Solutions to Homework 5

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Problems Assigned

- 13.2: 10, 21, 24, 27, 37, 49
- 13.3: 2, 13, 18, 25, 48, 49, 55, 56, 57 (for unit tangent vector only), 58

13.2, #10

Take the derivative of each part of the vector function and get $r'(t) = \langle \boldsymbol{-e^{-t}, 1 - 3t^2, \frac{1}{t}} \rangle$

13.2, #21

Given $r(t) = \langle t, t^2, t^3 \rangle$, and noting that the unit tangent vector $T(1) = \frac{r'(1)}{|r'(1)|}$

we get

$$r'(t) = \langle \mathbf{1, 2}\boldsymbol{t}, \mathbf{3}\boldsymbol{t^2} \rangle$$
$$T(1) = \frac{r'(1)}{|r'(1)|} = \frac{\mathbf{1}}{\sqrt{\mathbf{14}}} \langle \mathbf{1, 2, 3} \rangle$$
$$r''(t) = \langle \mathbf{0, 2, 6}\boldsymbol{t} \rangle$$
$$r'(t) \times r''(t) = \begin{bmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{bmatrix} = \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} i - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} j + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} k =$$
$$\langle \mathbf{6}\boldsymbol{t^2}, \mathbf{-6}\boldsymbol{t}, \mathbf{2} \rangle$$

13.2, #24

Given parametric equations and a point, find parametric equations for the tangent line from the curve to the point.

$$x = \ln(t+1),$$
$$y = t\cos 2t,$$
$$z = 2^t$$
$$(0,0,1)$$

The vector equation for the curve is $r(t) = \langle \ln(t+1), t\cos 2t, 2^t \rangle$. The equation for a tangent line from this curve is $r'(t) = \langle \frac{1}{t+1}, \cos 2t - 2t\sin 2t, 2^t \ln 2 \rangle$. The point $(0,0,1)$ corresponds to $t = 0$, so the tangent vector is $r'(0) = \langle 1, 1, \ln 2 \rangle$.

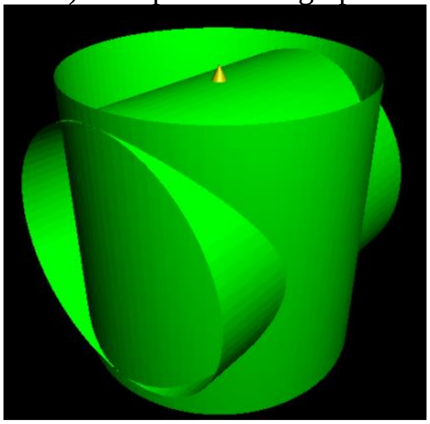
The parametric equations are therefore:

$$\boldsymbol{x = 0 + 1t = t}$$
$$\boldsymbol{y = 0 + 1t = t}$$
$$\boldsymbol{z = 1 + (\ln 2)\, t}$$

13.2, #27

Find the vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point $(3,4,2)$.

To get practice thinking about these types of problems, I recommend graphing it using an online grapher. (All Macs have a built-in program called Grapher, where you can do 3D graphs, and I imagine PCs have something similar.) This produces a graph that looks like this:



Because all the points in the curve of intersection also have to be on the cylinder $x^2 + y^2 = 25$, we know that we can parametrize the x and y values as $x = 5\cos t$ and $y = 5\sin t$. To find an equation for z, manipulate the second curve to find that $z^2 = 20 - y^2$. We are interested in the point $(3,4,2)$, so we are interested in positive z-values. We find $z = \sqrt{20 - y^2} = \sqrt{20 - 25\sin^2 t}$

A vector equation for the curve of intersection is therefore

$$r(t) = \langle 5\cos t, 5\sin t, \sqrt{20 - 25\sin^2 t}\, \rangle$$

To find the equation for the tangent line to a point, we find

$$r'(t) = \langle -5\sin t, 5\cos t\,, \frac{1}{2}(20 - 25\sin^2 t)^{-\frac{1}{2}}(-50\sin t\cos t)\, \rangle$$

The point $(3,4,2)$ corresponds to $t = \cos^{-1}\frac{3}{5}$, which you can find by looking at the x-term of $r(t)$. From here on out, draw a $3-4-5$ right triangle and use it to solve for the inverse quantities. We find

$$r'\left(\cos^{-1}\frac{3}{5}\right) = \langle -5\left(\frac{4}{5}\right), 5\left(\frac{3}{5}\right), \frac{1}{2}(20 - 25\left(\frac{4}{5}\right)^2)^{-\frac{1}{2}}(-50\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\, \rangle$$

$$= \langle \mathbf{-4}, \mathbf{3}, \mathbf{-6} \rangle$$

13.2, #37

First break the integral up by component.

