}$. Therefore, our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 1 - 4 \cdot \frac{A}{100}, \qquad A(0) = 0.$$

2. A tank has pure water flowing into it at 12 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water. How much salt will there be in the tank after 30 minutes?

   Let t be time in minutes and $A(t)$ be the amount of salt in the mixture at time t, measured in kilograms. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = (\text{amount of salt in}) - (\text{amount of salt out})$, where there amounts are measured over one minute. Notice that the volume of solution in the tank is not constant, since every minute 12 liters flow in and only 10 liters flow out, for a net increase of 2 liters per minute. First let's find the "amount of salt in." The solution entering the tank is pure water, so there is no salt coming in. The amount of salt out is $10 \cdot (\text{concentration of salt at time } t)$, where concentration of salt is the amount of salt divided by the the volume of solution. At time t, the amount of salt is A, and the volume is $100 + 2t$. Therefore our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -10 \cdot \frac{A}{100 + 2t}, \qquad A(0) = 10.$$

3. A large tank is filled to capacity with 100 gallons of pure water. Brine containing 3 pounds of salt per gallon is pumped into the tank at a rate of 4 gal/min. The well-mixed solution is pumped out of the tank at the rate of 5 gal/min. Find the amount of salt after 30 minutes.

   Let t be time in minutes and $A(t)$ be the amount of salt in the mixture at time t, measured in pounds. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = (\text{amount of salt in}) - (\text{amount of salt out})$, where there

amounts are measured over one minute. Notice that the volume of solution in the tank is not constant, since every minute 4 gallons are pumped in but 5 gallons are pumped out, for a net decrease of 1 gallon per minute. First let's find the "amount of salt in." There are 4 gallons of solution entering the tank each minute, and each gallon contains 3 lb of salt, so there are 12 pounds of salt entering every minute. The amount of salt out is $5 \cdot$ (concentration of salt at time t), where concentration of salt is the amount of salt divided by the the volume of solution. At time t, the amount of salt is A, and the volume is $100 - t$. Therefore our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 12 - 5 \cdot \frac{A}{100 - t}, \qquad A(0) = 0.$$

4. A tank with 200 gallons of brine solution contains 40 lbs of salt. A concentration of 2 lb/gal is pumped in at a rate of 4 gal/min. The concentration leaving the tank is pumped out at a rate of 4 gal/min. How much salt is in the tank after 1 hour?

   Let t be time in minutes and $A(t)$ be the amount of salt in the mixture at time t, measured in points. Our formula will be $\frac{\mathrm{d}A}{\mathrm{d}t} =$ (amount of salt in) $-$ (amount of salt out), where there amounts are measured over one minute. Notice that the volume of solution in the tank is constant, since every minute 4 gallons are pumped in and 4 gallons are pumped out. First let's find the "amount of salt in." There are 4 gallons of solution entering the tank each minute, and each gallon contains 2 lb of salt, so there are 8 pounds of salt entering every minute. The amount of salt out is $4 \cdot$ (concentration of salt at time t), where concentration of salt is the amount of salt divided by the the volume of solution. At time t, the amount of salt is A, and the volume is always 200. Therefore our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 8 - 4 \cdot \frac{A}{200}, \qquad A(0) = 40.$$

5. A 1000L tank starts out with 200L of fluid containing 10g/L of dye. Pure water is poured in at 20 L/min and the tank is being drained at a rate of 15L/min. Write the equation for the amount of dye in the tank at any time t.

   Let t be time in minutes and $A(t)$ be the amount of dye in the mixture at time t, measured in grams. Our formula will be $\frac{\mathrm{d}A}{\mathrm{d}t} =$ (amount of dye in) $-$ (amount of dye out), where there amounts are measured over one minute. Notice that the volume of solution in the tank is not constant, since every minute 20 liters are pumped in and 15 liters are pumped out, for a net increase of 5 liters per minute. First let's find the "amount of dye in." There are 20 liters of pure water entering the tank each minute, so there is no dye entering the tank. The amount of dye out is $15 \cdot$ (concentration of dye at time t), where concentration of dye is the amount of salt divided by the the volume of solution. At time t, the amount of dye is A, and the volume is always $200 + 5t$. Therefore our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -15 \cdot \frac{A}{200 + 5t}, \qquad A(0) = 2000.$$

6. The rate at which a certain drug is eliminated from the bloodstream is proportional to the amount of the drug in the bloodstream. A patient now has 45 mg of the drug in his blood-

stream. The drug is being administered to the patient intravenously at a constant rate of 5 milligrams per hour. Write a differential equation modeling the situation.

