9/3

Math 763
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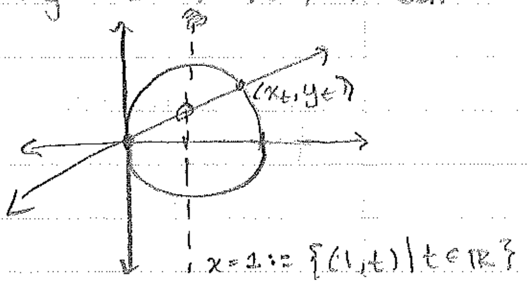
VV 605

① Enumerate all points sat. $(x-1)^2+y^2=1$ in k^2, k an arbitrary field.



(x_t,y_t) is the 2^{nd} pt of int, besides $(0,0)$ of the line through $(0,0)$ & $(1,t)$.

$y_t = t x_t$

$\Rightarrow (x_t-1)^2+(tx_t)^2=1$

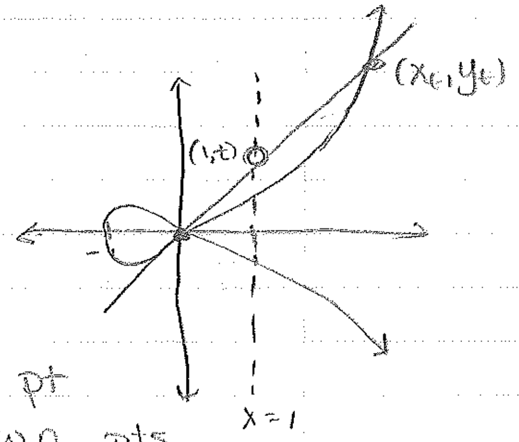
$x^2-2x_t+1+t^2x_t^2=1$

$\cancel{x_t}(x_t-2+t^2x_t)=0$

$x_t = \frac{2}{1+t^2}$, $y_t=\frac{2t}{1+t^2}$

We have constructed a 1-1 corresp btwn pts on the line & pts on the circle, almost. Problem: origin! So we need to add a pt to the line at ∞, i.e. compactify the line.

<u>Cor</u>: Any smooth quadric <u>with a pt</u> (in $\mathbb{P}^2$) is $\cong$ to $\mathbb{P}^1$.



② Same problem w/ $C: y^2=x^2(x+1)$

Over $\mathbb{R}$: $y=\pm|x|\sqrt{x+1}$

Draw line through $(0,0)$ & $(1,t)$, & denote 2^{nd} int pt (x_t,y_t).

[<u>Note</u>: given eqn of line & cubic, has double root at $(0,0)$, so has another pt of int.] Hope to get corresp btwn pts on $x=1$ & sol'ns of $y^2=x^2(x+1)$ [i.e., $(1,t) \Leftrightarrow (x_t,y_t)$]

$t^2\cancel{x_t^2}=\cancel{x_t^2}(x_t+1) \Rightarrow x_t = t^2-1$, $y_t=t^3-t$. But $t=\pm1$ gives $(0,0)$, since there are 2 tangents at $(0,0)$

__Cor__: Any cubic w/ a double pt (in $\mathbb{P}^2$) is $\cong$ after removing the double pt to $\mathbb{P}^1 \setminus \{2 \text{ pts}\}$. There is an open set on C which is $\cong$ to an open set in $\mathbb{P}^1$.

③ Try the same for $y^2 = x(x-1)(x+1)$. Will fail miserably - not a rational problem (i.e. can't represent sol'n as rat'l fcn of one variable). Not a __rational curve__.

- In algebraic terms, if $R_i = k[x,y]/(f_i)$, & $f_1 = y^2 - x^2(x+1)$, $f_2 = y^2 - x(x-1)(x+1)$, then the field of fracs L_i; $L_1 \cong k(t)$, but $L_2 \not\cong k(t)$.
- We can define an alg. invariant to distinguish the 2 cases: arithmetic genus $= 0$ in 1st case & $= 1$ in 2nd.

__Def__: Let k be an alg closed field.

- __Affine n-space__ over k, denoted $\mathbb{A}^n$, is the set of n-tuples of elts of k. $\mathbb{A}^n = k^n$ (but don't think of it as a vector sp - $\mathbb{A}^n$ does not have a distinguished origin.
- The set of alg. fcns on $\mathbb{A}^n$ is $k[x_1, \dots, x_n]$.
- If f is a poly in n vars, $Z(f) = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid f(x_1, \dots, x_n) = 0\} \subseteq \mathbb{A}^n$ is the __zero-locus__ of f. More generally, if $T \subseteq k[x_1, \dots, x_n]$, $Z(T) = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid f(x_1, \dots, x_n) = 0 \ \forall f \in T\}$.
- A subset $Y \subseteq \mathbb{A}^n$ is __algebraic__ iff $\exists\, T \subseteq k[x_1, \dots, x_n]$ s.t. $Y = Z(T)$.

__Ex__: In $\mathbb{A}^1$:

- $Y = \{a_1, \dots, a_\ell\} \subseteq \mathbb{A}^1 \Rightarrow Y = Z((x-a_1)\cdots(x-a_\ell))$, so Y is algebraic.
- $\mathbb{Z} \subseteq \mathbb{C} = \mathbb{A}^1_{\mathbb{C}}$ is not algebraic:

__Lemma__: $Z(T) = Z(\underline{a})$, $\underline{a}$ = ideal gen by T in $k[x_1, \dots, x_n]$

__Pf__: $T \subseteq \underline{a} \Rightarrow Z(T) \supseteq Z(\underline{a})$ [b/c added more polys to get $\underline{a}$]. But $\forall f \in \underline{a}$, $f = \sum g_i f_i$, $f_i \in T$. So if $\underline{x} \in Z(T)$, then $f_i(\underline{x}) = 0 \ \forall i \Rightarrow f(\underline{x}) = 0 \Rightarrow \underline{x} \in Z(\underline{a}) \Rightarrow Z(\underline{a}) \supseteq Z(T)$. $\square$

If $Z \subseteq \mathbb{C}$ were algebraic, $Z = Z(T)$ for some $T \subseteq \mathbb{C}[x]$
$\Rightarrow Z = Z(\underline{a})$, but $\underline{a} = (f)$, b/c $\mathbb{C}[x]$ a PID, for
$f \in \mathbb{C}[x]$. But f has fin. many zeros $\Rightarrow Z$ finite $\lightning$.
Thus, algebraic sets in $\mathbb{A}^1$ are finite or all of $\mathbb{A}^1$
(from $f \equiv 0$).

Prop: Alg. sets in $\mathbb{A}^n$ form the closed sets of a topology on $\mathbb{A}^n$, called the Zariski topology.

9/5 Pf: (1) $\emptyset$, $\mathbb{A}^n$ are alg. sets:

$\phi = Z(1)$
$\mathbb{A}^n = Z(0)$

(2) If $\underset{\underset{Z(T_1)}{\|}}{Y_1}, \underset{\underset{Z(T_2)}{\|}}{Y_2}$ are alg, then $Y_1 \cup Y_2$ alg. Claim: $Y_1 \cup Y_2 = Z(T_1T_2)$,
$T_1T_2 = \{fg \mid f \in T_1, g \in T_2\}$

Pf: Let $x \in Y_1 \cup Y_2$. WTS: $\forall fg \in T_1T_2$, $(fg)(x) = 0$
If $x \in Y_1$, then $f(x) = 0$ $\forall f \in T_1 \Rightarrow fg(x) = 0$ $\forall fg \in T_1T_2$.
Sim. if $x \in Y_2$.
Let $x \in Z(T_1T_2)$. Assume $x \notin Y_1$. Then $\exists f_0 \in T_1$
s.t. $f_0(x) \neq 0$. But $\forall g \in T_2$, since $x \in Z(T_1T_2)$, i.e.
$(f_0g)(x) = 0 \Rightarrow g(x) = 0 \Rightarrow x \in Z(T_2) = Y_2$.

(3) If $Y_i = Z(T_i)$, Then $\bigcap_i Y_i = Z(\bigcup_i T_i)$. Clear. $\square$

Def: This top. is called the Zariski topology on $\mathbb{A}^n$ (or on alg. sets by the induced topology).

Ex: • On $\mathbb{A}^1$, closed sets are finite or $\mathbb{A}^1$.
• On $\mathbb{A}^2$ x-axis $= Z(\{y\})$ a closed subset but infinite.

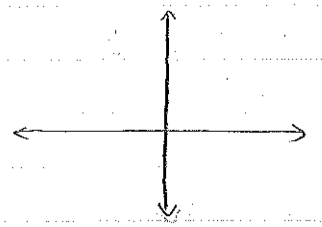
Def: A top. sp. $X^{\neq \emptyset}$ is irreducible $\Leftrightarrow$ if $X = X_1 \cup X_2$ w/ X_1, X_2 closed, then $X_1 = X$ or $X_2 = X$.

Ex: $Z(xy) \subseteq \mathbb{A}^2$

$\quad \|$

$Z(x) \cup Z(y)$

$\Rightarrow xy$ is reducible



Prop: If $U \subseteq X$ is nonempty & open & X irred, then U irred & dense.

Pf: We have $X = (X \setminus U) \cup \overline{U}$

$\uparrow$ closed b/c U open $\quad \uparrow$ closed

$(X \setminus U) \neq X$ b/c $U \neq \emptyset$.

So $\overline{U} = X$, i.e. U is dense.

[In $\mathbb{A}^1$, all open sets are ∞, so if U open, then $\overline{U} \supseteq U \Rightarrow \overline{U}$ ∞, but the only ∞ cl. set is $\mathbb{A}^1$].

Assume $U = Z_1 \cup Z_2$ w/ Z_1, Z_2 cl. in U. Let $Z_1 = Y_1 \cap U$, $Z_2 = Y_2 \cap U$, Y_1, Y_2 cl. in X.

$X = Y_1 \cup Y_2 \cup (X \setminus U)$ (since $U \subseteq Y_1 \cup Y_2$) all closed, & $X \setminus U \neq X \overset{X \text{ irred}}{\Rightarrow} Y_1 = X$ or $Y_2 = X \Rightarrow Z_1$ or $Z_2 = U \Rightarrow U$ irred. $\square$

*exer: Prop: If $Y \subseteq X$ is any subset & Y irred, then $\overline{Y}$ is irred.

Ex: If $X = \mathbb{A}^2$, $Y_1 = Z(\{y\})$ irred. $Y = Y_1 \setminus \{x_1, \ldots, x_k\}$ irred.

Then $\overline{Y_1}$ also irred.

Def: An affine variety is a closed irred. subset of $\mathbb{A}^n$.

Def: A quasi-affine variety is an open set in an affine variety.

We have seen:

$$\{T \subseteq k[x_1,\dots,x_n]\} \longrightarrow \{\text{alg. set in } \mathbb{A}^n\}$$
$$T \longmapsto Z(T).$$

Can we go the other way? ← arbitrary, not nec. alg.

$$\{\text{ideal in } k[x_1,\dots,x_n]\} \longleftarrow \{Y \subseteq \mathbb{A}^n\}$$
$$I(Y) \longleftarrow\!\shortmid\; Y$$

For $Y \subseteq \mathbb{A}^n$ arbitrary, define $I(Y) = \{f \in k[x_1,\dots,x_n] \mid f(x) = 0 \ \forall x \in Y\}$, an ideal of $k[x_1,\dots,x_n]$.

<u>Prop</u>: (1) I, Z are inclusion reversing (not strictly) ✓

(2) $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ ✓

(3) $I(Z(\underline{a})) = r(\underline{a})$ (Nullstellensatz)

(4) $Z(I(Y)) = \overline{Y}$

<u>Pf</u>: (4) $Y \subseteq Z(I(Y))$ and $Z(I(Y))$ cl $\Rightarrow$ $\overline{Y} \subseteq Z(I(Y))$

Let W be cl. & $W \supseteq Y$. <u>WTS</u>: $Z(I(Y)) \subseteq W$.

W cl $\Rightarrow$ $W = Z(\underline{a})$ for some ideal $\underline{a} \subseteq k[x_1,\dots,x_n]$.

$Z(\underline{a}) \supseteq Y \Rightarrow \underline{a} \subseteq I(Z(\underline{a})) \subseteq I(Y) \Rightarrow \underset{\substack{\shortparallel \\ W}}{Z(\underline{a})} \supseteq Z(I(Y))$ □

<u>Note</u>: Props 1, 2 & 4 work if k not alg cl.

<u>Ex</u>: Let $\underline{a} = (x^2+y^2+1) \subseteq \mathbb{R}[x,y]$.

$Z(\underline{a}) = \emptyset \Rightarrow I(Z(\underline{a})) = \mathbb{R}[x,y] \neq r(\underline{a})$.

In (3), always have $I(Z(\underline{a})) \supseteq r(\underline{a})$, even over non-alg. cl. fields.

↑ a radical ideal that contains $\underline{a}$

(b/c if $f^n(x) = 0$, $f(x) = 0$

(i.e. an ideal that's = to its radical)

<u>Cor</u>: I, Z give an inclusion reversing 1-1 corresp b/twn alg sets of $\mathbb{A}^n$ & radical ideals of $k[x_1,\dots,x_n]$.

Irreducible ⟺ prime ideals.

<u>Pf</u>: By (3), (4) IZ & ZI are id. on rad ideals & closed sets, resp. Then just restrict the bij.

Let $Y \subseteq \mathbb{A}^n$ be irred & cl. $\underline{a} = I(Y)$. Assume $fg \in \underline{a}$. $(fg) \subseteq \underline{a}$.

$\Rightarrow \underset{\substack{\shortparallel \\ Z(f) \cup Z(g)}}{Z(fg)} \supseteq Y \Rightarrow Y = (Y \cap Z(f)) \cup (Y \cap Z(g)) \Rightarrow$ wlog,

Since Y irred, $Y \subseteq Z(f)$. Apply I: $\underline{a} = I(Y) \supseteq I(Z(f)) = \sqrt{(f)}$
$\Rightarrow f \in \underline{a}$, $\Rightarrow$ $\underline{a}$ prime.

Conversely, assume $\underline{a}$ is prime. Write $Y = Z(\underline{a})$ as
$Y_1 \cup Y_2$ w/ Y_i closed. $Y_i = Z(\underline{a}_i)$ Assume $\underline{a}_i$ radical.
$Z(\underline{a}_1) \cup Z(\underline{a}_2) = Z(\underline{a}_1 \cdot \underline{a}_2) \Rightarrow \sqrt{\underline{a}_1} \cdot \sqrt{\underline{a}_2} \subseteq \underline{a} \Rightarrow \underline{a}_1$ or $\underline{a}_2 \subseteq \underline{a}$
$\Rightarrow Y_1 \supseteq Y$ or $Y_2 \supseteq Y$. D.

So, $\mathbb{A}^n$ irred $\Leftrightarrow$ $(0) \subseteq k[x_1, \dots, x_n]$ prime.
If $f \in k[x_1, \dots, x_n]$ irred, then (f) prime, so $Z(f)$ is an affine variety in $\mathbb{A}^n$.

<u>Def</u>: Let $Y \subseteq \mathbb{A}^n$ be an alg. set. Define the <u>affine coordinate ring</u> of Y to be $A(Y) = \mathcal{O}(Y) = \mathcal{O}_Y = k[x_1, \dots, x_n] / I(Y)$.

9/10 f irred in $k[x_1, \dots, x_n]$, $Z(f) \subseteq \mathbb{A}^n$ a variety. Called a <u>hypersurface</u>. If $n = 2$, <u>curve</u>; $n = 3$, <u>surface</u>.

