My research area is geometric group theory, which is the study of groups through their actions on metric spaces. Historically, the actions of groups were used to obtain information about the space, as with the action of the fundamental group of a manifold on its universal cover. Geometric group theory, in contrast, uses the metric properties of spaces on which a group acts to gain information about the group itself. As a field, geometric group theory became prominent with the work of Gromov in the 1980s, and combines the theory and techniques of algebra, topology, geometry, and functional analysis.

In his seminal paper [Gro87], Gromov made the important observation that groups that have cocompact proper isometric actions (i.e., actions that have a compact fundamental domain and no accumulation points) on δ–hyperbolic spaces (see Section 1) have many properties similar to those of surface groups, which were studied extensively by Dehn [Deh87] in the early 1900s. Gromov called these groups hyperbolic groups. For the next thirty years, the study of hyperbolic groups and their generalization, relatively hyperbolic groups, was a central theme of geometric group theory. Recently, the work of Gromov has been extended further by considering classes of groups that satisfy only certain fundamental properties of hyperbolic groups, that is, groups that admit some aspects of negative curvature but are not themselves hyperbolic ([Ham08, BF02, DGO16, Osi16, BBF15, BHS19, Bow12, MM00]).

Broadly, my research interests lie in the study of groups that admit interesting actions on hyperbolic spaces, all of which exhibit aspects of negative curvature to varying degrees. A single group may admit many different actions on different hyperbolic spaces, and my research focuses on understanding the relationships between these different actions, building new such actions for the group, and using these actions to obtain algebraic information about the group.

One of the main classes of groups which I study is the class of *acylindrically hyperbolic groups*, one generalization of hyperbolic groups suggested by Osin [Osi16]. For these groups, we require an acylindrical action on a hyperbolic space (see Section 1). This notion unified a large, seemingly disparate, collection of groups, including free groups, finitely presented groups with one relation and at least three generators, the fundamental groups of many non-hyperbolic 3-manifolds, and the mapping class group of most punctured closed surfaces, all of which exhibit some aspects of negative curvature. A close study of the acylindrical actions of such groups on hyperbolic spaces has yielded new insights into these classical groups.

My research program has four main components, which I will address in more detail below.

- §1. **Acylindrically hyperbolic groups.**
  - Define a poset of *acylindrical* actions of a single group on different hyperbolic spaces; this is a first step towards establishing an analogue of Teichmuller theory for acylindrically hyperbolic groups. Determine which groups admit *largest acylindrical actions* on hyperbolic spaces. ([1, 2])
  - Study *random walks* using the dynamics of an action of the group on a hyperbolic space to find subgroups which split as free products. ([6])
- §2. **Relating actions on hyperbolic spaces.** Define a poset of (not necessarily acylindrical) actions of a single group on different hyperbolic spaces, and completely describe this poset for certain classes of groups, including solvable Baumslag-Solitar groups. ([9, 13, 14])
- §3. **Hierarchically hyperbolic groups.** This class is another generalization of hyperbolic groups.
  - Study the structure of the poset of acylindrical actions of hierarchically hyperbolic groups, and give a complete characterization of contracting elements in such groups. ([4])
  - Give a linear bound on the length of shortest conjugators between certain classes of elements of such groups, a first step towards solving the *conjugacy problem* for this class of groups. ([3])
  - Give conditions under which such groups have *uniform exponential growth*. ([12])
- §4. **Big mapping class groups.** The *mapping class group* of a surface S is the group of orientation-preserving homeomorphisms of S up to isotopy. When $\pi_1(S)$ is infinitely generated the mapping class group is called *big*. Study the action of big mapping class groups on a hyperbolic graph associated to the surface, and construct the first examples of an *infinite-type* loxodromic isometry. This is a first step towards finding a Nielsen–Thurston classification of elements, in analogy with mapping class groups of finite-type surfaces. ([11])

1. Acylindrically hyperbolic groups

As mentioned above, my research focuses on groups that act on δ–hyperbolic spaces. A metric space is *δ–hyperbolic* if there exists a constant $\delta \geq 0$ such that every geodesic triangle in the space is δ–thin (see Figure 1). Such spaces generalize the metric properties of trees and the hyperbolic plane.