Let t be time in hours and $A(t)$ be the amount of drug in the patient at time t, measured in milligrams. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = $ (amount of drug in) $-$ (amount of drug out), where there amounts are measured over one hour. First let's find the "amount of drug in." The drug is entering the patient at a rate of 5 mg per hour, so in one hour, 5 mg of the drug will enter the patient. The amount of drug leaving the patient is proportional to the amount of drug in the patient at time t, so this is kA, where k is the constant of proportionality. Therefore our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 5 - kA, \qquad A(0) = 45.$$

7. Ten thousand dollars is deposited in a bank account with a nominal annual interest rate of 5% compounded continuously. No further deposits are made. Write a differential equation reflecting the situation if money is withdrawn continuously at a rate of $4000 per year.

Let t be time in years and $A(t)$ be the amount of money in the bank at time t, measured in dollars. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = $ (amount of money in) $-$ (amount of money out), where there amounts are measured over one year. First let's find the "amount of money in." This will come from the interest rate, which is 5% of the amount of money in the bank at time t, so the amount of money being added to the account is $0.05A$. The amount of money leaving the bank is the amount withdrawn, which is $4000 (since we're looking at one year at a time). Therefore our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0.05A - 4000, \qquad A(0) = 10000.$$

8. In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population and its population would double every three weeks. There are 250,000 mosquitoes in the area initially when a flock of birds arrives that eats 80,000 mosquitoes per week. How many mosquitoes remain after two weeks?

Let t be time in weeks, and $A(t)$ be the number of mosquitos. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = $ (number of mosquitos in) $-$ (number of mosquitos out), where the amounts are measured over one week. First let's find the "number of mosquitos in." Mosquitos increase at a rate proportional to their current population, which can be expressed as an increase of kA each week, where k is the constant of proportionality. To determine k, we use the fact that if there were no predators, our differential equation would be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = kA$, so $A = Ce^{kt}$ (this is a separable equation; you can check this). Now, $A(0) = 250,000$, and $A(3) = 500,000$, so using both we have that $C = 250,000$ and $k = \frac{\ln 2}{3}$. Thus the number of mosquitos in is $\frac{\ln 2}{3} \cdot A$.

Now, since there are predators, the "number of mosquitos out" is not 0, it is 80,000. Thus our differential equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\ln 2}{3} \cdot A - 80000, \qquad A(0) = 25000.$$

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9. (Challenge!) A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gal/hr while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t.

We need two differential equations for this, one that will hold before the amount of pollution in the tank reaches 500 ounces, and one that will hold after that time.

- First differential equation

  Let t be time measured in hours, and $A_1(t)$ be the amount of pollution measured in ounces. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = (\text{amount of pollution in}) - (\text{amount of pollution out})$, where the amounts are measured over one hour. Notice that the volume of solution in the tank is constant, since every hour 3 gallons are pumped in and 3 gallons are pumped out. First let's find the "amount of pollution in." There are 3 gallons of solution entering the tank each hour, and each gallon contains 5 ounces of pollution, so there are 15 ounces of pollution entering every hour. The amount of pollution out is $3 \cdot (\text{concentration of pollution at time } t)$, where concentration of pollution is the amount of pollution divided by the the volume of solution. At time t, the amount of pollution is A_1, and the volume is always 800. Therefore our differential equation is

  $$\frac{\mathrm{d}A_1}{\mathrm{d}t} = 15 - 4 \cdot \frac{A}{800}, \qquad A(0) = 2.$$

- Second differential equation

  Let t be time measured in hours, and $A_2(t)$ be the amount of pollution measured in ounces. Our formula will be $\dfrac{\mathrm{d}A}{\mathrm{d}t} = (\text{amount of pollution in}) - (\text{amount of pollution out})$, where the amounts are measured over one hour. Notice that the volume of solution in the tank is no longer constant, since every hour 2 gallons are pumped in and 4 gallons are pumped out, for a net decrease of 2 gallons per hour. First let's find the "amount of pollution in." There are 2 gallons of pure water entering the tank each hour, so no pollution is entering. The amount of pollution out is $4 \cdot (\text{concentration of pollution at time } t)$, where concentration of pollution is the amount of pollution divided by the the volume of solution. At time t, the amount of pollution is A_2, and the volume is always $800 - 2t$. Therefore our differential equation is

  $$\frac{\mathrm{d}A_2}{\mathrm{d}t} = -2 \cdot \frac{A_2}{800 - 2t}, \qquad A_2(t_0) = 500.$$

Now we need to know at what time t_0 the amount of pollution stops being determined by $A_1(t)$ and starts being determined by $A_2(t)$. To do this, you would solve $A_1(t_0) = 500$.