<u>Ex</u>: max. ideals in $k[x_1, \dots, x_n]$ $\longleftrightarrow$ min. alg. sets in $\mathbb{A}^n$,
ie. points

$$\underline{m} = (x_1 - a_1, \dots, x_n - a_n) \longleftrightarrow P = (a_1, \dots, a_n)$$

<u>Affine Coordinate Ring</u>
If $Y = Z(\underline{a})$ is an alg. set in $\mathbb{A}^n$, define ($\underline{a}$ radical)
$A(Y) = \mathcal{O}(Y) = \mathcal{O}_Y = k[x_1, \dots, x_n] / \underline{a}$.
↑ all poly fcns → restricted to Y, get too many b/c some are same on Y, those whose difference is 0 on Y ie, those whose diff is in $\underline{a}$.

__Note__: (1) Y is a variety $\Leftrightarrow$ $\mathcal{O}_Y$ is an int. dom.
(2) $\mathcal{O}_Y$ is always a f.g. k-alg.

[set of fcns on variety completely determine the variety]

__Def__: A top sp. X is called __noetherian__ if any descending seq. of cl. subsets eventually stabilizes.
I.e. if $Y_1 \supseteq Y_2 \supseteq \cdots$ is a seq. of cl. sets, $\exists$ an i s.t. $\forall j \geq i$ $Y_j = Y_i$.
- $\mathbb{R}$ w/ usual top. not a noeth. sp.

__Ex__: $\mathbb{A}^n$ & any alg. set w/ Zariski top. are noeth.
- given a noeth. sp., any cl. subsp. also noeth.
- b/c 1-1 incl. reversing corresp. to ideals in quotient of poly. ring, which is noeth.

__Thm__: Let X be noeth. Then every closed set Y in X can be written as $Y = Y_1 \cup \cdots \cup Y_k$ w/ Y_i irred. If $Y_i \not\subseteq Y_j$, this writing is unique. The Y_i's are called the __irred components__ of Y.
__Pf__: (In Hartshorne) Peel off one irred. component at a time & take closure. Repeating yield descending chain, so stabilizes & $\therefore$ finite.

__Def__: If Y is a Noeth top sp, then $\dim Y = \sup\{n \mid \exists\ \emptyset \neq Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_n,\ Y_i \text{ irred \& cl}\}$, ie longest strictly incr. chain of irred. cl. subsets. (Y_n not nec. $= Y$)
- need irred else always ∞ (add one pt at a time)

__Ex.__: $\dim \mathbb{A}^1 = 1$ $\quad \{pt\} \subsetneq \mathbb{A}^1$

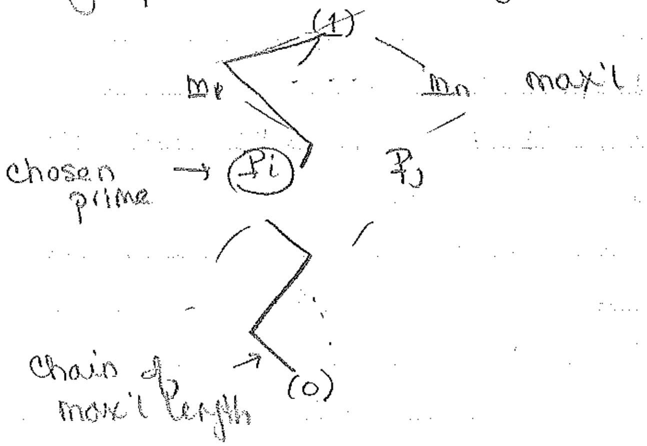
__Obs__: $\dim Y$ may be ∞ even though Y noeth (b/c can be longer & longer finite chains)

__Def__: If R is a ring, the __Krull dimension__ is
$\dim R = \sup\{n \mid \exists\ \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n,\ \mathfrak{p}_i$ prime ideals$\}$

__Prop__: If Y is an alg. set, $\dim Y = \dim \mathcal{O}_Y$.
__Pf__: Let $Y = Z(\underline{a})$, $\underline{a} = \sqrt{\underline{a}}$. Then $\{$closed, $\overset{\text{irred}}{}$ subsets of $Y\}$
$\xleftrightarrow{1\text{-}1}$ $\{$prime ideals $\mathfrak{q} \supseteq \underline{a}\}$ $\xleftrightarrow{1\text{-}1}$ $\{$prime ideals of
$k[x_1, \ldots, x_n]/\underline{a} = \mathcal{O}_Y\}$.

__Thm__: Let k = field, B a f.g. k-alg (quotient of poly ring in fin. vars / k) & an int. dom.
(a) $\dim B = \text{tr.deg}\ B_{(0)}/k$ [transcendence degree]
$\uparrow$ field of fracs.
[: $B_{(0)}$ a fin. but not nec. alg ext'n of k. How many ind. elts can you find in $B_{(0)}/k$?]
(b) For every prime $\mathfrak{p} \subseteq B$, height $\mathfrak{p} + \dim B/\mathfrak{p} = \dim B$.
ht $\mathfrak{p}$ = length of longest chain of primes cotaining $\mathfrak{p}$.

__Ex__: You can get the longest chain to go through any prime ideal, by (b).



$\}$ $\dim B/\mathfrak{m}_i = 0$ b/c field.

Cor: $\dim \mathbb{A}^n = n$

Pf: Field of fracs of $k[x_1,\ldots,x_n]$ is $k(x_1,\ldots,x_n)$, & $\text{tr.deg.}\left(k(x_1,\ldots,x_n)/k\right) = n$. [the ind. elts are $x_1,\ldots,x_n$]

Prop: If $Y \subseteq \mathbb{A}^m$ is quasi-affine (ie Y = open subset of cl. $\overset{\text{irred}}{\wedge}$ set in $\mathbb{A}^m$): then $\dim Y = \dim \overline{Y}$.

Pf: $\dim Y \leq \dim \overline{Y}$: Let $Y_0 \subset Y_1 \subset \cdots Y_n$ be a chain of cl. $\overset{\text{irred}}{}$ subsets in Y. Then $\overline{Y_0} \subset \overline{Y_1} \subset \cdots \subset \overline{Y_n}$ is a chain of cl. $\overset{\text{irred}}{}$ subsets in $\overline{Y}$

(Note: $\overline{Y_i} \cap Y = Y_i$, so incl. still strict)

& recall closure of irred. is irred. $\dim Y < \infty$ (b/c $\overline{Y}$ has fin dim). Pick a max'l length chain $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ in Y, $n = \dim Y$. Y_0 is a pt, $\overset{P \in \mathbb{A}^n}{}$ b/c chain cannot be descended down.

b/c $\mathbb{A}^m$ does & $\overline{Y}$ cl. subset & so dim is same as that of $\mathcal{O}_{\overline{Y}}$, so chain less than that of all ideals in $\mathbb{A}^m$, which is m.

Let $P \Leftrightarrow \underline{m} \subseteq k[x_1,\ldots,x_n]$ max'l.

Take $\overline{Y_0} \subset \cdots \subset \overline{Y_n}$ cl. irred in $\overline{Y}$.

Claim: This chain cannot be made longer

B/c: If I could put $\overline{Y_i} \subset Z \subset \overline{Y_{i+1}}$ w/ Z cl. & irred, then in Y, $Y_i = \overline{Y_i} \cap Y \subset Z \cap Y \subset Y_{i+1} = \overline{Y_{i+1}} \cap Y$

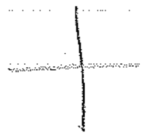
But all dense in their closure, & $\therefore$ inclusions are strict. [Intersecting w/ open set doesn't lose any info.]

$\Rightarrow$ n = ht of $\underline{m}$, since have longest chain of ideals contained in $\underline{m}$. But $\dim k[x_1,\ldots,x_n]/\underline{m} = 0$ (b/c field), so $\dim \overline{Y} = n$. $\square$

(*) needs work (*)

Prop: (Krull's Hauptidealsatz) Let A be a noeth. ring, $f \in A$ which is neither a zero-div. nor a unit. Then every minimal prime $\mathfrak{p} \ni f$ has ht 1.

i.e. every component of $Z(f)$ has codimension 1

ex: $f = xy$, in $k[x,y]$. $Z(f) =$  , two components, each of dim. 1 (b/c $\mathbb{A}^2$ dim 2)

Prop: A ring A is a ufd $\Leftrightarrow$ every ht 1 prime ideal is principal.

Cor: $Y \subseteq \mathbb{A}^n$ a variety has dim $n-1$ $\Leftrightarrow$ $Y = Z(f)$ for an irred. poly f.

Prop:

9/12 (*) If R is an int. dom s.t. $\forall \mathfrak{p}$ prime, $\text{ht } \mathfrak{p} + \dim R/\mathfrak{p} = \dim R$ then every nonextendable chain of primes has length dim R.

Pf: By induction on dim R. If $\dim R = 0$, done.

Let $0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a nonextendable chain.

$\text{ht } \mathfrak{p}_1 = 1$ (b/c can't fit anything below it)

$\Rightarrow \dim R/\mathfrak{p}_1 = \dim R - 1$

So $R/\mathfrak{p}_1$ is a ring for which we already proved the prop. we want:

$0 = \bar{\mathfrak{p}}_1 \subset \bar{\mathfrak{p}}_2 \subset \cdots \subset \bar{\mathfrak{p}}_n$ is a nonext. chain in $R/\mathfrak{p}_1$

So by ind. hyp, $n-1 = $ length of this chain $= \dim R/\mathfrak{p}_1$

$\Rightarrow n = \dim R$. $\square$

[Clears up previous pf]

Projective Space

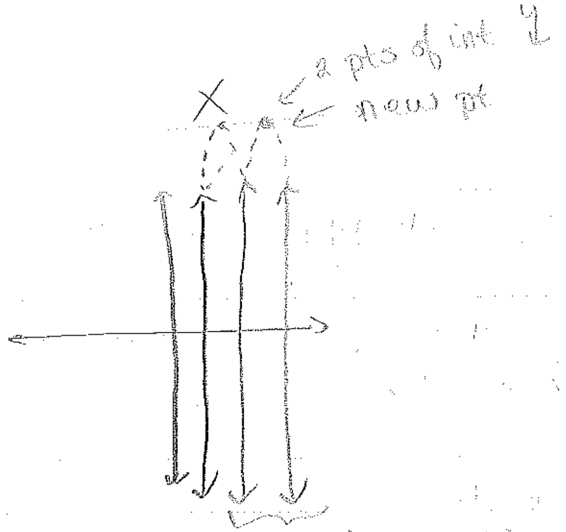
(~1600)

Problem: (1) In the plane, through every 2 pts $\exists!$ line.

(2) Two distinct lines meet in 0 or 1 pts.

Why 0 or 1? Unpleasant that it's not always 1.

Goal: Define a new space $\mathbb{P}^2$ s.t. $\mathbb{A}^2 \subseteq \mathbb{P}^2$, lines make sense, & a line restricted to $\mathbb{A}^2$ is a line in the classical sense, & axiom (1) holds, & 2 distinct lines meet in exactly one pt (i.e., no $\parallel$ lines).



need to add a pt "far away" where they intersect [i.e., like looking at railroad tracks]

- all lines that are parallel meet at same pt.

Def 1: $\mathbb{P}^2 = \mathbb{A}^2 \cup \{$one pt for each direction of $\parallel$ lines$\}$

equivalence classes of lines in $\mathbb{A}^2$ under $l_1 \sim l_2$ if $l_1 \parallel l_2$.

set of lines through origin
(or set of pts on the $\odot$ w/ antipodal pts identified)

All aded pts "pts at ∞" form a line (b/c btwn any 2 pts is a line)

→ a line in $\mathbb{P}^2$

So a line in $\mathbb{P}^2$ = usual line w/ one pt added, or the set of lines through origin in $\mathbb{A}^2$.

Want def that doesn't distinguish pts at ∞.

Def: Let V be a vect. sp / field k. Define

$\mathbb{P}V = \{$Lines in V through the origin$\}$

OR $= (V \setminus \{0\})/k^*$, ie (nonzero vectors acted upon by nonzero elts of k via scalar mult) orbits of this action

OR $= \{(0 \subset L \subset V) \mid L$ a 1-dim'l vect subsp$\}$ (rel. to flag varieties)

$\longleftrightarrow (0,1,n)$, $n = \dim V$

$\mathbb{P}^n = \mathbb{P}(\underset{\substack{\shortparallel \\ V}}{k^{n+1}})$ w/ specific choice of basis

<u>Def</u>: If $\overset{0\neq}{(x_0, x_1, \ldots, x_n)} \in V$ (Identify $V = k^{n+1}$ by picking a basis) denote its image in $\mathbb{P}V$ by $[x_0 : x_1 : \cdots : x_n]$

<u>E.g.</u>: $[1:2] = [2:4]$ in $\mathbb{P}^1$ (since pts in $\mathbb{P}$ are equiv. classes)

$[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ $\forall \lambda \neq 0, \lambda \in k$.

- So, a pt in V yields a ! pt in $\mathbb{P}V$, but a pt in $\mathbb{P}V$ does not give a ! pt in V.

- What is $\mathbb{P}^0$? $[x_0] = [1]$ b/c $[x_0] = [1/x_0 \cdot x_0] = [1]$

  $\mathbb{P}^0 = \{pt\}$.

- what is $\mathbb{P}^1$? $[x_0 : x_1] = P$ if $x_1 \neq 0$, $[\frac{x_0}{x_1} : 1] = P$

  If $x_1 = 0$, $x_0 \neq 0$, $P = [1:0]$

  $\Rightarrow \mathbb{P}^1 = \mathbb{A}^1 \sqcup \{[1:0]\}$

  $\uparrow$ i.e., pt at ∞.

- What is $\mathbb{P}^2$?

  $\mathbb{P}^2 = \underset{\substack{\uparrow \\ [x_0 : x_1 : 1]}}{\mathbb{A}^2} \sqcup \mathbb{P}^1 = \mathbb{A}^2 \amalg \mathbb{A}^1 \amalg \mathbb{A}^0$

  (divide by last coord)

  $\nwarrow$ if last coord $= 0$, have case of $\mathbb{P}^1$ b/c $[x_0 : x_1 : 0]/\sim$

- $\mathbb{P}^n = \coprod_{0 \leq k \leq n} \mathbb{A}^k$.

<u>Claim</u>: $\mathbb{RP}^n$, $\mathbb{CP}^n$ (ie $\mathbb{P}^n / k = \mathbb{R}$, $\mathbb{CP}^n = \mathbb{P}^n / k = \mathbb{C}$) are compact.

[projective sp. is compactification of $\mathbb{R}^n$, $\mathbb{C}^n$]

they are quotients of S^n for $\mathbb{R}^n$

$\hookrightarrow$ by identifying antipodal pts

$\mathbb{RP}^n \cong S^n/\text{antipode}$.

<u>Def</u>:

If $W \subset V$ is a linear subsp of dim $k+1$, then the image under the proj. map $V \setminus \{0\} \to \mathbb{P}V$ is what we call a <u>linear k-space</u> in $\mathbb{P}V$.

<u>Ex</u>: Check that (a) through every 2 pts of $\mathbb{P}^2$ $\exists !$ line.

(b) Every 2 lines meet at a pt.

a line is a pt in $\mathbb{P}^2$)

What are functions on $\mathbb{P}^n$?

- only constant fcns (b/c only const. entire fcns on cpt)
- If f is a homogeneous poly in $k[x_0, \dots, x_n]$, it makes sense to ask where is $f = 0$ on $\mathbb{P}^n$. (doesn't make sense to ask what is $f(\bar{x})$ $\bar{x} \in \mathbb{P}^n$ b/c can rescale pt & get different answer) If f is homogeneous of deg d (all mon's have deg $= d$), then $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$ & both $= 0$ at same time since $\lambda \neq 0$.