A group is *hyperbolic* if it admits a proper cocompact action on a hyperbolic metric space. Osin [Osi16] generalizes this to the class of *acylindrically hyperbolic groups* by weakening the hypothesis of a proper cocompact action to a non-elementary acylindrical action, defined below. An action $G \curvearrowright X$ of G on a metric space X is *acylindrical* if for every $\varepsilon > 0$ there exist $R, N \geq 0$ such that for all $x, y \in X$ with $d(x,y) \geq R$, $|\{g \in G \,|\, d(x,gx) \leq \varepsilon \text{ and } d(y,gy) \leq \varepsilon\}| \leq N$; this notion is defined by Bowditch [Bow08]. Here, non-elementary can be taken to mean that the group does not have a cyclic subgroup of finite index and the action has unbounded orbits. This extra assumption rules out trivial cases, as every group acts acylindrically on a point.

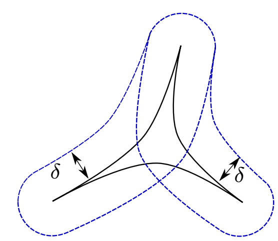

FIGURE 1. A δ–thin triangle.

1.1. **Poset of acylindrical structures.** A fixed acylindrically hyperbolic group will admit many different acylindrical actions on different hyperbolic spaces, and my first goal is to understand the relationship between these actions. Towards this end, in [2] Osin, Balasubramanya, and I introduce a partial order on the set of all actions of a given group, where, roughly speaking, one action is larger than another if the smaller space can be formed by equivariantly collapsing some subspace of the larger. The motivation is that the larger an action is in this partial order, the more information about the geometry of the group it should provide. For example, under this partial order, the action of any group on a point (which provides no information about the group) is always the smallest action, while the action on its Cayley graph (which recovers all of the algebraic information about the group) is the always the largest action. We use this partial order to define the *poset of acylindrically hyperbolic structures* on a given group G, denoted $\mathcal{AH}(G)$. This is a poset of equivalence classes of cobounded acylindrical actions of G on hyperbolic metric spaces. Essentially, two actions are equivalent if a slight perturbation of one yields the other. Formally,

$$\mathcal{AH}(G) = \{G \curvearrowright R \mid G \curvearrowright R \text{ is cobounded, acylindrical and } R \text{ is hyperbolic}\}/\sim,$$

where $G \curvearrowright R$ denotes the action of G on the metric space R and $\sim$ is the equivalence mentioned above.

The long-term goal of this project is to understand the structure of this poset with the hope of endowing it with a metrizable topology and thus turning it into a metric space. This space would then play the role of a "moduli space" of acylindrically hyperbolic structures of the group, in rough analogy with Teichmuller space of mapping class groups.

Our first step toward understanding the structure of this poset is to show that $\mathcal{AH}(G)$ is large.

Theorem 1.1 ([2])**.** *If G is acylindrically hyperbolic, then there exist chains and anti-chains in $\mathcal{AH}(G)$ of cardinality $2^{\aleph_0}$.*

We actually prove a more detailed version of the above theorem, which gives a more precise description of the structure of $\mathcal{AH}(G)$, building upon work of Hume, Osin, and I in [9] (see Section 2.1) and I. Kapovich in [Kap16] to construct new acylindrical actions on hyperbolic spaces.

We are at the beginning of the study of this poset, and many basic questions have yet to be answered. For example, when can this poset distinguish between non-isomorphic groups? As a first step towards an answer to this question, we consider the existence of a *largest* element in $\mathcal{AH}(G)$, that is, an element which is comparable to and larger than every element of $\mathcal{AH}(G)$; such an element must be unique. We often refer to the largest element (if it exists) as a *largest acylindrical action.* Every non-acylindrically hyperbolic group admits such an action: this is the action on a line, if the group if virtually cyclic, or on a point, otherwise.