$$\int_0^1 \frac{1}{t+1}dt\, \boldsymbol{i} + \int_0^1 \frac{1}{t^2+1}dt\, \boldsymbol{j} + \int_0^1 \frac{t}{t^2+1}dt\, \boldsymbol{k}$$

$= \ln(t+1)]_0^1 \boldsymbol{i} + \tan^{-1} t]_0^1 \boldsymbol{j} + \frac{1}{2}\ln(t^2+1)]_0^1 \boldsymbol{k}$

$= \ln 2\ \boldsymbol{i} + \frac{\pi}{4}\ \boldsymbol{j} + \frac{1}{2}\ln 2\ \ \boldsymbol{k}$

13.2, #49

Use the vector product rule to find that $f'(t) = u(t) * v'(t) + u'(t) * v(t)$

$u(2) = \langle 1, 2, -1\rangle$, $u'(2) = \langle 3,0,4\rangle$, $v(t) = \langle t, t^2, t^3\rangle$, and $v'(t) = \langle 1, 2t, 3t^2\rangle$

$v(2) = \langle 2, 4, 8\rangle$ and $v'(2) = \langle 1, 4, 12\rangle$,

$f'(t) = \langle 1, 2, -1\rangle * \langle 1, 4, 12\rangle + \langle 2, 4, 8\rangle * \langle 3,0,4\rangle = \mathbf{35}$

13.3, #2

Find the length of the curve $r(t) = \langle 2t, t^2, \frac{1}{3}t^3\rangle$, $0 \le t \le 1$. Recall that arc length is defined by

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

Arc length is also denoted with the letter L. Here we have.

$\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 2, 2t, t^2 \right\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{2^2 + (2t)^2 + (t^2)^2} = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = 2 + t^2$ for $0 \le t \le 1$.

$L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 (2+t^2)\, dt = 2t + \frac{1}{3}t^3\big]_0^1 = \frac{7}{3}$.

13.3, #13

a) We first need to find the arc length. We have

(a) $\mathbf{r}(t) = (5-t)\,\mathbf{i} + (4t-3)\,\mathbf{j} + 3t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = -\mathbf{i} + 4\,\mathbf{j} + 3\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{1+16+9} = \sqrt{26}$. The point $P\,(4,1,3)$ corresponds to $t = 1$, so the arc length function from P is

$s(t) = \int_1^t |\mathbf{r}'(u)|\, du = \int_1^t \sqrt{26}\, du = \sqrt{26}\, u\big|_1^t = \sqrt{26}\,(t-1)$. Since $s = \sqrt{26}\,(t-1)$, we have $t = \frac{s}{\sqrt{26}} + 1$.

To reparametrize with respect to arc length, you simply substitute $\frac{s}{\sqrt{26}}+1$ in for our previous parameter t. This is

$$\mathbf{r}(t(s)) = \left[5-\left(\frac{s}{\sqrt{26}}+1\right)\right]\mathbf{i} + \left[4\left(\frac{s}{\sqrt{26}}+1\right)-3\right]\mathbf{j} + 3\left(\frac{s}{\sqrt{26}}+1\right)\mathbf{k}$$
$$= \left(4-\frac{s}{\sqrt{26}}\right)\mathbf{i} + \left(\frac{4s}{\sqrt{26}}+1\right)\mathbf{j} + \left(\frac{3s}{\sqrt{26}}+3\right)\mathbf{k}$$

b) This part asks us to substitute 4 in for s, because s measures "units along the curve." We do so and get

(b) The point 4 units along the curve from P has position vector

$$\mathbf{r}(t(4)) = \left(4-\frac{4}{\sqrt{26}}\right)\mathbf{i} + \left(\frac{4(4)}{\sqrt{26}}+1\right)\mathbf{j} + \left(\frac{3(4)}{\sqrt{26}}+3\right)\mathbf{k}, \text{ so the point is } \left(4-\frac{4}{\sqrt{26}}, \frac{16}{\sqrt{26}}+1, \frac{12}{\sqrt{26}}+3\right).$$

13.3, #18

a) Recall that $T(t) = \frac{r'(t)}{|r'(t)|}$ and $N(t) = \frac{T'(t)}{|T'(t)|}$

(a) $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle \quad \Rightarrow$

$\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t, \ -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\cos^2 t + \sin^2 t)} = \sqrt{5t^2} = \sqrt{5}\,t$ [since $t > 0$]. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$. $\quad \mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \quad \Rightarrow$

$|\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$. Thus $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

b) The curvature is found below.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$

13.3, #25

We want to find the curvature of $r(t) = \langle t, t^2, t^3 \rangle$ at $(1,1,1)$. Recall that curvature can be found three separate ways.