Def: If $T \subseteq k[x_0, \dots, x_n]$ is a collection of homogeneous polys, then $Z(T) = \{P \in \mathbb{P}^n \mid f(P) = 0 \ \forall f \in T\}$. Such a set is called an algebraic set in $\mathbb{P}^n$.

Thm: The alg. sets form the closed sets of a topology on $\mathbb{P}^n$, called the Zariski topology

Fact: $\mathbb{P}^n$ is a noetherian top. sp., so all notions of dim, irred, etc. are same.

Def: An algebraic set in $\mathbb{P}^n$ is called a proj. variety if it is irred. A quasiprojective variety is an open set of a proj. var.

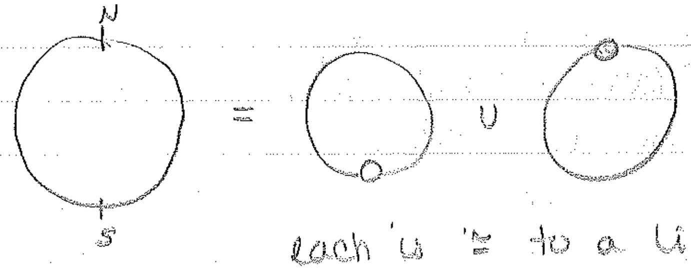
Ex: $Z(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$ (note: always one more var. than size of sp.)

- If restrict to $\mathbb{A}^2$, can declare $z = 1$, so is a circle
- Pts at ∞ are when $z = 0$. Solving $x^2 + y^2 = 0$ 2 sol'ns b/c quadrics & lines int at 2 pts here $(1, i), (1, -i)$, so $\odot$ w/ 2 pts at ∞.

Def: Let $U_i \subseteq \mathbb{P}^n$ be the locus of pts where $x_i \neq 0$.
$U_i = \mathbb{P}^n \setminus Z(x_i)$. U_i open b/c $Z(x_i)$ cl.
Obviously, $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ ...

Thm: $U_i \underset{\text{homeomorphic}}{\cong} \mathbb{A}^n$ wrt the Zariski top

Similar to:



each is $\cong$ to a line,
so $\odot \simeq$ ⤢ $\cup$ ⤢ .

Notation: • Let $f \in k[t_1, \dots, t_n]$ be an arbitrary poly. & pick a var x_i. The homogenization of f wrt x_i is $\overline{f}(x_0, \dots, x_n) = x_i^{\deg f} \cdot f\left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$
(or $\left(\frac{x_0}{x_1}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x^n}{x_i}\right)$)

Ex: If $f = t_1^2 + t_2^2 - 1$. Homog wrt x:
$\overline{f} = t_1^2 + t_2^2 + x^2$ b/c
$\overline{f}(t_1, t_2, x) = x^2 \cdot f(t_1/x, t_2/x) = x^2\left(\frac{t_1^2}{x^2} + \frac{t_2^2}{x^2} - 1\right) = t_1^2 + t_2^2 - x^2$
$\overline{f}$ hom. & $f(t_1, t_2) = \overline{f}(t_1, t_2, 1)$.

9/17 <u>From HW</u>: Look at $X=\{(t^3,t^4,t^5) \mid t\in k\} \subseteq \mathbb{A}^3$. Let $I(X)=\mathfrak{p}$ be the corresp ideal in $k[x,y,z]$. Show $\mathfrak{p}$ is a prime of ht 2 not gen'd by 2 poly's.

* Why (geom.) not 2 gen: find one hypersurface that contains curve. & for every second hypersurface that contains the curve intersects the first in the given curve & a second curve. Then need a third curve to isolate the curve we want. *

$f: \mathbb{A}^1 \to \mathbb{A}^3$

$t \mapsto (t^3,t^4,t^5)$ cts wrt Zariski top.

$\mathbb{A}^1$ irred $\Rightarrow$ $f(\mathbb{A}^1)$ irred. (generally true for cts maps)

[so its closure corresp. to prime ideal]

Why 2 gen's:

Incorrect but intuitive: A general hyperplane intersects the curve 5 times. If 2 hypersurf's cut out the curve, of deg d_1 & d_2, then their int. has deg $= d_1 \cdot d_2 = 5$ (in this case), so one must have deg 1, so the curve lies in a plane ↯ (only works in hyperspace)

Correct but clodgy: let f be a poly in x,y,z s.t. f vanishes on X, $f = \sum a_{ijk} x^i y^j z^k$, $i,j,k \in \mathbb{N}$

$\sum_{3i+4j+5k=d} a_{ijk} = 0$

$$\left.\begin{array}{ll} a_{0,0,0} = 0 & (d=0) \\ a_{1,0,0} = 0 & (d=3) \\ a_{0,1,0} = 0 & (d=4) \\ a_{0,0,1} = 0 & (d=5) \end{array}\right\} \text{no const \& no lin. terms}$$

$a_{2,0,0} = 0$ (no x^2)

$a_{1,1,0} = 0$ (no xy)

$$\left.\begin{array}{ll} d=8: & y^2 = xz \\ d=9: & x^3 = yz \\ d=10: & x^2y = z^2 \end{array}\right\} \begin{array}{l} \text{in our ideal b/c } (t^4)^2 = (t^3)(t^5) \\ \text{—''—} \\ \text{—''—} \end{array}$$

Cannot get 3rd from any 2. If choose 1st 2, cannot get a z^2. Sim for others [lin. alg. arg.]

$\to$ deg 2 parts must span a 3 dim vect. sp.

Let $\phi_i : \mathbb{A}^n \longrightarrow \mathbb{P}^n$ for $i = 0, 1, \dots, n$ be def. as

$(y_1, \dots, y_n) \longmapsto [y_1 : y_2 : \cdots : y_{i-1} : 1 : y_{i+1} : \cdots : y_n]$

Conversely, let $U_i \subseteq \mathbb{P}^n$ be the open set

$U_i = \{(x_i \neq 0)\} \subseteq \mathbb{P}^n$ (complement of $Z(x_i)$, so open)

[$\mathbb{A}^n$ lands in U_i under ϕ_i]

Map $U_i \to \mathbb{A}^n$ by

$[x_0 : x_1 : \cdots : x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{\hat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}\right)$

<u>Claim</u>: These maps are inverse homeomorphisms.

- clearly inverses [left until def. morphisms

<u>Cor</u>: $\mathbb{P}^n$ is covered by $n+1$ copies of $\mathbb{A}^n$.

<u>Cor</u>: (1) $\dim \mathbb{P}^n = n$

(2) $\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \cdots \amalg \mathbb{A}^0$ "stratification"

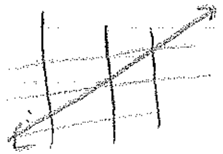
(2) ⇒ (1) by induction: $\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{P}^{n-1}$

Let Z be closed in $\mathbb{P}^n$. $Z = \underbrace{\overline{(Z \cap \mathbb{A}^n)}}_{\dim \leq n} \cup \underbrace{(Z \cap \mathbb{P}^{n-1})}_{\dim \leq n-1}$

$\Rightarrow \dim Z = n$

ex → ($\dim A \cup B = \max(\dim A, \dim B)$) [b/c closures can't incr. dim]

Products of Projective Spaces

- As a set, $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$, but the top. is <u>not</u> the prod. top. In prod. top, only get  , b/c cl. sets in $\mathbb{A}^1$ are fin. sets of pts.

  $Z(x+y)$ is not cl. in prod. top, but is in Zariski prod.

  → actually should be taking tensor prod. of coord ring of each.

- $\mathbb{P}^n \times \mathbb{P}^m \neq \mathbb{P}^{n+m}$, even as a set. (check # of coords) (∃ rescale coords ind on left but not right)

We need a closed embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^r$, called the <u>Segre embedding</u>:

$([x_0 : x_1 : \cdots : x_n], [y_0 : y_1 : \cdots : y_m]) \longmapsto [x_0y_0 : x_0y_1 : \cdots : x_0y_m : x_1y_0 : \cdots : x_ny_m]$

(all possible products)

$\Rightarrow r = (n+1)(m+1) - 1$ (one less than # of coords) $= nm+n+m$

- map cannot be surj.
- Well-def: if rescale 1st coord of preimage, then whole image gets rescaled. Same for 2nd coord.
- Inj: given pt in $\mathbb{P}^r$, $x_iy_j \neq 0$. look at x_iy's, get 2nd pt. look at xy_j's & get 1st pt.
- Image is a closed set.

ex

<u>Important case</u>: $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$([x:y], [u:v]) \longmapsto [xu : xv : yu : yv]$

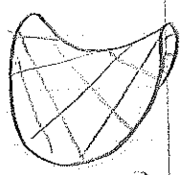
$[a : b : c : d]$

<u>Note</u>: ~~ac = bd~~ ad = bc ← quadric surface, Q

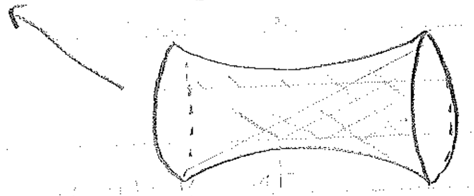
$\mathrm{Im}\,\phi = Z(~~ac-bd~~\ ad-bc) \subseteq \mathbb{P}^3_{a:b:c:d}$

"quadric surface is ruled in 2 different ways"

- there are 2 families of lines, L's, L′'s, any 2 L's don't meet, any 2 L′'s don't meet, any $L \cap L' =$ pt.



(picture pg. 14)



- The lines are $\mathrm{Im}(\mathbb{P}^1 \times \{pt\})$ & $\mathrm{Im}(\{pt\} \times \mathbb{P}^1)$
- fix a & b, then solve eqn & get a line.

- ~~ac = bd~~ ad = bc $\Leftrightarrow a^2+b^2+c^2+d^2=0$ (full rank ⇒ smooth)

[symm. matrices are diagonalizable ⇔ all quadrics can be written w/ only square terms]

$$\begin{array}{c|c} & x_1 \cdots x_n \\ \hline x_1 \\ \vdots & \begin{pmatrix} \text{coeffs} \end{pmatrix} \\ x_n \end{array}$$

. Full rank ⇒ need all square terms (i.e. no zero entries on diagonal)

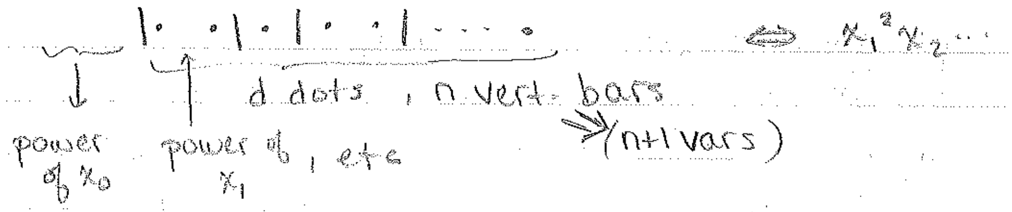
The Veronese Embedding (the d-tuple embedding)

$\phi : \mathbb{P}^n \hookrightarrow \mathbb{P}^r_{y_0 \cdots y_I}$

$[x_0 : \cdots : x_n] \longmapsto [\underbrace{x_0^d : x_0^{d-1}x_1 : \cdots : x_n^d}_{\text{all deg } d \text{ monomials in } x_0, \dots, x_n}]$

- well def b/c scale preimage by k scale image by k^d

$r = \binom{n+d}{d} - 1$



n+d slots, d dots ⇒ $\binom{n+d}{d}$ monomials

- If $Y = Z(f)$, f hom. of deg d in $\mathbb{P}^n$, $f = \sum a_I x^I$ $(x^I = x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n})$

  $Y = (\bar{f} = 0) \cap (\operatorname{Im} \phi)$

  $\downarrow$ hyperplane

  $\Downarrow$ Linear hyperplane $\bar{f} = \sum a_I y_I$

Ex: ① 3-uple embedding of $\mathbb{P}^1$

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$

$[s:t] \longmapsto [s^3 : s^2t : st^2 : t^3]$

(setting $t = 1$, $s \mapsto (s^3, s^2, s, 1)$, the twisted cubic)

② 2-tuple embedding of $\mathbb{P}^2$

$\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

$[x:y:z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2]$ the Veronese surface

* all quadrics in $\mathbb{P}^2$ arise by int. of the surface in $\mathbb{P}^5$ & a hyperplane in $\mathbb{P}^5$ (there is a $\mathbb{P}^5$'s - worth of hyperplanes ⇒ $\mathbb{P}^5$'s worth of quadrics in $\mathbb{P}^2$)

9/19 Def: $I \subseteq K[x_0,\dots,x_n] := S$ ideal. Then
$Z(I) = \{[x_0:\cdots:x_n] \in \mathbb{P}^n \mid f(x_0,\dots,x_n) = 0 \ \forall$ homogeneous $f \in I\}$

Def: A graded ring S is a ring together with a decomposition $S = \bigoplus_{d \geq} S_d$ (we say $f \in S_d$ is a homogeneous elt of deg d), s.t. $S_d \cdot S_{d'} \subseteq S_{d+d'}$

Ex: $S = k[x_0,\dots,x_n]$ is a graded ring w/
$S_d = \{f \in S \mid f$ homog. of deg $d\}$.

Note: $\underline{m} = S_+ = \bigoplus_{d>0} S_d$ is an ideal in S called the irrelevant ideal. (In poly. ring, these are polys w/ 0 const. term) S_+ a max. ideal.

Def: An ideal $I \subseteq S$ (S a graded ring) is homogeneous if $I = \bigoplus_{d \geq 0} (I \cap S_d)$.

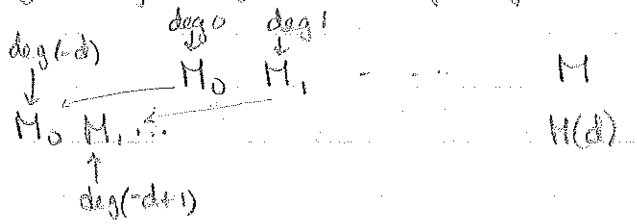
Lemma: I is homog. $\Leftrightarrow$ we can find a set of gen's of I all of which are homog.

Ex: (x,y^2) is homogeneous. So not all gen's must
$\quad \shortparallel$
(x,y^2+x) be homog.

Lemma: • If S is graded & I homog, then S/I graded:
$(S/I)_d = \pi(S_d)$, $\pi : S \to S/I$.
($\uparrow$ will not intersect b/c I homog.)
• If $f: S \to S'$ is a map of graded rings (i.e. ring map & $f(S_d) \subseteq f(S'_d)$). Then $\ker f$ is a homog. ideal.

Def: Let S be a graded ring. A graded module $/S$ is a module M & a decomp $M = \bigoplus_{d \in \mathbb{Z}} M_d$ s.t.
$S_d \circ M_{d'} \subseteq M_{d+d'}$.

Twisting Operation: If M is a graded S-mod, define $M(d) \cong M$ as an S-mod. but $(M(d))_k = M_{d+k}$.
(grading shift always goes to the left)



Serre's Module (Line Bundle): $S(1)$

* If M f.g., there is a minimum deg. – the d's cannot go arly far to the left.

* If M, N are graded S-mods, then $M \otimes_S N$ is also graded: $(M \otimes_S N)_d = \bigoplus_{k+l=d} M_k \otimes_S N_l$. (i.e. $\otimes$ comm. w/ $\oplus$)

ex * $M(1) = M \otimes S(1)$. , $M \otimes [N(d)] \cong (M \otimes N)(d)$

in book {
* $+, \cdot, \cap, \sqrt{\ }$ of homog. ideal is homog.
* to check $\overset{\text{homog.}}{\wedge}$ ideal prime, check only for prod. of homog. elts.