Theorem 1.2 ([2])**.** *When G is the mapping class group of surface, a right-angled Artin group, or the fundamental group of a 3–manifold, $\mathcal{AH}(G)$ has a largest element.*

We also investigate the existence of a largest acylindrical action for the class of relatively hyperbolic groups, another generalization of hyperbolic groups, which have been extensively studied ([Gro87, Far98, Osi06, GM08]). Intuitively, a group G is hyperbolic relative to subgroups $H_1, \ldots, H_n$ if these subgroups are the only obstruction to G being a hyperbolic group; for example the fundamental group of a finite-volume hyperbolic 3–manifold is relatively hyperbolic. Relatively hyperbolic groups are acylindrically hyperbolic, and thus one can ask about the structure of $\mathcal{AH}(G)$. We obtain the strongest possible result in this setting:

Theorem 1.3 ([2, 9])**.** *Let G be hyperbolic relative to $H_1, \ldots, H_n$. If each $\mathcal{AH}(H_i)$ admits a largest element for each i, then so does $\mathcal{AH}(G)$.*

While the previous two theorems give positive results about the existence of a largest acylindrical action, it is not the case that *every* finitely generated group admits such an action.

Theorem 1.4 ([1])**.** *There exist finitely generated groups which do not admit largest acylindrical actions. In particular, Dunwoody's inaccessible group displays this phenomenon.*

With this theorem, I give the first example of a finitely generated group without a largest action. However, this group is infinitely presented. Balasubramanya, Osin, and I give the first example of a finitely presented group without a largest action in [2], using Brady's construction of a non-hyperbolic subgroup of a hyperbolic group [Bra99]. Moreover, in [7] Hume and I construct uncountably many (quasi-isometry classes of) finitely generated groups without largest actions using infinitely presented small cancellation groups.

It is natural to wonder which properties can distinguish between groups that do and do not have a largest acylindrical action. While we have shown that finite presentation is not sufficient, it may be possible to distinguish such groups using higher finiteness properties.

Beyond understanding when $\mathcal{AH}(G)$ has a largest element, there are many additional questions about the structure of $\mathcal{AH}(G)$ to be explored. For example:

Question 1.5. *For which groups G (if any) is $\mathcal{AH}(G)$ a lattice?*

1.2. **Random walks and free products.** One way to understand groups is to study the behavior of "typical" elements and subgroups. The study of *random walks* allows one to define precisely what is meant by a typical element. For simplicity, I will restrict this discussion to *simple* random walks, though all results in this section hold in much greater generality. By a *simple random walk of length n* on a group G with finite generating set S, we mean a *random element* w_n that is equal to the product $w_n = s_1 ... s_n$, where each $s_i \in S$ is chosen uniformly at random with probability $\frac{1}{|S|}$. A *random subgroup* of G is a subgroup generated by (finitely many) independent random walks.

In [6], Hull and I investigate the algebraic and geometric relationships between random elements of a group and certain fixed subgroups by studying the dynamics of an acylindrical action of the group on a hyperbolic metric space. Historically, random walks have been studied using analytic and probabilistic tools, focusing on, for example, the speed and entropy of the random walk and establishing various central limit theorems ([KV83, Mat15, KL06, KT17]). While several authors have recently employed geometric group theory to understand random walks ([MS16, MT18, TT16]), they have focused on the geometry of a single random walk or multiple independent random walks, and thus our work takes this theory in a new direction.

Given a fixed action $G \curvearrowright X$ on a hyperbolic space, we are particularly interested in subgroups whose orbits in X have a nice geometry. Specifically, we investigate subgroups which are are *quasi-convex* in X (i.e., subgroups such that geodesics in X between points of the orbit stay within a uniform neighborhood of the orbit), *quasi-isometrically embedded* in X (i.e., distances in a Cayley graph of the subgroup and distances in its image under the orbit map differ by at most a uniform linear amount), and/or *geometrically separated* in X (i.e., orbits of distinct cosets spread out quickly from each other in X). We will consider only subgroups whose orbits are not too large in the space X, which we make precise by requiring the existence of an element f that acts as translation on X (called a *loxodromic isometry*) and is *transverse to H*, that is, an axis of f "avoids" all translates of the orbit of H in X.