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Here, we use the third variant.

25. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. The point $(1,1,1)$ corresponds to $t = 1$, and $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \Rightarrow$ $|\mathbf{r}'(1)| = \sqrt{1+4+9} = \sqrt{14}$. $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle \Rightarrow \mathbf{r}''(1) = \langle 0, 2, 6 \rangle$. $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 6, -6, 2 \rangle$, so $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{36+36+4} = \sqrt{76}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

13.3, #48

Recall that $T(t) = \frac{r'(t)}{|r'(t)|}$, $N(t) = \frac{T'(t)}{|T'(t)|}$, and $B(t) = T(t) \times N(t)$. The point $(1,0,0)$ corresponds to $t = 0$. We have:

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln\cos t\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \frac{-\sin t}{\cos t}\mathbf{k} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} - \tan t\,\mathbf{k},$$

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + (-\tan t)^2} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|.$$

Here we can assume $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and then $\sec t > 0 \Rightarrow |\mathbf{r}'(t)| = \sec t$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -\sin t, \cos t, -\tan t\rangle}{\sec t} = \langle -\sin t\cos t, \cos^2 t, -\sin t\rangle \quad \text{and} \quad \mathbf{T}(0) = \langle 0, 1, 0\rangle.$$

$\mathbf{T}'(t) = \langle -[(\sin t)(-\sin t) + (\cos t)(\cos t)], 2(\cos t)(-\sin t), -\cos t\rangle = \langle \sin^2 t - \cos^2 t, -2\sin t\cos t, -\cos t\rangle$, so

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = \frac{\langle -1, 0, -1\rangle}{\sqrt{1+0+1}} = \frac{1}{\sqrt{2}}\langle -1, 0, -1\rangle = \left\langle -\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right\rangle.$$

Finally, $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \langle 0, 1, 0\rangle \times \left\langle -\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right\rangle = \left\langle -\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right\rangle$.

13.3, #49

Recall that a plane with a point (x_1, y_1, z_1) and a normal vector $\langle a, b, c\rangle$ can be defined as

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

The normal plane of $r(t)$ has a normal vector $r'(t)$ and a point at $(0, 1, 2\pi)$, so we can write the following.

$\mathbf{r}(t) = \langle \sin 2t, -\cos 2t, 4t\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 2\cos 2t, 2\sin 2t, 4\rangle$. The point $(0, 1, 2\pi)$ corresponds to $t = \pi/2$, and the normal plane there has normal vector $\mathbf{r}'(\pi/2) = \langle -2, 0, 4\rangle$. An equation for the normal plane is $-2(x - 0) + 0(y - 1) + 4(z - 2\pi) = 0$ or $-2x + 4z = 8\pi$ or $x - 2z = -4\pi$.

The oscillating plane has a normal vector $B(t)$, so we must first find that. Once we find $B(t)$, we use a multiple of that vector as the normal vector to the plane and we use the point on the plane $(0, 1, 2\pi)$ to find the equation of the plane.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2\cos 2t, 2\sin 2t, 4\rangle}{\sqrt{4\cos^2 2t + 4\sin^2 2t + 16}} = \frac{1}{2\sqrt{5}}\langle 2\cos 2t, 2\sin 2t, 4\rangle = \frac{1}{\sqrt{5}}\langle \cos 2t, \sin 2t, 2\rangle \quad \Rightarrow$$

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\sin 2t, 2\cos 2t, 0\rangle \quad \Rightarrow \quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{4\sin^2 2t + 4\cos^2 2t} = \frac{2}{\sqrt{5}}$, and

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin 2t, \cos 2t, 0\rangle$. Then $\mathbf{T}(\pi/2) = \frac{1}{\sqrt{5}}\langle -1, 0, 2\rangle$, $\mathbf{N}(\pi/2) = \langle 0, -1, 0\rangle$, and

$\mathbf{B}(\pi/2) = \mathbf{T}(\pi/2) \times \mathbf{N}(\pi/2) = \frac{1}{\sqrt{5}}\langle 2, 0, 1\rangle$. Since $\mathbf{B}(\pi/2)$ is normal to the osculating plane, so is $\langle 2, 0, 1\rangle$, and an equation of the plane is $2(x - 0) + 0(y - 1) + 1(z - 2\pi) = 0$ or $2x + z = 2\pi$.