If $Z \subseteq \mathbb{P}^n$ is a closed subset, define
$I(Z) = \langle f \in S \mid f$ homog. & f vanishes along $Z \rangle$.
By def, $I(Z)$ is a homog. ideal in f (b/c gen by homog. elts.

Prop: There is a 1-1, inclusion reversing correspondence btwn radical $\overset{\text{homog.}}{\wedge}$ ideals I, s.t. $I \neq S_+$, and alg sets in $\mathbb{P}^n$. Primes $\longleftrightarrow$ varieties (irred. alg sets).

ex
- Problem w/ S_+: $Z(S_+) = \phi$ (b/c only zero at 0, but $0 \notin \mathbb{P}^n$), & $I(\phi) = S$, so operations I & Z being inverses will work as long as get at least 1 pt. in zero locus.

Pf goes thru usual pf for affine sp.
any ~~nonempty~~ ? open set has covering by affine sets, take ideal of that, the homogenize that.

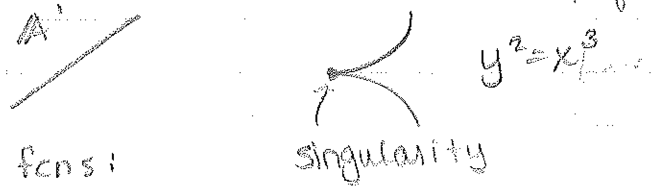
Maps

Ex: The affine line $\mathbb{A}^1$ & the curve $C: (y^2 = x^3) \subseteq \mathbb{A}^2$ are homeomorphic.

$$\begin{array}{l} \mathbb{A}^1 \longrightarrow C \\ t \longmapsto (t^2, t^3) \end{array} \quad \text{inj., cts, bij.}$$

- given $(x,y) \neq 0$, $y/x = t$ & $(0,0) \to 0 = t$.

But map backwards not poly, it's rational. - shouldn't be map of varieties.



} shouldn't be ISO. as varieties

ring of fcns: $k[x]$

ring of fcns: $k[x]/(x^3 - y^2)$

[one int. cl. & one not in its field of fracs]

Def: Let $X \subseteq \mathbb{A}^n$ be a quasi-affine variety & let $U \subseteq X$ be an open set. A fcn $f: U \to k$ is said to be regular if it satisfies:
$\forall x \in U$, $\exists V \subseteq U$ a nbhd of x & $g, h \in k[x_1, \dots, x_n]$ s.t. $h(y) \neq 0 \ \forall y \in V$ & $f(y) = \frac{g(y)}{h(y)} \ \forall y \in V$.

* A regular fcn must locally look like the ratio of 2 polys.

(tricky) ex: Regular fcns on $\mathbb{A}^2 \setminus \{(0,0)\}$ are $k[x,y]$.
(sim. to Hartzog's Thm in complex anal.)

Prop: A regular fcn f on X is cts when regarded as a map $X \to \mathbb{A}^1$. ↑ i.e., a global regular fcn.
↑ identify elts of k w/ $\mathbb{A}^1$, so has top.

Pf: WTS $f^{-1}(\text{cl.}) = \text{cl.}$

- Enough to check $\overset{Y}{f^{-1}(a)}$ is cl. in X $\forall a \in k$, since cl $\subseteq \mathbb{A}^1$ is finite set of pts & preimage of fin set is union of preimage of each pt.
- Y is cl $\Leftrightarrow$ $Y \cap U_i$ is cl. in U_i for a cover $\{U_i\}_{i \in I}$ of X by open sets (b/c cl. is a local prop)
  $\Leftrightarrow \forall x \in Y, \exists U_x \subseteq X$ open s.t. $x \in U_x$ & $U_x \cap Y$ cl. in U_x.
- Since f regular, $\forall x \in Y$ $\exists U_x$ s.t. $f = g/h$ at all pts of U_x. $Y \cap U_x = \{y \in U_x \mid f(y) = a\}$
  $= \{y \in U_x \mid g(y)/h(y) = a\} = \{y \in U_x \mid g(y) - ah(y) = 0\}$
  $= \underbrace{U_x \cap \underbrace{Z(g - ah)}_{\text{cl. in } \mathbb{A}^n}}_{\text{cl. in } U_x}$

□

Def: Let X, Y be quasi-affine varieties. A cts map $f: X \to Y$ is said to be *regular* if whenever $U \subseteq Y$ open & $g: U \to k$ is regular, then $g \circ f$ is regular on $f^{-1}(U)$.

Note: If $Y \subseteq \mathbb{A}^n$, there are regular fcns $x_1, \dots, x_n : Y \to k$, namely the coord fcns (proj onto ith coord). It is enough to check $g \circ f$ regular for $g = x_i$, $i \in \{1, \dots, n\}$.

Ex: $\mathbb{A}^1 \to \mathbb{A}^2$ reg. b/c comp w/ x_1 & x_2 is
$t \longmapsto (t^2, t^3)$ t^2 or t^3, resp, which are regular b/c polys (& so rat'l)

9/24 Def: Let X be a quasi-proj. var, $X \subseteq \mathbb{P}^n$, $U \subseteq X$ open, $f: U \to k$ is regular if $\forall x \in U$, $\exists\ x \in V \subseteq U$ open & homog. polys $g, h \in k[x_0, \dots, x_n]$ of the same deg s.t. $f(y) = \frac{g(y)}{h(y)}$ $\forall y \in V$. & $h(y) \neq 0$ $\forall y \in V$.

Consider the fcn x_0/x_1, def $\forall x$ s.t. $x_1 \neq 0$. Well-def b/c rescaling coords doesn't change fcn.

Also $\frac{x^2 - 2x_1x_2}{x_3^2 + x_4^2}$. $\Rightarrow$ need ratio of 2 homog. polys of same deg for this fcn to be well-def.

Note: $g(y)$ & $h(y)$ don't make sense alone, but their ratio does.

Note: same as for $\mathbb{A}^n$, w/ homog. & of same deg.

Prop: A reg. fcn on a qproj var is also cts.

Def: A variety is any (proj, affine, qproj, qaffine) variety.

Def: Let X & Y be varieties, $f: X \to Y$ cts map. Then f is regular (or a map/morphism of varieties) if $\forall U \subseteq Y$ open, $\forall g: U \to k$ regular, then $g \circ f : f^{-1}(U) \to k$ is also regular.

Def: The category Var has objects varieties & morphisms maps of varieties.
$\Rightarrow$ notion of isomorphism of varieties, ie maps (regular) in both directions s.t. composites are identities

Ex: $\mathbb{A}^1 \cong$ twisted cubic

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[d, equal] \arrow[r, phantom, "\cong"] & \text{twisted cubic} \arrow[d, equal] \\
\{(t) \in \mathbb{A}^1\} \arrow[r, shift left] & \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\} \arrow[l, shift left] \\
t \arrow[r, mapsto] & (t, t^2, t^3) \\
x & (x, y, z) \arrow[l, mapsto]
\end{tikzcd}$$

$*$ Check both maps regular

*

Ex: $U_i \subseteq \mathbb{P}^n$ is $\cong$ to $\mathbb{A}^n$

-check the maps we def. are reg.:

$$\mathbb{A}^n \longrightarrow \mathbb{P}^n$$

$$(y_1,\dots,y_n) \longmapsto [y_1;\dots;\overset{i}{1};\dots;y_n]$$

$$\mathbb{A}^n \longleftarrow U_i$$

$$\left[\tfrac{x_0}{x_i};\dots;\widehat{\tfrac{x_i}{x_i}};\dots;\tfrac{x_n}{x_i}\right] \longleftarrow\!\shortmid [x_0;\dots;x_n]$$

Ex: $\mathbb{A}^1 \cong \mathbb{P}^1 \setminus \{pt\}$, as varieties

Ex: $\mathbb{A}^1 \setminus \{0\} \cong \{xy=1\} \subseteq \mathbb{A}^2$

$$x \longmapsto (x, 1/x)$$

$$x \longleftarrow\!\shortmid (x,y)$$

$\mathbb{A}^1 \setminus \{0\}$ q affine var.

$\{xy=1\}$ closed (affine var.)

Ex: $\mathbb{A}^2 \setminus \{0\}$ is not affine (ie not $\cong$ to any affine var)

Cor: Let X be a var, $f,g : X \to k$ regular. Assume $\exists$ $U \subseteq X$ open, $U \neq \emptyset$, s.t. $f|_U = g|_U$. Then $f=g$.

Pf: Let $Z = (f-g)^{-1}(\{0\})$. $f-g$ regular.

Z is closed. $U \subseteq Z$ (b/c $f-g=0$ on U).

X irred $\Rightarrow$ U dense $\Rightarrow$ $Z=X$

$\uparrow$ b/c then Z dense & Z cl $\Rightarrow Z=X$

Def: Let X be a var, $P \in X$. A germ of a fcn is a pair (U,f) s.t. $U \ni P$ open & f reg $: U \to k$; up to the equiv. rel. $(U,f) \sim (V,g) \Leftrightarrow \exists$ W open, $P \in W$, & $W \subseteq U$, $W \subseteq V$ s.t. $f|_W = g|_W$.

- germs can be $+, \times, -, \div$ (under $\neq 0$ assump.)
- germs make sense for any sp. w/ notion of "good fcn"

Def: Let X be a var, $P \in X$. The local ring of X at P, $\mathcal{O}_{X,P} = \{$germs of regular fcns at $P\}$

- local in sense of only one max'l ideal.

<u>Def</u>: Let X be a variety. A rational fcn on X is a pair (U,f), U open $\overset{\neq\emptyset}{\wedge}$ $f: U \to k$ reg up to equivalence rel. $(U,f) \sim (V,g) \iff \exists\, W$ open, $\neq \emptyset$, $W \subseteq U \cap V$, s.t. $f|_W = g|_W$.

**
- nothing but a fcn defined somewhere (some open set) up to equiv that if shrink open set, get same fcn.

- Rat'l fcns on X form a field, $K(X)$:

Add'n:

$$\underset{(U\cap V,\, f|_{U\cap V})}{\underset{\wr}{(U,f)}} + \underset{(U\cap V,\, g|_{U\cap V})}{\underset{\wr}{(V,g)}} = (U \cap V,\ f|_{U\cap V} + g|_{U\cap V})$$

Mult, subtr ok.

$\neq\emptyset$ b/c $Z(f)$ not all of U & cl. in U

Let $(U,f) \in K(X)$ s.t. $f \not\equiv 0$. Then $(U \setminus Z(f), f|_{U\setminus Z(f)}) \sim (U,f)$ and is nowhere 0, so can take open inv:

$(U \setminus Z(f), \frac{1}{f}|_{U\setminus Z(f)})$ is an inv.

$\Rightarrow K(X)$ a field.

<u>Claim</u>: If $X \simeq Y$, then $K(X) \simeq K(Y)$ & $\mathcal{O}_{X,P} \simeq \mathcal{O}_{Y,f(P)}$.

obvious. (see next pg for exact invariance statement)

<u>Ex</u>: $X = \mathbb{A}^1$, $P = 0$.

<u>Germs</u>: $(\mathbb{A}^1, x^2)$ a germ, $\underset{\uparrow \text{open set containing } 0}{(\mathbb{A}^1 \setminus \{3\}, \frac{1}{x-3})}$ $\big\}^{\text{germ}} \Rightarrow \frac{1}{x-3} \in \mathcal{O}_{X,P}$

$(\mathbb{A}^1 \setminus Z(g), \frac{f}{g})$ $g(0) \neq 0$ a germ.

$\mathcal{O}_{X,P} = \underbrace{k[x]}_{\text{taken ratios of polys s.t. } g \notin (x).}{}_{(x)}$ ↖ prime → max'l corresp to $P=0$.

<u>Rat'l Fcns</u>: $\frac{1}{x-3}$, $\frac{1}{x}$, $\frac{5x-7}{15x^2-3x+5}$, any quotient of polys as long as $g \not\equiv 0$.

(all maps that are def. somewhere)

$K(X) = k(x)$

Prop: If $\phi: X \xrightarrow{\sim} Y$, then ϕ induces an iso $\phi_p : \mathcal{O}_{Y,\phi(p)} \xrightarrow{\sim} \mathcal{O}_{X,P}$ $\forall P \in X$, and $K(Y) \xrightarrow[\sim]{\phi} K(X)$

- the equiv rel in rat'l fcns comes from this attempt: $(U,f) \sim (V, f|_V)$ $\forall V \subseteq U$. But this isn't equiv. rel, so extend to an equiv. rel & get rel. in def.

Def: If $U \subseteq X$ is open, $\mathcal{O}(U)$ = ring of reg. fcns on U.
In particular, $\mathcal{O}(X)$ = ring of global reg. fcns.

Thm: Let $X \subseteq \mathbb{A}^n$ be an affine variety w/ affine coord. ring $A(X) = k[x_1, \dots, x_n]/I(X)$. Then

(1) $\mathcal{O}(X) \cong A(X)$

(2) There is a 1-1 corresp btwn pts $P \in X$ of X & max'l ideals $\underline{m}_P \subseteq A(X)$. Under this corresp, $\mathcal{O}_{X,P} \cong A(X)_{\underline{m}_P}$, and so $\dim X = \dim \mathcal{O}_{X,P}$. ⟶ [note localizing at max ideal does not change dim & $\dim X = \dim A(X)$

(3) $K(X) \cong A(X)_{(0)}$ (i.e. field of fracs)
$\Rightarrow \dim X = \text{tr.deg.}\ K(X)/k$.

[(2) says can get dim of variety from any local ring]

Pf: $k[x_1, \dots, x_n] \longrightarrow \mathcal{O}(X)$ ⟶ or ($\bar{f}: X \to k$ s.t. $\bar{f}(x) = f(x)$ $\forall x$)
sum of monos. → $f \longmapsto f$ ← map (i.e. polys are reg fcn at any pt in X)

$\ker = I(X) \Rightarrow$ get injective map $A(X) \longrightarrow \mathcal{O}(X)$

$\mathcal{O}(X) \hookrightarrow \mathcal{O}_{X,P} \hookrightarrow K(X)$.

reg fcn def. everywhere, so def. in nbhd of P & inj b/c if 2 fcns agree on nbhd of P, they agree everywhere

def in nbhd of $P \Rightarrow$ def. somewhere & inj b/c agree on an open set containing P

So think of $\mathcal{O}(X)$ & $\mathcal{O}_{X,P}$ as subrings of $K(X)$

(2): The 1-1 corresp. we've seen (pts ⟺ max'l ideals)
$\underline{m}_P = \{ f \in A(X) \mid \bar{f}(P) = 0 \}$

In $\mathbb{A}^n$, we had $P=(a_1,\dots,a_n)\in\mathbb{A}^n$

$$\updownarrow$$

$$\underline{m}_P=(x_1-a_1,\dots,x_n-a_n)$$
$$=\{f\in k[x_1,\dots,x_n] \mid f(a_1,\dots,a_n)=0\}$$

[proven for affine sp, then proven for any affine var.]

$A(X)_{\underline{m}_P} \xrightarrow{\psi} \mathcal{O}_{X,P}$

$\frac{f}{g} \longmapsto (X\setminus Z(g), \frac{f}{g})$

$f\in A(X)$

$g\in A(X)\setminus\underline{m}_P$

$\uparrow$ open, contains P b/c $g\notin\underline{m}_P$, since $\underline{m}_P=\{g \mid g(P)=0\}$

$\Updownarrow$

$g(P)\neq 0 \Leftrightarrow P\notin Z(g)$

$\frac{f}{g}$ regular [b/c locally can be written as f/g]

Inj: b/c if 2 reg fcns agree on open, then everywhere $\hookrightarrow$ already done.