Theorem 1.6 ([6]). *Let G be a group with a non-elementary acylindrical action hyperbolic metric space X, and let H be a subgroup of G which is quasi-convex or quasi-isometrically embedded in X. Suppose there exists a loxodromic element $f \in G$ which is transverse to H. Then a random subgroup R will satisfy $\langle H, R\rangle \cong H * R$ and $\langle H, R\rangle$ is quasi-convex or quasi-isometrically embedded, respectively, in X.*

Moreover, if H is quasi-isometrically embedded and geometrically separated in X, then $\langle H, R\rangle$ is quasi-isometrically embedded and geometrically separated in X.

This theorems has several nice applications, all of which are new.

Corollary 1.7 ([6]). *Let G and H be one of the following:*

(1) G a non-elementary hyperbolic group with no finite normal subgroups and H an infinite index quasi-convex subgroup;
(2) G the fundamental group of a complete, finite volume Riemannian manifold with pinched negative sectional curvature and H a cusp subgroup of G;
(3) G a relatively hyperbolic group and H an infinite index, relatively quasi-convex *subgroup;*
(4) G a non-elementary subgroup of $MCG(S)$ for a non-exceptional surface S with no non-trivial finite normal subgroups, and H a convex cocompact subgroup*;*
(5) G the fundamental group of a closed, orientable, non-prime 3–manifold and H the fundamental group of a prime submanifold.

Then a random subgroup R satisfies $\langle H, R\rangle \cong H * R$. Moreover, in (3) and (4), we additionally have that $\langle H, R\rangle$ is relatively quasi-convex and convex cocompact in G, respectively.

In addition to the above corollaries, this theorem uses random walks to give a new proof that an acylindrically hyperbolic group G (with no non-trivial normal subgroups) has *property P_{naive}*, that is, for any finite collection collection of elements $h_1, \dots, h_n \in G$, there exists an infinite order element $g \in G$ such that $\langle h_i, g\rangle \simeq \langle h_i\rangle * \mathbb{Z}$ for all $i = 1, \dots, n$. Dahmani and I originally proved this result in [5].

2. Poset of hyperbolic structures

Many non-acylindrically hyperbolic groups admit interesting actions on hyperbolic spaces. In [2] we generalize the poset $\mathcal{AH}(G)$ defined in Section 1.1 to the *poset of hyperbolic structures of G* by removing the requirement that an action is acylindrical:

$$\mathcal{H}(G) = \{G \curvearrowright R \mid G \curvearrowright R \text{ is cobounded and } R \text{ is hyperbolic}\}/\sim .$$

By construction, $\mathcal{AH}(G)$ is a sub-poset of $\mathcal{H}(G)$, and thus $\mathcal{H}(G)$ is at least as complicated as $\mathcal{AH}(G)$.

As for $\mathcal{AH}(G)$, we first investigate for which finitely groups G the poset $\mathcal{H}(G)$ has a largest element. Rasmussen and I use mapping class groups to show that the existence of a largest element in $\mathcal{AH}(G)$ does not guarantee that $\mathcal{H}(G)$ has a largest element, as well; thus $\mathcal{H}(G)$ may indeed be a much more complicated poset. Moreover, Hume and I use small cancellation theory to construct uncountably many (quasi-isometry classes of) groups for which neither poset has a largest element in [7].

2.1. Extending group actions.

In general, for groups G which exhibit many aspects of negative curvature (for example, the mapping class group of a finite-type surface), the poset $\mathcal{H}(G)$ is quite complicated and giving a complete description seems fairly intractable. For such groups, the work of Hume, Osin, and I in [9] allows one, in many situations, to study the hyperbolic structures of certain subgroups of G to obtain information about $\mathcal{H}(G)$. These subgroups are often much less complicated than G itself, which simplifies the problem considerably. To do this, we first address the following general question:

Question 2.1. *Given a group G, a subgroup $H \leq G$, and an action $H \curvearrowright S$ on a metric space S, does there exist an action of G on a (possibly different) metric space that extends $H \curvearrowright S$?*

Much of the inspiration for this question (and our answer) comes from representation theory, in particular the notion of an induced representation, which is an important tool for constructing new representations of a group. Roughly, if H is a subgroup of G, we say that an action $G \curvearrowright R$ is *an extension* of $H \curvearrowright S$ if there is a quasi-isometric embedding $S \to R$ which is (coarsely) compatible with the actions of H on S and R. We say that *the extension problem is solvable for the pair $H \leq G$* if there is a positive answer to the above question for *every* action of H on a metric space.