Note that we can use $\langle 2,0,1\rangle$ as the normal vector because it is parallel to the vector $\frac{1}{\sqrt{5}}\langle 2,0,1\rangle$.

13.3, #55

First we parametrize the curve of intersection. We can choose $y = t$; then $x = y^2 = t^2$ and $z = x^2 = t^4$, and the curve is given by $\mathbf{r}(t) = \langle t^2, t, t^4 \rangle$. $\mathbf{r}'(t) = \langle 2t, 1, 4t^3 \rangle$ and the point $(1, 1, 1)$ corresponds to $t = 1$, so $\mathbf{r}'(1) = \langle 2, 1, 4 \rangle$ is a normal vector for the normal plane. Thus an equation of the normal plane is

$2(x-1) + 1(y-1) + 4(z-1) = 0$ or $2x + y + 4z = 7$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4t^2+1+16t^6}} \langle 2t, 1, 4t^3 \rangle$ and

$\mathbf{T}'(t) = -\frac{1}{2}(4t^2+1+16t^6)^{-3/2}(8t+96t^5)\langle 2t, 1, 4t^3 \rangle + (4t^2+1+16t^6)^{-1/2}\langle 2, 0, 12t^2 \rangle$. A normal vector for the osculating plane is $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1)$, but $\mathbf{r}'(1) = \langle 2, 1, 4 \rangle$ is parallel to $\mathbf{T}(1)$ and

$\mathbf{T}'(1) = -\frac{1}{2}(21)^{-3/2}(104)\langle 2, 1, 4 \rangle + (21)^{-1/2}\langle 2, 0, 12 \rangle = \frac{2}{21\sqrt{21}}\langle -31, -26, 22 \rangle$ is parallel to $\mathbf{N}(1)$ as is $\langle -31, -26, 22 \rangle$, so $\langle 2, 1, 4 \rangle \times \langle -31, -26, 22 \rangle = \langle 126, -168, -21 \rangle$ is normal to the osculating plane. Thus an equation for the osculating plane is $126(x-1) - 168(y-1) - 21(z-1) = 0$ or $6x - 8y - z = -3$.

13.3, #56

The trick here is to first find $B(t)$ and notice that the vector $B(t) = \langle t^2+2, t^2+2, 0 \rangle$ will, when evaluated at any t, appear in the form $\langle c, c, 0 \rangle$, which is parallel to $\langle 1,1,0 \rangle$.

$\langle 1,1,0 \rangle$ is therefore an normal vector to the oscillating plane, and can be used along with the general point $\left(t+2, 1-t, \frac{1}{2}t^2\right)$ on the curve to find the equation of the oscillating plane. This equation reduces to $x + y = 3$, which is independent of t. The equation of the oscillating plane is therefore independent of t, which means the oscillating plane is the same for all points on the line.

In more detail,

$$\mathbf{r}(t) = \left\langle t+2, 1-t, \tfrac{1}{2}t^2 \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, -1, t \rangle, \quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2+t^2}}\langle 1, -1, t \rangle,$$

$$\begin{aligned}\mathbf{T}'(t) &= -\tfrac{1}{2}(2+t^2)^{-3/2}(2t)\langle 1, -1, t \rangle + (2+t^2)^{-1/2}\langle 0, 0, 1 \rangle \\ &= -(2+t^2)^{-3/2}\left[t\langle 1, -1, t \rangle - (2+t^2)\langle 0, 0, 1 \rangle\right] = \frac{-1}{(2+t^2)^{3/2}}\langle t, -t, -2 \rangle\end{aligned}$$

A normal vector for the osculating plane is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, but $\mathbf{r}'(t) = \langle 1, -1, t \rangle$ is parallel to $\mathbf{T}(t)$ and $\langle t, -t, -2 \rangle$ is parallel to $\mathbf{T}'(t)$ and hence parallel to $\mathbf{N}(t)$, so $\langle 1, -1, t \rangle \times \langle t, -t, -2 \rangle = \langle t^2+2, t^2+2, 0 \rangle$ is normal to the osculating plane for any t. All such vectors are parallel to $\langle 1, 1, 0 \rangle$, so at any point $\left(t+2, 1-t, \frac{1}{2}t^2\right)$ on the curve, an equation for the osculating plane is $1[x-(t+2)] + 1[y-(1-t)] + 0\left(z - \frac{1}{2}t^2\right) = 0$ or $x + y = 3$. Because the osculating plane at every point on the curve is the same, we can conclude that the curve itself lies in that same plane. In fact, we can easily verify that the parametric equations of the curve satisfy $x + y = 3$.