$\frac{f}{g}=\frac{h}{k}$, then $f/g=h/k$ on open sets $\Rightarrow$ $f/g-h/k=0$ on int of open sets $\Rightarrow$ done.

Surj: If h reg at P, then $\exists$ some (smaller) open set s.t. $h=f/g$ $\Rightarrow$ $h\in$ Im of map :

Let (U,h) be a germ around P. Then $\exists V\subseteq U$ w/ $P\in V$ s.t. $h(y)=f(y)/g(y)$, $\forall y\in V$ for some $f,g\in A(X)$, $g(P)\neq 0$

$\Rightarrow (U,h)=(V,f/g)=\psi(f/g)$.

$\Rightarrow \psi$ an iso.

9/26 (c) $\exists$ map $A(X)_{(0)} \xrightarrow{\psi} K(X)$

$f/g,\ g\neq 0 \longmapsto (X\setminus Z(g), f/g)$

$\uparrow$ open set, $\neq\emptyset$

& $\exists$ map $K(X)\longrightarrow A(X)_{(0)}$

$(U, f/g) \longmapsto f/g$

$\uparrow$ if reg, can be written as f/g for some U

ψ inj for same reason - if agree on open set, agree everywhere!

(a) $A(X) \subseteq \mathcal{O}(X) \subseteq \underset{\text{inside } K(X)}{\bigcap_{P \in X}} \mathcal{O}_{X,P} = \underset{\text{inside } A(X)_{(0)}}{\bigcap_{\underline{m} \subseteq A(X) \text{ max'l}}} A(X)_{\underline{m}} = A(X)$

$\uparrow$ fact about int. dom's

$\square$

<u>Thm</u>: $X \subseteq \mathbb{P}^n$ proj. Let $S = S(X) =$ proj. coord ring

$= k[x_0, \dots, x_n]/\underset{\sim}{I(X)}$

$= \langle f \mid f \text{ homog } \& \ f(X) = 0 \rangle$

(1) $\mathcal{O}(X) \cong k$ [i.e. only const. fcns are reg. everywhere]

(2) To a pt $P \in X$, assoc. $\underline{m}_P \subseteq S(X)$, $\underline{m}_P = \langle f \in S(X) \mid f(P) = 0 \rangle$

$\mathcal{O}_{X,P} = S(X)_{(\underline{m}_P)}$ (actually a 1-1 corresp)

<u>Notation</u>: If S is a graded ring, $\mathfrak{p} \subseteq S$ prime & homog,

$S_{(\mathfrak{p})} := \{ \frac{f}{g} \in S_{\mathfrak{p}} \mid \deg f = \deg g \}$

$= (S_{\mathfrak{p}})^0 =$ deg 0 elts of $S_{\mathfrak{p}}$.

If $f \in S$ is homog, then

$S_{(f)} := \{ \frac{g}{f^n} \in S_f \mid \deg g = \deg(f^n) \} \Rightarrow$ deg 0 part

$\uparrow$ careful, in comm. alg meant localize at prime ideal gen by f (if possible). Never means this here.

(3) $K(X) = S(X)_{((0))} \leftarrow$ deg 0 part of $S(X)_{(0)} =$ field of fracs.

<u>Pf</u>: (2): Let P be in U_i for some i. $\mathcal{O}_{X,P} = \mathcal{O}_{X_i,P}$,

$X_i = X \cap U_i$. X_i is affine, b/c $X_i \subseteq U_i \cong \mathbb{A}^n$.

So $\mathcal{O}_{X_i,P} = A(X_i)_{\underline{m}'_P}$, $\underline{m}'_P =$ ideal of $A(X_i)$ corresp. to P.

$\mathcal{O}(U_i)$ [$= k[x_1, \dots, x_n]$ b/c $\cong \mathbb{A}^n$]

$\cong k[x_0, \dots, x_n]_{(x_i)}$ [from HW]

$I(X_i) \longleftrightarrow I(X) \cdot k[x_0, \dots, x_n]_{(x_i)}$

localization commute w/ quotients

$\Rightarrow A(X_i) = k[x_0, \dots, x_n]_{(x_i)} / I(X) \cdot k[x_0, \dots, x_n]_{(x_i)} \overset{\swarrow}{=} S(X)_{(x_i)}$

($\uparrow$ b/c should be $\mathcal{O}(U_i)/I(X_i)$)

$\Rightarrow \mathcal{O}_{X,P} = \left((S(X)_{x_i})_{\underline{m}'_P} \right)^0 = \left((S(X)_{\underline{m}_P})_{x_i} \right)^0$

But $x_i \notin \underline{m}_P$, i.e. x_i does not vanish at P, so x_i inverted in localizing at $\underline{m}_P$

$\Rightarrow \quad = (S(X)_{\underline{m}_P})^0 = S(X)_{(\underline{m}_P)}$

(3): Essentially same proof.
$K(X_i)$ = field of fracs of $A(X_i)$, so need to take f. of fracs of $S(X)$ (b/c $A(X_i) = S(X)$).

(1): Intuitively, a global regular fcn will be of form f/g w/ $\deg f = \deg g$. If $\deg g \geq 1$, then $Z(g) \neq \phi \Rightarrow f/g$ not defined at $Z(g)$.
$\Rightarrow \deg g$ must be 0, so $\deg f = 0$, as well
$\Rightarrow f/g$ = constant.

- Why not x on $\mathbb{P}^1$? B/c x takes ∞ at ∞.
True of all poly's on affine sp - blow up at ∞.

Problem: Know only locally $= f/g$; can be written so that won't blow up - i.e. is an inessential singularity.

Pf: Pick $f \in \mathcal{O}(X)$ (i.e. regular everywhere). $(f \in S(X)_{((0))})$
In particular, $f|_{X_i}$ is regular. (X_i affine)
$\Rightarrow f|_{X_i} = \frac{g_i}{x_i^{N_i}}$, where $\underbrace{g_i \in S(X)}_{\text{homog}}$ & $\deg g_i = N_i$.

$$\begin{array}{c} S(X)_{((0))} \\ \cap| \\ S(X)_{(0)} \\ \cup| \\ S(X) \end{array}$$

$\Rightarrow x_i^{N_i} \cdot f = g_i \in S(X)_{N_i}$ (N_ith deg. piece).
Pick $N \geq \Sigma N_i$.
$\forall h \in S(X)_N$ (so one x_i will have power $\geq N_i$)
$hf \in S(X)_N$ b/c $h = \Sigma a_{i_1 \cdots i_n} \underbrace{x_1^{i_1} \cdots x_n^{i_n}}_{\text{one } i_k \geq N_k}$ & b/c $\deg f = 0$
$\Rightarrow S(X)_N \cdot f^q \subseteq S(X)_N \quad \forall q \geq 0$
$\Rightarrow x_0^N \cdot f^q \in S(X)_N \quad \forall q \geq 0$
$\Rightarrow S(X)[f] \subseteq x_0^{-N} S(X) \leftarrow$ a f.g. $S(X)$-mod (gen by x_0^{-N})
& $S(X)$ Noeth $\Rightarrow S(X)[f]$ is f.g. b/c submod.
$\Rightarrow f$ is int $/S(X)$. (b/c adjoined elt of f. of fracs)
$\Rightarrow f^m + a_1 f^{m-1} + \cdots + a_m = 0$ for some $a_1, \ldots, a_m \in S(X)$
$\Rightarrow$ take degree 0 piece$^{(=0)}$: $f^m + (a_1)^0 f^{m-1} + \cdots + (a_m)^0 = 0$
$(a_k)^0$ = deg 0 piece of a_k. $\Rightarrow (a_k)^0 \in (S(X))^0 = k$
$\Rightarrow f \in \overline{k}$ (f alg $/k$)
$\overline{k} = k \Rightarrow f$ a const.
(k alg. cl.) $\square$

- The comm. alg. problem is: Let S be a graded domain ($S_0 = k$, $S_i = 0$, $i<0$) gen/k by $x_0, \dots, x_n$ w/ $\deg x_i = 1$. $f \in S_{(0)}$ w/ $\deg f = 0$ s.t. $\forall i\ \exists N_i$ s.t. $x_i^{N_i} f \in S$. Prove f alg /k.

<u>Thm</u>: Let X be any variety, & Y be affine. Then $\mathrm{Hom}_{Var}(X,Y) \cong \mathrm{Hom}_{k\text{-alg}}(A(Y), \mathcal{O}(X))$.

<u>Pf</u>: $\phi \longmapsto (f \in A(Y) \longmapsto f \circ \phi)$

$$\begin{tikzcd} X \arrow[r, "\phi", "\text{reg}"'] \arrow[dr, "\substack{f\circ\phi \\ \text{reg} \\ \cap \\ \mathcal{O}(X)}"'] & Y \arrow[d, "\text{reg } f \in A(Y)"] \\ & k \end{tikzcd}$$

(finish next time)

<u>Cor</u>: $\underset{\substack{\uparrow \\ \text{category of} \\ \text{Affine vars}}}{\mathcal{A}ff} \simeq \left(\underset{\substack{\uparrow \\ \text{category of f.g.} \dots}}{\substack{\text{f.g. } k\text{-alg} \\ \text{int dom.}}} \right)^{op} \leftarrow$ opposite category; morphisms reverse dir of objects

<u>Pf</u>: Funtor: $X \longmapsto \mathcal{O}(X)$ (surj.)

op b/c roles of X, Y are reversed, i.e. $\mathcal{O}$ is a contravariant fcn.

$\cong$ in thm $\Rightarrow$ functor fully faithful

surj $\Rightarrow$ equivalence.

<u>Aside</u>: Affine schemes category (Aff Sch) $\simeq$ (Comm. Ring)^{op}

10/1 Read from 3.5 on. (finish chapter)

Blowing-up

- Blow-up of $\mathbb{A}^2$ at the origin:

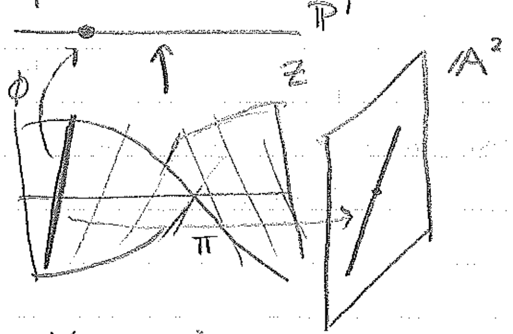
Consider the variety $\underset{(x,y)\ [s:t]}{Z \subseteq \mathbb{A}^2 \times \mathbb{P}^1}$ given by $xt = ys$.

Z is the blowup of $\mathbb{A}^2$ at origin: $Z = Bl_0 \mathbb{A}^2$.
(think of $\mathbb{A}^2$ as open subset of $\mathbb{P}^2$, so $\mathbb{A}^2 \times \mathbb{P}^1$ subset of $\mathbb{P}^2 \times \mathbb{P}^1$, which can be embedded in $\mathbb{P}^5$ via Segre)

$$\begin{tikzcd}
\mathbb{P}^1 & & \\
& Z \arrow[ul, "\phi"'] \arrow[d, "\pi", "reg."'] & \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \arrow[dl] \\
& \mathbb{A}^2 &
\end{tikzcd}$$

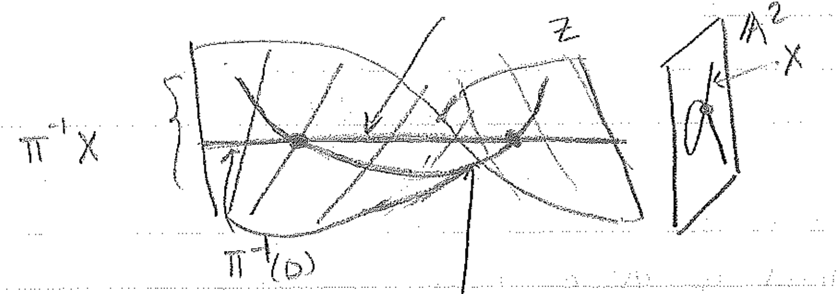
Claim: If $p=(x,y) \in \mathbb{A}^2$ is not $(0,0)$, then $\pi^{-1}(P) = \{Q\}$.
$xt = ys \Rightarrow \frac{x}{y} = \frac{s}{t}$, so x & y fixed determines s & t,
$Q = [x:y]$.
$\pi^{-1}(0) = \mathbb{P}^1$ ($\Rightarrow \pi$ only 1-1 away from origin)
$\Rightarrow$ a.e. an iso, but not at 0.

- Define $\Psi : \mathbb{A}^2 \setminus \{0\} \to Z \setminus \pi^{-1}(0)$, an iso. w/ inverse π.
$(x,y) \longmapsto ((x,y), [x:y])$
- For z, replaced pt w/ entire line ($\{0\}$ w/ $\mathbb{P}^1$)



- What is $\pi(\phi^{-1}([s:t]))$? Always a line in $\mathbb{A}^2$ through the origin.
Take line in plane & put it at a ht proportional to its slope.
At 0, no well-def. line through 0 & it, so

- Let $X \subseteq \mathbb{A}^2$ be a subvariety, $0 \in X$. Define the $Bl_0 X = \overline{\pi^{-1}(X) \setminus \pi^{-1}(0)}$



this curve alone is $Bl_0 X$
(remove line, get 2 holes, close them up)

→ $Bl_0 X$ uncrosses the curve → stretches it up.

Ex: $X = \{y^2 = x^2(x+1)\}$ "nodal curve in plane"

Write $Y = Bl_0 X$ in 2 patches by writing $\mathbb{P}^1 = \underset{(t \neq 0)}{\mathbb{A}^1} \cup \underset{(s \neq 0)}{\mathbb{A}^1}$

In first patch, U_1, we are looking inside $\mathbb{A}^3$

(b/c cover $\mathbb{A}^2 \times \mathbb{P}^1$ w/ $(\mathbb{A}^2 \times \mathbb{A}^1) \cup (\mathbb{A}^2 \times \mathbb{A}^1)$).

In $\mathbb{A}^3_1$, coords are (x, y, s) – 3rd coord really s/t b/c $t \neq 0$, but let $t = 1$.

$Z \cap U_1$? $xt = ys$ w/ $t = 1$, i.e. $x = ys$.

$U_1 \cap \pi^{-1}(X)$ is cut out by $y^2 = x^2(x+1)$ and $x = ys$, i.e.

$\begin{cases} y^2 = y^2 s^2(ys+1) \\ x = ys \end{cases}$ → can have $y = 0$, $s =$ anything, $x = 0$ → get the line in Z (i.e. the exceptional line)

→ or $y \neq 0 \Rightarrow 1 = s^2(ys+1)$, the remaining pts on $\pi^{-1}X$

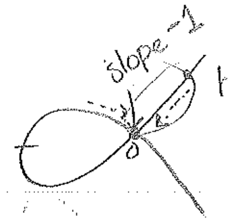
Considering ↗ alone gives the closure.

Thus $Bl_0 X =$ cut out by $s^2(ys+1) = 1$ and $x = ys$ inside U_1.

What are the 2 pts that were removed then added?

Let $\chi : Bl_0(X) \to X$ be the restriction of π. Then what is $\chi^{-1}(0)$? $y = 0 \Rightarrow s^2 = 1 \Rightarrow s = \pm 1$.