In [9], Hume, Osin, and I completely characterize the pairs $H \leq G$ for which the extension problem is solvable when G is finitely generated by explicitly constructing an extension of the given action of H, which we call an *induced action* of G. When building the induced action, one may start with *any* metric space, hyperbolic or not. However, for applications we are most interested in extending actions $H \curvearrowright S$ where S is a hyperbolic metric space; in this case, we would like the construction of the induced action of G to also yield a hyperbolic metric space. The following theorem shows that for an acylindrically hyperbolic group G, this will be the case whenever H is a hyperbolically embedded subgroup. The formal definition is technical, but this is a class of subgroups that have a particularly nice geometry in G; the existence of such a subgroup is equivalent to the group being acylindrically hyperbolic by [Osi16].

Theorem 2.2 ([9])**.** *There is an injective map of posets $\mathcal{H}(H) \hookrightarrow \mathcal{H}(G)$ whenever H is a hyperbolically embedded subgroup of G.*

Therefore we can use the geometry of $\mathcal{H}(H)$, which in many cases will be a simpler poset, to understand the geometry of $\mathcal{H}(G)$. We have already seen one instance in which this result allows us to understand the geometry of the larger poset in Theorem 1.1.

2.2. **Complete descriptions of $\mathcal{H}(G)$.** In contrast to acylindrically hyperbolic groups, it is sometimes possible to give a simple and complete description of $\mathcal{H}(G)$ for groups G which have just enough negative curvature to admit interesting actions on hyperbolic spaces but are still quite far from being hyperbolic themselves. For example, one can use direct products to show the following.

Theorem 2.3 ([2])**.** *For every $n \in \mathbb{N}$, there is a group G such that $|\mathcal{H}(G)| = n$.*

A more complicated example involves solvable Baumslag–Solitar groups $BS(1,n) := \langle a, t \mid tat^{-1} = b^n \rangle$, which splits as the semidirect product $\mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$.

Theorem 2.4 ([14])**.** *For any n, if G is the Baumslag–Solitar group $BS(1,n)$, then $\mathcal{H}(G)$ has the structure given in Figure 2. In particular, $\mathcal{H}(G)$ does not have a largest element.*

To prove the above theorem, Rasmussen and I give a (somewhat surprising) correspondence between equivalence classes of cobounded actions of $BS(1,n)$ on hyperbolic spaces and certain ideals of the ring of n–adic integers $\mathbb{Z}_n$. The (equivalence classes of) actions can easily be described geometrically. They consist of the action on a point, the action on a line coming from the natural projection to the second factor $\mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z} \to \mathbb{Z}$, the action on the hyperbolic plane coming from the representation of $BS(1,n)$ into $PSL_2(\mathbb{Z})$, and finitely many actions on Bass–Serre trees corresponding to splittings of $BS(1,n)$ as HNN extensions over (non-obvious) subgroups of $\mathbb{Z}[\frac{1}{n}]$.

A *quasi-isometry* is a quasi-isometric embedding that is also coarsely surjective. If there is a quasi-isometry between the Cayley graphs of two groups, we say the groups are *quasi-isometric*; such groups have the same large-scale geometry and thus are considered "the same" from the point of view of geometric group theory. We do not know whether the poset $\mathcal{AH}(G)$ or, more generally, $\mathcal{H}(G)$ is a quasi-isometry invariant. However, using Farb–Mosher's quasi-isometry classification of Baumslag–Solitar groups [FM99], we obtain the following corollary.

Corollary 2.5 ([14])**.** *If G, G' are quasi-isometric solvable Baumslag–Solitar groups, then $\mathcal{H}(G) = \mathcal{H}(G')$.*

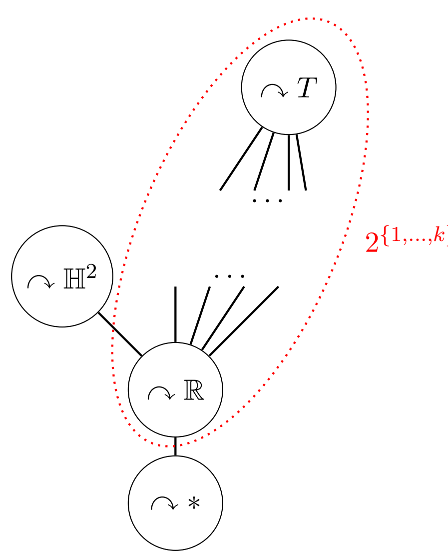

FIGURE 2. The poset $\mathcal{H}(G)$ for $G = BS(1,n)$. The subposet circled in red is isomorphic to the poset $2^{\{1,\dots,k\}}$, which is a lattice.