13.3, #57, only unit tangent vector

First, find the unit tangent vector.

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \rangle \text{ so}$$

$$|\mathbf{r}'(t)| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}}$$

$$= \sqrt{e^{2t}\left[2(\cos^2 t + \sin^2 t) - 2\cos t \sin t + 2\cos t \sin t + 1\right]} = \sqrt{3e^{2t}} = \sqrt{3}\, e^t$$

$$\text{and } \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{3}\, e^t} \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \rangle = \frac{1}{\sqrt{3}} \langle \cos t - \sin t, \cos t + \sin t, 1 \rangle.$$

The unit vector for the z-axis is $\langle 0,0,1 \rangle$. Recall that the formula for the dot product between vector u and vector v is

$$u * v = |u||v| \cos\theta, \text{ where } \theta \text{ is the angle between vector } u \text{ and vector } v$$

This gives us $\cos\theta = \frac{u*v}{|u||v|}$.

Inputting the two vectors in here, we find,

$$\cos\alpha = \frac{\mathbf{T}(t) \cdot \mathbf{k}}{|\mathbf{T}(t)|\ |\mathbf{k}|} = \frac{\frac{1}{\sqrt{3}} \langle \cos t - \sin t, \cos t + \sin t, 1 \rangle \cdot \langle 0,0,1 \rangle}{\frac{1}{\sqrt{3}} \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1}\ \sqrt{1}} = \frac{1}{\sqrt{2(\cos^2 t + \sin^2 t) + 1}} = \frac{1}{\sqrt{3}}.$$ Thus the angle is constant; specifically, $\alpha = \cos^{-1}(1/\sqrt{3}) \approx 54.7°$.

13.3, #58

N is orthogonal to both T and B. Therefore, if both T and B lie in a plane, N will be orthogonal to that plane, which is the same as saying it is a normal vector to that plane. However, N is unwieldy to deal with, so we will instead use $T'(t)$. Because $N(t) = \frac{T'(t)}{|T'(t)|}$, and $|T'(t)|$ is a scalar, we know that $T'(t)$ is parallel to $N(t)$, and can also serve as the normal vector. The calculations are below.

point $\left(\sqrt{2}/2, \sqrt{2}/2, 1\right)$ corresponds to $t = \pi/4$, so we can take $\mathbf{T}'(\pi/4)$ as a normal vector for the plane [since it is parallel to $\mathbf{N}(\pi/4)$]. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}$ and

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + \sec^4 t} = \sqrt{1 + \sec^4 t}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1 + \sec^4 t}}\left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right)$. By Formula 3 of Theorem 13.2.3,

$$\mathbf{T}'(t) = -\frac{2\sec^4 t\tan t}{(1 + \sec^4 t)^{3/2}}\left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right) + \frac{1}{\sqrt{1 + \sec^4 t}}\left(-\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2\sec^2 t\tan t\,\mathbf{k}\right) \text{ and}$$

$$\begin{aligned}\mathbf{T}'(\pi/4) &= -\frac{2(\sqrt{2})^4(1)}{[1 + (\sqrt{2})^4]^{3/2}}\left(\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + (\sqrt{2})^2\,\mathbf{k}\right) + \frac{1}{\sqrt{1 + (\sqrt{2})^4}}\left(-\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 2(\sqrt{2})^2(1)\,\mathbf{k}\right) \\ &= -\frac{8}{5\sqrt{5}}\left(\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 2\,\mathbf{k}\right) + \frac{1}{\sqrt{5}}\left(-\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 4\,\mathbf{k}\right) = -\frac{13\sqrt{2}}{10\sqrt{5}}\,\mathbf{i} + \frac{3\sqrt{2}}{10\sqrt{5}}\,\mathbf{j} + \frac{4}{5\sqrt{5}}\,\mathbf{k}\end{aligned}$$

We can take the parallel vector $-13\sqrt{2}\,\mathbf{i} + 3\sqrt{2}\,\mathbf{j} + 8\,\mathbf{k}$ as a normal for the plane, so an equation for the plane is

$-13\sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right) + 3\sqrt{2}\left(y - \frac{\sqrt{2}}{2}\right) + 8\,(z - 1) = 0$ or $-13\sqrt{2}\,x + 3\sqrt{2}\,y + 8z = -2$ or $13x - 3y - 4\sqrt{2}\,z = \sqrt{2}$.