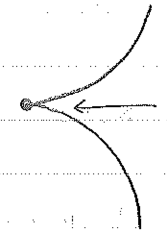
$\chi^{-1}(0) = \{(0,0,1), (0,0,-1)\}$



ht's here are all slightly bigger than 1 – as approach 0, get ht 1.

- could separate curve b/c the tangents at 0 are distinct.

<u>Ex</u>: $y^2 = x^3$



Limit is 0 from both directions, so preimage $\pi^{-1}(0)$ just one pt.

$X = \{y^2 = x^3\}$, $t \neq 0$, use s as a coord.

$$\pi^{-1} \begin{cases} y^2 = x^3 \\ x = ys \end{cases} \Leftrightarrow \begin{cases} y^2 = y^3 s \\ x = ys \end{cases} \begin{array}{l} \to y = 0 \Rightarrow \text{exceptional line.} \\ \to y \neq 0 \Rightarrow \end{array}$$

$$Bl_0(X) = \begin{cases} 1 = ys \\ x = ys \end{cases}$$ but preimage of 0 is empty: $X^{-1}(0) = \emptyset$, so must be in other patch ↪ (since if $y=0$, $1 \neq ys$)

In second patch, $s \neq 0$, use t as a coord:

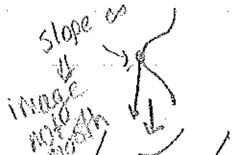
$$\pi^{-1}: \begin{cases} y^2 = x3 \\ tx = y \end{cases} \Leftrightarrow \begin{cases} t^2x^2 = x^3 \\ tx = y \end{cases} \longrightarrow x = 0 \Rightarrow \text{exceptional line}$$

$$Bl_0(X) = \begin{cases} t^2 = x \\ tx = y \hookrightarrow y = t^3 \end{cases}$$ If $x = y = 0$, then $t = 0$, so $X^{-1}(0) = (0,0,0)$

so

$$\begin{tikzcd} \mathbb{A}^3_{(x,y,z)} \arrow[d] & \supseteq (t,t^2,t^3) = C \arrow[dl] \\ \mathbb{A}^2_{(y,z)} & (t^2,t^3) \end{tikzcd}$$

, so the Blowup is the twisted cubic



- Blow-up of $\mathbb{A}^n$:

$Z \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$

$Z \text{ "="} \{((x_1, \dots, x_n), \overset{y_1, \dots, y_n}{[x_1 : \dots : x_n]})$

↑ i.e., want coord in $\mathbb{P}^n$ to be same as in $\mathbb{A}^n$ but this not well-def at 0.

- want vectors $\vec{x}$ & $\vec{y}$ to be prop:

$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}$ w/ rank = 1 $\Rightarrow x_i y_j - x_j y_i = 0$

Z = cut out by $\{x_i y_j - x_j y_i = 0 \ \forall i j\}$

$$\begin{tikzcd} Z \arrow[d, "\pi"'] & Z \setminus E & ((x_1, \dots, x_n), [x_1, \dots, x_n]) \\ \mathbb{A}^n & \mathbb{A}^n \setminus \{0\} \arrow[u, "\psi"'] & (x_1, \dots, x_n) \arrow[u, mapsto] \end{tikzcd}$$

Let $E = \pi^{-1}(0)$, called the exceptional divisor

$E \cong \mathbb{P}^{n-1}$

$\mathbb{A}^n \setminus \{0\} \underset{\psi}{\overset{\pi}{\rightleftarrows}} Z \setminus E$ inverse maps.

- If $X \subseteq \mathbb{A}^n$ is a subvariety, then
  $Bl_0 X = \overline{\pi^{-1}(X) \setminus E}$

Hironaka proved Resolution of Singularities in char 0
- blowing up enough times removes singularities.
- → unproven in char p.

Def: Let X & Y be varieties. A rational map $\phi : X \dashrightarrow Y$ is an equivalence class of pairs (U, f) where $U \subseteq X$ open & $\neq \emptyset$, & $f : U \to Y$ regular, under

$(U, f) \sim (V, g) \Leftrightarrow \exists W \subseteq U \cap V, \overset{\text{open}}{W} \neq 0$ s.t. $f|_W = g|_W$.

* ϕ is not a fcn $X \to Y$.

__Ex__: {Rational maps $X \to \mathbb{A}^1$} = {Rat'l fcns on X}

__Ex__: $X \dashrightarrow Y \dashrightarrow Z$
- Image of 1st map may land in a cl. set where 2nd map not def.
- But if contains big open set, ok b/c can intersect that open set w/ domain of def of 2nd map.

__Def__: $\phi : X \dashrightarrow Y$ is called __dominant__ if for some (U, f), $f(U) \supseteq V$, $V \subseteq Y$ open, $\neq \emptyset$.
- then such maps can be composed.

__New Category__: Objects: varieties
Rat Morphisms: Dominant rat'l maps

__Def__: An __isomorphism__ in this category is called a __birational map__.
- 2 rat'l maps st composition in both dir's is $= \mathbb{1}$.

__Ex__: Z and $\mathbb{A}^n$, w/ ψ & ϕ from before are inverse rat'l dominant maps
$\mathbb{A}^n \overset{\psi}{\dashrightarrow} Z$ (rat'l b/c only def. somewhere)

- $X \underset{\text{rat'l dom}}{\overset{\psi}{\dashrightarrow}} Y \underset{\overset{\wedge}{k(Y)}}{\overset{f}{\dashrightarrow}} \mathbb{A}^1$, $f \circ \psi$ is a rational map on X, so $f \circ \psi \in k(X)$ (b/c defined somewhere)

So have fcn $k(Y) \to k(X)$

__Thm__: There is a bijective corresp. btwn
{rat'l dom. maps $X \dashrightarrow Y$} $\xleftrightarrow{1\text{-}1}$ {k-alg. maps $k(Y) \to k(X)$}
Moreover, this gives an equivalence
$$\mathcal{R}at \cong \{\text{field ext'ns } K/k \text{ f.g., } k\text{-alg's as a field}\}^{op}$$
$$\Downarrow$$
$$\{\text{f.g. field ext'ns } K/k \text{ w/ maps of } k\text{-alg's}\}$$

$$\left.\begin{aligned} X &\longmapsto k(X) \\ X \dashrightarrow Y &\longmapsto k(Y) \to k(X) \end{aligned}\right\} \text{ done above.}$$

10/3 **Ex**: $X = \mathbb{A}^2 \setminus \{(0,0)\}$, $Y = \mathbb{A}^2$

$\mathcal{O}(X) = \underset{\curvearrowright}{k[x,y]} = \mathcal{O}(Y)$.

If $X \cong Y$, then$^{\Phi}$ X is affine. ⇒ then Φ would induce an iso $X \cong Y$. Φ induces the standard inclusion which is not surj ⚡.

Ex: Can a quasi affine be also affine? Yes.

$X = \mathbb{A}^1 \setminus \{0\}$, q-affine, but $X \cong Y = Z(xy-1) \subseteq \mathbb{A}^2$, & Y affine.

$$\begin{tikzcd} Z \arrow[d, "\wr"'] & \subseteq \mathbb{A}^2 \arrow[d] & (x,y) \arrow[d, mapsto] \\ X & \subseteq \mathbb{A}^1 & x \end{tikzcd}$$

Def: We'll say X is affine ⟺ X is ≅ to an affine variety. (even if X is not presented as affine)

Lemma: Y,X any variety, $U \subseteq X$ open, f,g $:X \to Y$, reg. on X but $f|_U = g|_U \Rightarrow f = g$ on X.

Pf: Wlog, can assume $Y \subseteq \mathbb{P}^n$. Replace Y by $\mathbb{P}^n$ (ie compose f & g w/ inclusion of $Y \hookrightarrow \mathbb{P}^n$ → still reg & still agree on U).

Let $\Delta \subseteq \overset{[x_0,\dots,x_n]}{\mathbb{P}^n} \times \overset{[y_0,\dots,y_n]}{\mathbb{P}^n}$ be the diagonal: $\Delta = \{(x,y) \in \mathbb{P}^n \times \mathbb{P}^n \mid x = y\}$

Δ is closed: it is cut out by:

NOT $x_i = y_i$ b/c not bihomogenous (same deg in x's & same deg in y's)

$\Delta = Z(x_i y_j - x_j y_i, \forall i,j)$

Look at $\Psi: X \to \mathbb{P}^n \times \mathbb{P}^n$

$x \longmapsto (f(x), g(x))$ regular b/c each coord. reg.

Let $Z = \Psi^{-1}(\Delta)$. $f|_U = g|_U \Rightarrow U \subseteq Z$.

Δ cl ⇒ Z cl. Therefore, $\overline{U} = X \subseteq Z \Rightarrow f = g$ on all of X. □

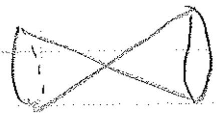
* Common Trick: Intersect smth w/ diagonal to determine where 2 things are =.

Prop: Let $f \in k[x_1, \dots, x_n]$. Then $\mathbb{A}^n \setminus Z(f) \simeq Z = Z(fy=1) \subseteq \mathbb{A}^{n+1}_{(x_1,\dots,x_n,y)}$

Also, $\mathcal{O}(Z) = k[x_1, \dots, x_n]_f$.

* i.e. remove codim. 1 set from $\overset{\text{(smooth) UFD)}}{\text{affine var.}}$, get affine var. (from prev. ex's know if codim ≥ 2, may not be affine)

Ex: $X = Z(xy - z^2) \subseteq \overset{\dim 2}{\mathbb{A}^3}$ [affine quadric cone]



-smooth quadric in $\mathbb{P}^2$ - then join all pts to origin.

Let $Z = Z(x=z=0) \subseteq X$ dim 1 (Line) $\Rightarrow$ codim 1

- Prove Z is not cut out by single poly.
- ~~I believe that $X \setminus Z$ may not be affine.~~ Actually affine.
  - If $x=0$, then $z^2=0$ so $z=0$
- the ideal $I = (x,z) \subseteq A(X)$ is not principal
  - the ideal (x) is not radical, b/c $z^2 \in (x)$ but $z \notin (x)$

* details

Pf: $Z \longrightarrow \mathbb{A}^n \setminus Z(f)$

[illegible] } clear.

$y = 1/f \longleftarrow\!\shortmid f$

$k[x_1, \dots, x_n]_f = k[x_1, \dots, x_n, f^{-1}] = k[x_1, \dots, x_n, y]/(yf=1)$

Prop: Let X be any variety. Then $\exists$ a basis for its topology consisting of affine varieties.

-given pt & nbhd, can find an affine var in nbhd

Pf: ETS: $\forall x \in X$, $\forall U \ni x$ open, $\exists V \ni x$, $V \subseteq U$, V affine.

X var $\Rightarrow$ U var, so can replace U by X.

ETS: $\forall X$ var, $\forall x \in X$ $\exists V \ni x$, $V \subseteq X$ s.t. V open & affine. [every variety is glued from affines]

Let $X \subseteq \mathbb{P}^n$ q-proj. (every var is q-proj). Pick some U_i s.t. $x \in U_i$ (in the standard cover of $\mathbb{P}^n$).

$x \in X \cap U_i$ open in X & $X \cap U_i \subseteq U_i \simeq \mathbb{A}^n$. wlog, X can be quasi-affine (NTS that inside can

If Z everything, $X = \emptyset$, affine. So can assume I not trivial $(I \neq (0$

Let $X = Y \setminus Z$, where Y cl. in $\mathbb{A}^n$, Z cl. in Y (or in $\mathbb{A}^n$)

(so Y is the affine & X open)

$I = I(Z)$, $I \subseteq k[x_1, \dots, x_n]$ ideal. Pick $\overset{\text{this}}{f} \neq 0$, $f \in I$,

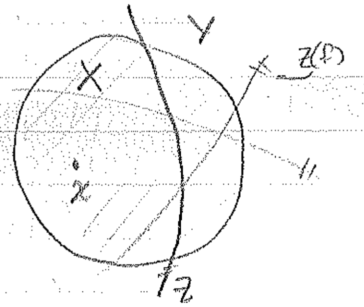
$\underset{\text{open in } X,\ x \text{ in it}}{U} = Y \setminus Z(f) \subseteq X$, since $(f) \subseteq I(Z) \Rightarrow Z \subseteq Z(f)$

$[x \in X \Rightarrow x \notin Z \Rightarrow \exists f \neq 0$ s.t. $f \in I$ & $f(x) \neq 0]$

But U is affine, since $\parallel$

$U = Y \cap \underbrace{(\mathbb{A}^n \setminus Z(f))}_{\text{affine var.}}$

i.e. $U = Z(\underset{k[x_1,\dots,x_n,y]}{\overset{\cap}{I(Y)}}, yf = 1) \subseteq \mathbb{A}^{n+1}_{x_1,\dots,x_n,y}$



[so if remove $\underset{\text{zero locus of a single poly.}}{Z(f)}$ from affine var, get an affine var] □

Thm: Let X, Y be varieties. Then $\exists$ a bijection

$\{$rat'l dominant maps $X \dashrightarrow Y\} \xleftrightarrow{1-1} \{$field maps $K(Y) \to K(X)$, respecting the k-alg. structure$\}$

↑ scalars map to scalars.

Pf: If $\underbrace{X \overset{\phi}{\underset{\text{rat'l dom}}{\dashrightarrow}} Y \overset{f}{\underset{\text{rat'l}}{\dashrightarrow}} \mathbb{A}^1}$, then $f \circ \phi : X \to \mathbb{A}^1$ is a rat'l map in $K(Y)$

$\Rightarrow f \in K(Y)$

so $\phi \longmapsto (\underset{K(Y)}{\overset{\cap}{f}} \mapsto \underset{K(X)}{\overset{\cap}{f \circ \phi}})$.

($\Leftarrow$): Let $\Psi : K(Y) \to K(X)$ be a k-alg map.

Want to define a rat'l dom. map $X \dashrightarrow Y$.

Enough to define $X \dashrightarrow U \subseteq Y$ open. Wlog, replace Y by an affine open in it (ok by prev. prop — since will have same $K(Y)$).

$A(Y)$ is a f.g. k-alg. Pick $y_1, \dots, y_n$ gen's.

$K(Y)$ is the f. of f's of $A(Y)$, so $y_i \in K(Y)$

f of f of any open
subset of a var is
the same as the f
of f of the var.

Then $\theta_1, \ldots, \theta_n$, $\theta_i = \psi(y_i)$ are in $k(X)$, i.e. rat'l fcns on X. (each def. on an open set)
There exists some $U \subseteq X$ s.t. all θ_i are def on U (just the $\cap$ of all θ_i's domains of def).
$\Rightarrow \theta_i \in \mathcal{O}(U)$
We have defined a map $\psi: A(Y) \to \mathcal{O}(U)$, a k-alg. map. Also injective (b/c $\psi: k(Y) \to k(X)$ inj. b/c field map & $A(Y) \subseteq k(Y)$ & $\mathcal{O}(U) \subseteq k(X)$).
We get a map $U \to Y$, (from prev. prop, since Y affine) which is dominant (ie image dense) b/c ψ injective
[injective ring maps $\overset{1-1}{\longleftrightarrow}$ dominant maps]
ie $U \to Y$ is a rat'l map $X \dashrightarrow Y$.
These 2 mappings are inverses. (Check) $\square$

<u>Cor</u>: $\mathcal{R}at \simeq \{\text{f.g. field ext'ns of } k \text{ w/ } k\text{-alg. maps}\}^{op}$
<u>Pf</u>: WTS if K a f.g field ext'n of k, then $K = k(X)$ for some variety X. (prev. thm say morphisms are same)
Let $y_1, \ldots, y_n$ be gen of K/k, Let $A = k[y_1, \ldots, y_n] \subseteq K$, a f.g. k-alg, an int. dom (b/c $\subseteq$ field), so $A = A(Y)$ for some affine var. Y (from before),
but $A_{(0)} = K \Rightarrow k(Y) = K$. $\square$

10/8 My def. of dominant map $f:X\to Y$: $f(X) \supseteq U$, open, $U \neq \emptyset$.

we will use this def. [Hartshorne's def (standard): $f:X\to Y$ is dominant if $f(X)$ is dense in Y.