Rasmussen and I use a similar strategy to completely describe $\mathcal{H}(G)$ for $G = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ when ϕ is an Anosov homeomorphism of the torus [13]. In this case, as well, $\mathcal{H}(\mathbb{Z}^2 \rtimes_\phi \mathbb{Z})$ does not have a largest element. As the groups $BS(1,n)$ and $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ are examples of ascending HNN extensions, we ask the following question.

Question 2.6. *Is it possible to completely describe $\mathcal{H}(G)$ when $G = \mathbb{Z}^n \rtimes_\phi \mathbb{Z}$ and $\phi \in SL_2(\mathbb{Z})$ or, more generally, when G is an ascending HNN extension of finitely generated abelian groups?*

3. Hierarchically hyperbolic groups

Hierarchically hyperbolic groups (HHGs) were defined by Behrstock–Hagen–Sisto in [BHS17, BHS19] as another generalization of hyperbolic groups that have a structure similar to mapping class groups and cubical groups. Every hierarchically hyperbolic group G comes with a *hierarchy structure* that generalizes the hierarchy structure of mapping class groups defined by Masur–Minsky in [MM00]. It is shown in [BHS17] that each such structure yields a (possibly elementary) acylindrical action of G on a hyperbolic space denoted $\mathcal{C}(G)$. The class of HHGs includes many acylindrically hyperbolic groups, such as mapping class groups and directly indecomposable right-angled Artin and Coxeter groups, as well as direct products of these groups, which are not acylindrically hyperbolic. Because of the additional structure present in HHGs, certain theorems may hold or be easier to prove in this setting compared to all acylindrically hyperbolic groups.

3.1. Building new hierarchical structures.

Given a hierarchically hyperbolic group G, Behrstock, Durham, and I construct a new hyperbolic space $\mathcal{C}_\mathfrak{T}(G)$ such that $G \curvearrowright \mathcal{C}_\mathfrak{T}(G)$ is a largest acylindrical action. This action recovers all previously known largest acylindrical actions for non-relatively hyperbolic groups and gives new examples of groups with such actions, getting us closer to completely classifying such groups.

Theorem 3.1 ([4])**.** *All HHGs admit largest acylindrical actions, including mapping class groups, the fundamental groups of 3–manifolds, and most CAT(0) cubical groups, including right-angled Artin and right-angled Coxeter groups and all compact special groups (in the sense of Haglund–Wise* [HW08]*).*

All (non-relatively hyperbolic) finitely generated groups that are known to admit largest actions are both acylindrically hyperbolic and HHGs. In light of this, we ask whether there exists a (non-relatively hyperbolic) acylindrically hyperbolic group that is not an HHG and admits a largest acylindrical action.

The new space $\mathcal{C}_\mathfrak{T}(G)$ that we construct is, in a natural sense, an analogue of the curve complex for mapping class groups, a hyperbolic space associated to a surface with an acylindrical action of the mapping class group of this surface [MM00, Bow08]. Many results about mapping class groups rely on the geometry of the curve complex, and thus investigating the geometry of $\mathcal{C}_\mathfrak{T}(G)$ may yield analogous results.

In particular, we use the geometry of $\mathcal{C}_\mathfrak{T}(G)$ to study Morse and contracting elements of an HHG. An infinite order element is *Morse* if its axis γ in the Cayley graph of G has the property that any quasi-geodesics with endpoints on γ stays within uniformly bounded distance of γ. It is *contracting* if (roughly) the diameter of the nearest-point projection of any ball disjoint from γ is bounded above by a uniform constant (see Figure 3). Contracting elements are always Morse, but the converse is false in general. Behrstock, Durham, and I establish the equivalence of these notions for HHGs and use $\mathcal{C}_\mathfrak{T}(G)$ to give a complete characterization of contracting elements in HHGs, generalizing a result for mapping class groups and CAT(0) groups.

Theorem 3.2 ([4])**.** *An infinite order element in a hierarchically hyperbolic group is contracting if and only if it is Morse.*

We also use the ideas in the proof of Theorem 3.2 to give a new proof of Maher–Sisto's result in [MS16] that a random subgroup of an HHG quasi-isometrically embeds into $\mathcal{C}_\mathfrak{T}(G)$ (see Section 1.2).

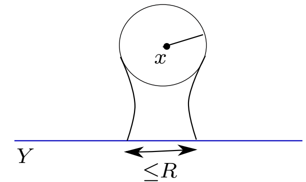

FIGURE 3. A contracting subset Y. The constant R depends only on Y.

3.2. **Short conjugators.** In [MM00] and [Tao13] it is shown that mapping class groups satisfy the *linear conjugator bound*. That is, if g, h are two conjugate elements of a mapping class group of a surface, then there exists an element t such that $g = t^{-1}ht$ with $|t| \leq K(|g| + |h|)$. One consequence of this bound is an exponential time algorithm to solve the conjugacy problem, one of Dehn's classic decision problems. In both papers, the authors rely heavily on the geometry of the curve complex.

Question 3.3. *Do hierarchically hyperbolic groups have solvable conjugacy problem?*

Extending the methods of [BD14], in which Behrstock–Druţu prove the linear conjugator bound for mapping class groups and many $\mathrm{CAT}(0)$ groups, Behrstock and I prove the following theorem. The result holds in greater generality, but to avoid technicalities we state only the following special case.

Theorem 3.4 ([3]). *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group. There exist constants K, C such that if $a, b \in G$ are Morse elements which are conjugate in G, then there exists $g \in G$ with $ga = bg$ and*

$$|g| \leq K(|a| + |b|) + C.$$

In the case of CAT(0) cubical groups that are hierarchically hyperbolic, we show in [3] that the above linear bound holds for *all* infinite order elements. This naturally leads to the following question:

Question 3.5. *What conditions on a hierarchically hyperbolic group are necessary to ensure that all elements satisfy the linear conjugator bound? In particular, for which hierarchically hyperbolic groups do finite order elements satisfy the linear conjugator bound?*

3.3. **Uniform exponential growth.** A finitely generated group has *(uniform) exponential growth* if the number of elements that can be written as words of bounded length grows (uniformly) exponentially fast with respect to *any* finite generating set. Exponential growth rates and uniform exponential growth rates are of interest in a broad range of areas, including differential geometry, dynamical system theory, and the theory of unitary representations (see [dlH02] and citations within).

Gromov asked if every finitely generated group with exponential growth has uniform exponential growth. However, this is not the case: the first counterexample was constructed by Wilson [Wil04b], and additional counterexamples have since been constructed [Wil04a, Nek10]. However, Gromov's question is still open for finitely presented groups.

Question 3.6 ([Gro81]). *Do all finitely presented groups with exponential growth have uniform exponential growth?*

In [12], Ng, Spriano, and I consider this question for the class of virtually torsion-free HHGs.

Theorem 3.7 ([12]). *Let G be a virtually torsion-free HHG. Then either G has uniform exponential growth or there is a space E such that the Cayley graph of G is quasi-isometric to $\mathbb{Z} \times E$.*

The first consequence of Theorem 3.7 is that if the Cayley graph of a hierarchically hyperbolic group G is not isometric to a (non-trivial) product, then G has uniform exponential growth. We give two interesting situations in which this is the case.

Corollary 3.8 ([12]).

(1) Every virtually torsion-free hierarchically hyperbolic group which is not virtually cyclic and has a cut point in some asymptotic cone has uniform exponential growth.

(2) Virtually torsion-free hierarchically hyperbolic groups which are acylindrically hyperbolic have uniform exponential growth.

All known examples of virtually torsion-free hierarchically hyperbolic groups are either virtually abelian (and so do not have exponential growth) or have uniform exponential growth by Theorem 3.7.