(My def ⇒ standard if Y a variety. But a priori there may be more dom. maps in H's def. But:

Thm: Let $f:X\to Y$ be a map of affine varieties, s.t. f dom. (in H's sense). Then $f(X)$ contains an open set.

(so H ⇒ my def for arb. var's b/c can pick affine open subset of Y, preimage open so contains affine. Thus there's a restriction of f from affine to affine & then contains open)

Auxilliary Thm: Let $A\subseteq B$ be an inclusion of Noetherian domains s.t. B is a f.g. A-alg. Then for every $b\in B$ $\exists a\in A$ s.t.:

$\forall \phi: A\to K$ s.t. $\phi(a)\neq 0$ (K an alg. cl. field),

$\exists$ an ext'n $\Phi: B\to K$ s.t. $\Phi(b)\neq 0$ extending ϕ.

(proven in 742)

Beginning of pf of Thm: Let $A=\mathcal{O}(Y)$, $B=\mathcal{O}(X)$. The map $f:X\to Y$ gives a map $\phi:A\to B$ of rings. Since f dominant, ϕ is injective. [If $g\in A$ & pullback to X. If $\phi(g)=0$, then zero on dense set ⇒ zero fcn]. So we can think of $A\subseteq B$. } check

Take $b=1$. The aux. thm gives an $a\in A$ s.t. ....

(I would try to prove that $f(X) \supseteq \underline{Y\setminus Z(a)}$)

guess for the open set we're trying to produce.

Thm: Let X, Y be varieties. Then TFAE:

(1) X is birational to Y.

(2) $K(X) \cong K(Y)$ as k-alg's

(3) $\exists\, U \subseteq X$ open $\neq \emptyset$ & $V \subseteq Y$ open $\neq \emptyset$ s.t. $\underset{\text{iso.}}{U \cong V}$.

[we've seen nodal curve in plane birat'l to twisted cubic]

Pf: (1)⟺(2): From last time w/ Rat $\cong \cdots$

(3) ⇒ (2): $\underbrace{K(X) \cong K(U)}_{\text{b/c } \varphi \text{ def somewhere in } X, \text{ def somewhere in } U} \underset{\text{b/c } U \cong V}{\cong} K(V) \cong K(Y)$

(1) ⇒ (3) Let $U \subseteq X$, $\phi: U \to Y$ a regular dom map,

$V \subseteq Y$, $\psi: V \to X$ regular dom. map.

Take $\left.\begin{array}{l} U' = U \cap \phi^{-1}(V) \\ V' = V \cap \psi^{-1}(U) \end{array}\right\}$ open & $\neq \emptyset$.

ϕ & ψ defined on these sets.

If $\phi|_{U'} \subseteq V'$ & $\psi|_{V'} \subseteq U'$, then their comp. must be id. & $U' \cong V'$.

NTS: $\phi(U') \subseteq V'$:

$\phi(U') \subseteq V \cap \phi(U) \subseteq \overset{= V'}{V \cap \psi^{-1}(U)}$ b/c

$V \cap \phi(U) \subseteq \psi^{-1}(U)$ b/c $\psi \circ \phi = \text{id}$:

b/c if $x \in V \cap \phi(U)$. $x \in V \Rightarrow \psi(x)$ def. ✓

& $\psi(x) \in U$ b/c $x \in \phi(U)$, ie. $x = \phi(y)$

for some $y \in U \Rightarrow \psi(x) = \psi(\phi(y)) = y$. ✓ □

Ex: 2 var's w/ diff dim are not birat'l b/c tr. deg of $K(X)$, $K(Y)$ $\neq$, so $K(X) \not\cong K(Y)$, ∃ dim. a birat'l invariant.

Ex: $\mathbb{P}^1$ and $Z(y^2 - x(x-1)(x+1)) \subseteq \mathbb{A}^2$

$K(\mathbb{P}^1) = k(x)$

$K(X)$ is a fin. ext'n (quadratic) of $k(x)$, but not $\cong k(x)$. [can't prove this yet]

Smoothness

Provisional def: Let $X \subseteq \mathbb{A}^n$ be a variety, $p \in X$. We say X is smooth at P iff

$$\operatorname{rank}\begin{pmatrix} \frac{\partial f_1}{\partial x_1}\Big|_p & \cdots & \frac{\partial f_m}{\partial x_1}\Big|_p \\ \vdots & & \\ \frac{\partial f_1}{\partial x_n}\Big|_p & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_p \end{pmatrix} = n - \dim X = \operatorname{codim} X.$$

where $(f_1, \dots, f_m) = I(X)$. Jacobean matrix.

Rule: $\frac{\partial x^m}{\partial x} = mx^{m-1}$. Extend by Leibnitz.

Can happen: $f'(x) \equiv 0$ but $f \not\equiv$ const. B/c in char p, $f'(x) \equiv 0 \Leftrightarrow f(x) = g(x^p)$
- x^{p-1} has no integral in char p.

(similar to separability of field extns)

Ex: Find the singular pts (pts at which not smooth) of $y^2 = x^2(x-1)$. So $f(x,y) = x^2(x-1) - y^2$

$$J = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 2x(x-1) + x^2 \\ -2y \end{pmatrix} = \begin{pmatrix} 3x^2 - 2x \\ -2y \end{pmatrix}.$$

$\dim X = 1$, so want $\operatorname{rank} J = 2 - 1 = 1 \Leftrightarrow J \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\Leftrightarrow (x, y) \neq (0,0)$.
($y = 0$, then $f(x,0) \Rightarrow x = 1, 0$, & $x = 1 \not\Rightarrow \operatorname{rank} J = 1$)
Seems like $(2/3, 0)$ singular, but not on X.

Cor: $\underbrace{\text{Sing } X}_{\text{singular locus of } X} \subseteq X$ is always closed.

Pf: $\text{Sing } X = Z(f_1, \dots, f_m, \text{ \& rank} < \operatorname{codim} X)$
[rank will never be more than $n - \dim X$]
Finish pf after a comm. alg result.

so just a list of eqn's $\left\{ \begin{array}{l} \text{det of the minors of} \\ \operatorname{codim} X \text{ (a poly in } x_i\text{'s} \\ \text{vanishes} \end{array} \right.$ □

Def: A local ring $(A, \underline{m})$ is regular if
$\dim_k \underline{m}/\underline{m}^2 = \dim A$, $k = A/\underline{m}$.
↑vect sp dim ↑krull dim [note: ht $\underline{m} = \dim A$ in local ring]

$\underline{m}/\underline{m}^2$ always a module over $A/\underline{m}$, a field, so can find dim as a vect sp.

Thm: Let X be a variety, $P \in X$. Then X is smooth at $P \Leftrightarrow \mathcal{O}_{X,P}$ is regular.

Cor: Smoothness is an intrinsic property.

Def: X is smooth at P $\Leftrightarrow \mathcal{O}_{X,P}$ is regular.
[prev. def has become a thm]

Pf of Thm: First, we'll prove that $\mathbb{A}^n$ smooth:
$\dim \mathbb{A}^n = n$, so NTS $\dim_k \underline{m}/\underline{m}^2 = n$. For $P \in \mathbb{A}^n$, $P = (a_1, \dots, a_n)$
define $\theta: k[x_1, \dots, x_n] \to k^n$
$$f \longmapsto \left(\partial f/\partial x_1|_P, \dots, \partial f/\partial x_n|_P\right)$$
Let $\underline{a}_p = (x_1 - a_1, \dots, x_n - a_n)$. Then $\theta(\underline{a}_p) = k^n$, b/c
$f_1 = x_1 - a_1 \longmapsto (1, 0, \dots, 0)$, $\dots$, $f_n = x_n - a_n \longmapsto (0, \dots, 0, 1)$.
k^n vect field, & hit all basis vectors.
$\theta(\underline{a}_p^2) = 0$. In fact, $\operatorname{Ker} \theta|_{\underline{a}_p} = \underline{a}_p^2$. (i.e. poly's that are at least quad
$\Rightarrow \theta' : \underline{a}_p/\underline{a}_p^2 \xrightarrow{\sim} k^n$ is an isomorphism.
Thus $\mathbb{A}^n$ is smooth at P. (b/c $\dim_k \underline{a}_p/\underline{a}_p^2 = n = \dim \mathbb{A}^n = \dim \mathcal{O}_{\mathbb{A}^n, P}$)

Let $\underline{b} = (f_1, \dots, f_m) = I(X)$. $\underline{b} \subseteq \underline{a}_p$ (i.e. $P \in X$)
$\theta(\underline{b}) \subseteq k^n$, & $\dim \theta(\underline{b}) = rk(J)$. w/ $k^n \simeq \underline{a}_p/\underline{a}_p^2$,
$(\underline{b} + \underline{a}_p^2)/\underline{a}_p^2 \subseteq \underline{a}_p/\underline{a}_p^2$ b/c $\underline{b} \subseteq \underline{a}_p$
↳ what gets killed under $\theta|_{\underline{b}} \to k^n$
image is $\underline{b}/\ker\theta = \underline{b}/\underline{b} \cap \underline{a}_p^2$
$\wr\wr$ Fund Iso Th
$(\underline{b} + \underline{a}_p^2)/\underline{a}_p^2$

$$\Theta(g\cdot f) = \left(\partial(gf)/\partial x_1 (p), \ldots, \partial(gf)/\partial x_n (p)\right)$$
$$= \left(g^{(p)}\,\partial f/\partial x_1 (p) + \underbrace{\partial g/\partial x_1^{(p)}\, f(p)}_{0} \;\ldots\ldots\right)$$
b/c f vanishes at P.
$$= g^{(p)}\Theta(f)$$

$\Rightarrow$ so when finding dim $\Theta(\underline{b})$, need only look at $\Theta(f_i)$, the gen's of $\underline{b}$. All other elt's of $\underline{b}$ are, by above calc, lin comb's of $\Theta(f_i)$'s.

10/10 **Thm:** (from yesterday): If $X \xrightarrow{f} Y$ dom, X, Y affine, then $\exists$ U open in Y, $U \neq \emptyset$, $f(X) \supseteq U$. [true for arbitrary varieties, but can reduce to affine sp].

Pf: Let $A = \mathcal{O}(Y)$, $B = \mathcal{O}(X)$. Get $\phi: A \to B$ inj b/c f dom. B a f.g. A-alg. Take $b = 1$, apply auxilliary thm: get $0 \neq a \in A$ s.t. every $\Psi: A \to k$ w/ $\Psi(a) \neq 0$ extends to $\overline{\Psi}: B \to k$ (automatic that $1 \mapsto 1$)

Claim: $f(X) \supseteq Y \setminus Z(a) = U$. [a a fcn on $Y \Rightarrow Y \setminus Z(a)$ open]
$a \neq 0 \Rightarrow U \neq \emptyset$.

Pf: Let $P \in Y \setminus Z(a)$. Then $ev_P: \mathcal{O}(Y) \twoheadrightarrow k$
$f \mapsto f(P)$.

(*) check

$\ker(ev_P) = \underline{m}_P \subseteq \mathcal{O}(Y)$
$\Rightarrow \mathcal{O}(Y)/\underline{m}_P \cong k$ } this actually true $\forall$ varieties by Nullstelle satz

Since $P \in Y \setminus Z(a)$, $ev_P(a) \neq 0$ (i.e. a doesn't vanish at P). Take Ψ to be ev_P. Then $\exists$ $\overline{\Psi}: \mathcal{O}(X) \to k$ extending ev_P. [$\mathcal{O}(X)$ bigger then $\mathcal{O}(Y)$]
$\overline{\Psi}$ is obviously surj (by $\to 1$), so
$\ker\overline{\Psi} = \underline{m}_Q \subseteq \mathcal{O}(X)$ max'l
$\Rightarrow \underline{m}_Q$ corresp. to a pt $Q \in X$. But
b/c $\overline{\Psi}$ extends ev_P $\longrightarrow$ $\underline{m}_Q \cap A = \underline{m}_P \Leftrightarrow f(Q) = P$.

(we've shown $P \in f(X)$, but P arbitrary in $Y \setminus Z(a)$
$\Rightarrow f(X) \supseteq U$) □

Def: A local ring $(A, \underline{m})$ is regular if $\dim_k \underline{m}/\underline{m}^2 = \dim A$,
$k = A/\underline{m}$, the residue field.

Thm: If $X \subseteq \mathbb{A}^n$ is cut out by $(f_1, \ldots, f_m)$ & $P \in X$, then
$\mathcal{O}_{X,P}$ is regular $\Leftrightarrow$ rank $J(P) =$ codim X. Then we say X is smooth at P.

$$J(P) = \begin{vmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_m/\partial x_1 \\ \vdots & & \vdots \\ \partial f_1/\partial x_n & \cdots & \partial f_m/\partial x_n \end{vmatrix}$$

Pf: (Recap) $\quad \theta : k[x_1, \ldots, x_n] \to k^n$

$$f \longmapsto \begin{pmatrix} \partial f/\partial x_1 (P) \\ \vdots \\ \partial f/\partial x_n (P) \end{pmatrix}$$

If $\underline{a}_p = (x_1 - a_1, \ldots, x_n - a_n)$, $P = (a_1, \ldots, a_n)$, then
$\theta' : \underline{a}_p/\underline{a}_p^2 \xrightarrow{\sim} k^n$ (clear $\underline{a}_p \twoheadrightarrow k^n$ & $\ker = \underline{a}_p^2$)
$\Rightarrow \mathbb{A}^n$ is smooth at P ($\dim_k \underline{a}_p/\underline{a}_p^2 = n$)

Let $\underline{b} = I(X) = (f_1, \ldots, f_n)$
$\theta(\underline{b}) = \mathrm{Im}(J(P))$
$\Rightarrow \mathrm{rk}(J(P)) = \dim \theta(b)$ [by def of rank]
$\underline{b} \subseteq \underline{a}_p$ b/c $P \in X \Rightarrow \theta|_{\underline{b}} : \underline{b} \to k^n$, $\ker \theta|_{\underline{b}} = \underline{b} \cap \ker \theta$
$= \underline{b} \cap \underline{a}_p^2$
$\Rightarrow \mathrm{Im}\, \theta(\underline{b}) \simeq \underline{b}/\ker \theta|_{\underline{b}} \simeq \underline{b}/\underline{b} \cap \underline{a}_p^2 \simeq \underline{b} + \underline{a}_p^2/\underline{a}_p^2$ (by Iso. Thm)
$\mathcal{O}_{X,P} = \mathcal{O}(X)_P = (A/\underline{b})_{\underline{m}_p}$ $\quad$ ($A = k[x_1, \ldots, x_n]$, $\underline{m}_p \subseteq A/\underline{b}$ is
$\Rightarrow \underline{m}_p/\underline{m}_p^2 \simeq \underline{a}_p/\underline{b} + \underline{a}_p^2$ $\quad$ the image of $\underline{a}_p$ under $A \to A/\underline{b}$)
There is a ses:

$$0 \to \frac{\underline{b} + \underline{a}_p^2}{\underline{a}_p^2} \xrightarrow[(**)]{} \underline{a}_p/\underline{a}_p^2 \xrightarrow[(*)]{} \underline{a}_p/\underline{b} + \underline{a}_p^2 \to 0$$

→ $\underline{a}_p$ is preimage of $\underline{m}_p$, $\underline{b} + \underline{a}_p^2$ is preimage of $\underline{m}_p^2$, so quotients are $\simeq$.