Question 3.9. *Is there virtually torsion-free hierarchically hyperbolic group which is not virtually abelian but does not have uniform exponential growth?*

As all hierarchically hyperbolic groups are finitely presented, a positive answer to the above question would provide a negative answer to Question 3.6. In [12], Ng, Spriano, and I give a more precise description of the hierarchical structure which any group that gives a positive answer to the above question must have. To avoid technicalities, we do not state it here.

4. Big mapping class groups

The *mapping class group* of a surface S, denoted $\mathrm{MCG}(S)$, is the group of orientation-preserving homeomorphisms of S up to isotopy. Mapping class groups have been an important tool in many different areas of mathematics, such as 3–manifold topology, differential geometry, and dynamics. While in the majority of this work the surfaces considered have finitely generated fundamental group, called *finite type*, there are many cases in which one must consider surfaces of *infinite type*, that is, with infinitely generated fundamental group; this occurs, for example, in real and complex dynamics. If S is infinite type, then $\mathrm{MCG}(S)$ is called a *big mapping class group*. While mapping class groups (of finite-type surfaces) are well-studied, the study of big mapping class groups is quite new.

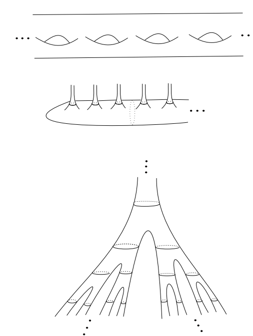

FIGURE 4. Examples of infinite-type surfaces.

There are several stark differences between mapping class groups and big mapping class groups. For example, while mapping class groups are always finitely presented, big mapping class groups are not even compactly generated. Additionally, and crucially for my research, mapping class groups are always acylindrically hyperbolic (or finite), while big mapping class groups are *never* acylindrically hyperbolic [BG18].

However, some of the machinery of mapping class groups may still yield interesting results in the context of big mapping class groups. One of the most important features of mapping class groups is the *Nielsen–Thurston classification* of elements, which states that every element of a mapping class group is either *finite-type*, *reducible*, or *pseudo-Anosov*. One major open question for big mapping class groups is whether there exists an analogue of the Nielsen–Thurston classification for elements of big mapping class groups. The same classification does not hold: Patel–Vlamis show in [PV18] that big mapping class groups contain elements that do not fall into any of these three categories. Such elements are examples of *infinite-type* elements, that is, homeomorphisms that are not compactly supported. More precisely, there is no compact subsurface such that these homeomorphisms restrict to the identity on its complement; this is a phenomenon which cannot occur in the finite-type setting. In order to encompass all infinite-type elements, any Nielsen–Thurston classification must either contain additional categories or consist of more broadly defined categories.

Taking the latter approach, Miller, Patel, and I propose a generalization of pseudo-Anosov elements based on their characterization as precisely the elements that act loxodromically on the curve graph of the surface; recall that this is an infinite-diameter hyperbolic graph with an acylindrical action of the mapping class group and a loxodromic isometry is one that acts as translation along an axis. While one cannot hope to find an *acylindrical* action of a big mapping class group $\mathrm{MCG}(\Sigma)$ on a hyperbolic space (as these groups are never acylindrically hyperbolic), one can still hope to use the dynamics of $\mathrm{MCG}(\Sigma)$ on *some* infinite-diameter hyperbolic space to generalize the definition of a pseudo-Anosov element. In [11], Miller, Patel, and I propose such a generalization and construct an infinite family of non-compactly supported homeomorphisms of an infinite-type surface which act loxodromically on a certain hyperbolic space constructed by Aramayona–Fossas–Parlier in [AFP17]. These are the first examples of such elements.

Theorem 4.1 ([11]). *Let Σ be an infinite-type surface with an isolated puncture p. Then there is an infinite collection of non-compactly supported elements of the mapping class group of Σ (fixing p) which act loxodromically on a hyperbolic graph.*

Moreover, we justify these elements as generalizations of pseudo-Anosovs by showing that the elements we construct satisfy two key properties of pseudo-Anosov elements in the finite-type setting, one involving quasi-morphisms and the other involving laminations on the surface.

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