(*) quotient map b/c $\underline{b} + \underline{a}_p^2 \supseteq \underline{a}_p^2$
(**) $\underline{b} + \underline{a}_p^2 \to 0$, but needs to be in $\underline{a}_p/\underline{a}_p^2$, so $\underline{b} + \underline{a}_p^2/\underline{a}_p^2$ is ker(*)
but $\dim \underline{a}_p/\underline{a}_p^2 = \dim \frac{\underline{b} + \underline{a}_p^2}{\underline{a}_p^2} + \dim \underline{a}_p/\underline{b} + \underline{a}_p^2$ $\quad$ b/c all vect. sp's

$\Rightarrow \boxed{\mathrm{rk}(J(P)) = n - \dim \underline{m}/\underline{m}^2}$

Then $\mathcal{O}_{X,P}$ regular $\Leftrightarrow \dim \underline{m}/\underline{m}^2 = \dim \mathcal{O}_{X,P} = \dim X$

$\Leftrightarrow n - \dim \underline{m}/\underline{m}^2 = \operatorname{codim} X \Leftrightarrow \mathrm{rk}\, J(P) = \operatorname{codim} X$.

(Due to Zariski in '40's.) $\square$

<u>Def</u>: The (Zariski) <u>tangent space</u> to X at P is

$T_{X,P} = (\underline{m}/\underline{m}^2)^\vee = \mathrm{Hom}_k(\underline{m}/\underline{m}^2, k)$, where $\underline{m}$ is the max'l ideal in $\mathcal{O}_{X,P}$.

- always a vect. sp / k.

How is this related to tan. sp. in diff. geom?

- Tangent vector to X at P is a direction in which to differentiate fcns on X at P.
  - in $\mathbb{R}^3$ at $(0,0,0)$, tan. sp. spanned by $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$.
- Also, if fcns analytic, its a potential first-order coefficient of the taylor expansions for fcns around P:

$$f(P) + \frac{\partial f}{\partial x_1}\cdot(x_1 - a_1) + \partial f/\partial x_2 (x_2 - a_2) + \cdots + \partial f/\partial x_n \cdot (x_n - a_n) + \text{H.O.T.}$$

$\rightarrow$ so it's a fcn:

$\partial/\partial x_1$: fcns $f \longrightarrow \partial f/\partial x_1$ = one of the 1st order taylor coeffs.

- Can restrict attn to fcns f s.t. $f(P) = 0$ (b/c $f(P)$ has nothing to do w/ 1st order coeffs).

$\underline{m}$ = fcns that vanish at P

$\underline{m}^2$ = fcns that vanish at P to 1st order at P ($f(P) = 0$ & $f'(P) = 0$

by Leibniz's rule

So $\underline{m}/\underline{m}^2$ = fcns that vanish at 0 & have HOT = 0.

the map from this to k says what lin comb. of 1st order coeffs you take

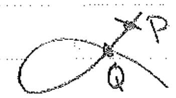
$(\underline{m}/\underline{m}^2)^\vee$ = space of all derivations, i.e. tangent vectors.

$\Rightarrow$ X is smooth at P $\Leftrightarrow$ $\dim X = \dim T_{X,P}$

Thm : $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.

$\Rightarrow$ $T_{X,P}$ may be greater than variety, & so not smooth, or can be as small as possible, ie $\dim X$, & then is smooth.

Ex: $X = Z(y^2 - x^2(x+1))$



$\dim T_{X,P} = 1$ (b/c smooth everywhere but origin & $\dim X = 1$)
$\dim T_{X,Q} = 2$
$\hookrightarrow$ if $X \hookrightarrow Y$, then $T_{X,P} \hookrightarrow T_{Y,P}$, so
since $X \subseteq \mathbb{A}^2$ & $\mathbb{A}^2$ smooth, so $T_{\mathbb{A}^2,Q} = 2$
& $T_{X,Q} \leq T_{\mathbb{A}^2,Q}$
OR $\operatorname{rk}(J(P)) = n - \dim \mathfrak{m}/\mathfrak{m}^2$
at Q, $J(Q) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so rank $= 0$
$n = 2$, so $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$
$\overset{\parallel}{\dim T_{X,Q}}$

* if a variety is embedded in $\mathbb{A}^n$, $\underset{(\dim X \leq)}{T_{X,P}} \leq n$ $\forall P$.

* Read from sec. 4:

Thm: Every variety X is birational to a hypersurface (X may not be cut out by one eqn, but can find an open set thats (—) to an open set of a hyperplane in $\mathbb{A}^{n+1}$)

check

Thm: Sing X is a proper closed subset of X.
(i.e. every variety is smooth a.e)
Pf : Question is local, so can reduce to X affine
$\searrow$ b/c set-closed local
(-if true $\forall$ affine open sets in X, true for X)

*
Local Q's $\Rightarrow$ can reduce to affine

Closed: Sing X cut out from $\mathbb{A}^n$ by $I(X)$ and determinants of the $(n-\dim X)$-minors of J
→ poly in the deriv's of f that all vanish at pt $\Rightarrow$ $\mathrm{rk}\, J < n - \dim X$
$\Rightarrow$ those pts are singular.

[Now, from $\dim A \leq \dim \underline{m}/\underline{m}^2$, can say P sing if $\mathrm{rk}\, J < \mathrm{codim}\, X$.]

$\neq X$: Assume Sing X = X. wlog, X is a hypersurface,
$X = Z(f) \subseteq \mathbb{A}^{n+1}$ (from prev. thm) (f irred)

Sing on open set $\Rightarrow$ sing everywhere, b/c Sing X is closed

$\mathrm{Sing}\, X = Z(f, \underbrace{\partial f/\partial x_1, \dots, \partial f/\partial x_n}_{\mathrm{rk}\, J = 0 \Rightarrow \text{all partials of } f \text{ vanish}}) = X$

$\Rightarrow (f, \partial f/\partial x_1, \dots, \partial f/\partial x_n) \subseteq (f)$

But $\deg \partial f/\partial x_i \lneq \deg f \Rightarrow \partial f/\partial x_i \in (f)$
$\Leftrightarrow \partial f/\partial x_i = 0$. (b/c $f \mid \partial f/\partial x_i$) $\forall i$

If Char = 0 $\Rightarrow$ f const $\lightning$

In char p, $\Rightarrow f = g(x_1^p, \dots, x_n^p) = (g_1(x_1, \dots, x_n))^p$
↖ $g_1 = g$ w/ p-th roots of coeffs ok b/c over $k = \bar{k}$.

$\lightning$ b/c f irred. $\square$

Ex: Look at $X = Z(\overset{f}{z^2 - yx^2 + 4y^{n+1}}) \subseteq \mathbb{A}^3$. (char 0), $n \geq 1$.

** Imp.

Find Sing X: want $\partial f/\partial x = 0$

$$\begin{aligned}
&\Rightarrow 2xy = 0 \Rightarrow x = 0 \text{ or } y = 0 \\
&\partial f/\partial y = 0 \Rightarrow -x^2 + 4(n+1)y^n = 0
\end{aligned}\left.\right\} \begin{aligned}&\text{if } x=0,\ y=0\\ &\text{if } y=0,\ x=0\end{aligned}$$

$\partial f/\partial z = 0 \Rightarrow 2z = 0 \Rightarrow \boxed{z = 0}$

$f = 0$

$\Rightarrow \boxed{x = 0, y = 0}$

So $\mathrm{Sing}\, X = \{0\}$.

HwExer: Blow this surface up until it becomes smooth.
Q: How many times do you need to do this?
→ in $\mathbb{A}^3_{xyz} \times \mathbb{P}^2_{uvw}$ have 3 patches:

$\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$
$xv = yu$ ← redundant
$xw = zu \Rightarrow x = zu$
$yw = zv \Rightarrow y = zw$

If $w \neq 0$, in $\mathbb{A}^3_{xyz} \times \mathbb{A}^2_{uv}$:

$$\begin{cases} x = zu \\ y = zv \\ z^2 - yx^2 + 4y^{n+1} = 0 \end{cases} \subseteq \mathbb{A}^5_{xyzuv}$$

$$\hookrightarrow \Rightarrow \{ z^2 - zvz^2u^2 + 4z^{n+1}v^{n+1} = 0$$

$$z^2 - z^3vu^2 + 4z^{n+1}v^{n+1} = 0$$

$$\cancel{z^2}(\underbrace{1 - zvu^2 + 4z^{n-1}v^{n+1}}) = 0$$

eqn of blow-up in patch $w \neq 0$ in coords u, v, z

$\swarrow$

$z = 0 \Rightarrow$

$\underbrace{x=0,\ y=0}$
exceptional divisor

$\hookrightarrow$ need to decide if smooth or not by taking partials.
$\Rightarrow$ will have determined sing. coords of blowup in this patch.

* 1^{st} Blowup will still have sing. pts.

10/15 <u>Def</u>: If A is a ring, $I \subseteq A$ an ideal, the <u>completion of A wrt I</u> is $\varprojlim_{n \geq 1} A/I^n = \hat{A}$.

- Assume you have a family $\{M_i\}_{i \in \mathbb{N}}$ of modules (gps, rings, etc) over a ring. Moreover, we <u>are given</u> maps $\phi_{ij}: M_j \to M_i$ where $i \leq j$, & if $i \leq j \leq k$,

$$\begin{tikzcd} M_k \arrow[r, "\phi_{jk}"] \arrow[rr, bend left, "\phi_{ik}"] & M_j \arrow[r, "\phi_{ij}"] & M_i \end{tikzcd}$$

$\Rightarrow \phi_{ik} = \phi_{ij} \circ \phi_{jk}$

- <u>Our ex</u>: $M_i = A/I^i$ $\quad M_j \twoheadrightarrow M_i$ for $i \leq j$ b/c modding out bigger set in M_i. $([a] \mapsto [a])$

- We can define the inverse limit

$$\varprojlim_n M_i = \{(m_1, m_2, \ldots) \mid m_i \in M_i \text{ \& } \phi_{ij}(m_j) = m_i \ \forall i \leq j\}$$

<u>Ex</u>: $A = k[x]$, $I = (x)$.

$\hat{A} = \{(f_1, f_2, \ldots)\} = k[[x]]$ b/c look like polys but can have ∞ powers

$\underset{+(x)}{f_1 = a_0}$, $\underset{+(x^2)}{f_2 = a_0 + a_1x}$, $\underset{+(x^3)}{f_3 = a_0 + a_1x + a_2x^2}$

Note: f_2 has same const as f_1 b/c $\phi_{ij}(m_j) = m_i$

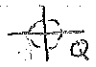
Def: The analytic local ring of a pt P on a variety X is $\widehat{\mathcal{O}}_{X,P}$ = completion of $\mathcal{O}_{X,P}$ wrt $\mathfrak{m}_P$.

- This is a complete local ring.
- in ex above, $k[\![x]\!]$ is the analytic local ring of the origin in the affine line
  - ↳ corresp to $\mathfrak{m}_0 = (x)$
  - ↳ corresp to $k[x]$

Def: Two points $P \in X$, $Q \in Y$ are said to have the same analytic type of singularity if $\widehat{\mathcal{O}}_{X,P} \simeq \widehat{\mathcal{O}}_{Y,Q}$ as k-alg's.

- If X not birational to Y, P & Q cannot have same local ring.

Cohen Structure Thm: Let P be a smooth point on a variety X, of dimension d. Then $\widehat{\mathcal{O}}_{X,P} \simeq k[\![x_1, \dots, x_d]\!]$ (i.e., this is the completion of a regular local ring)

Ex: Look at $X = Z(y^2 - x^2(x+1))$  and $Y = Z(xy)$ 

$\mathcal{O}_{X,P} \not\simeq \mathcal{O}_{Y,Q}$: $\mathcal{O}_{Y,Q}$ has zero divisors ($xy = 0 \cdots$, even in local ring)
↑domain (always true if X a variety)

Claim: $\widehat{\mathcal{O}}_{X,P} \simeq \widehat{\mathcal{O}}_{Y,Q}$

Pf: ① ∃ power series $g(x,y), h(x,y) \in k[\![x,y]\!]$ s.t.

$g = (y+x) + g_2 + g_3 +$ h.o.t. (ie no const & deg 1 part = x+y)
$h = (y-x) + h_1 + h_2 +$ hot. and $gh = y^2 - x^2(x+1)$.
→ so poly. no longer irreducible in completion.

Pf: deg 2 of gh is $y^2 - x^2$
deg 3 of gh is $(y+x)h_2 + (y-x)g_2$ & rhs is $-x^3$
deg 4: $(y+x)h_3 + (y-x)g_3 + g_2h_2 = 0$
possible b/c know $g_2 h_2$ & same arg

} possible (ideal gen) $(y+x, y-x) = (x,y)$ ⇒ any poly of deg ≥ 1 can be gen max ideal of origin

② There is an automorphism $k[[x,y]] \circlearrowleft \phi$ s.t. $\phi(x)=g$ & $\phi(y)=h$.

ϕ descends to an iso

$$\begin{tikzcd} k[[x,y]]/(xy) \arrow[r, "\sim"] \arrow[d, equal] & k[[x,y]]/(gh) \arrow[d, equal] \\ \hat{\mathcal{O}}_{Y,Q} & \hat{\mathcal{O}}_{X,P} \end{tikzcd} \qquad \square$$

$gh = y^2 - x^2(x+1)$

(uses fact that taking quotients commutes w/ completions, i.e. completion is exact)

Read: Miles Reid, "La Correspondance de McKay".

Let $G \leq SL(2,\mathbb{C})$ a finite subgp.

<u>ex</u>: $A_{2n} = \langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \rangle$, $\varepsilon = e^{2\pi i/2n}$ $(\simeq \mathbb{Z}/2n\mathbb{Z})$

$D_{4n} = \langle \underset{\substack{x \to \varepsilon x \\ y \to \varepsilon^{-1} x}}{\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}}, \underset{xy \to y(-x)}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \rangle$ "binary dihedral gp"

E_6, E_7, E_8 (exceptional gps). These are all the gps.

$\Rightarrow$ relation to classification of Dynkin diagrams of s.s. Lie gps.

<u>McKay</u>: $G \curvearrowright \mathbb{C}^2$ (b/c all 2x2 matrices do). Form the quotient $\mathbb{C}^2/G$. [a fin. gp acting by reg. auto's on sp then quotient is variety]

To understand $X = \mathbb{C}^2/G$ as an affine variety, we want to know $\mathcal{O}(X)$.

$$\begin{tikzcd} \mathbb{C}^2 \arrow[d, two heads, "\pi", "\substack{\text{surj} \\ \text{dom}}"'] \\ \mathbb{C}^2/G \end{tikzcd} \Rightarrow \mathcal{O}_X \overset{\pi^*}{\hookrightarrow} \mathbb{C}[x,y]$$

$\leftarrow$ pullback of fcns $\rightarrow$ the set of fcns that are G-invariant, b/c they're constant on orbits

$\Rightarrow \mathcal{O}_X = \mathbb{C}[x,y]^G$.

<u>Ex</u> If $G = D_{4n}$; $u = x^2y^2$ is an invariant polynomial

$v = x^{2n} + y^{2n}$ — " —

$w = xy(x^{2n} - y^{2n})$ — " —

$\mathbb{C}[u,v,w] \xrightarrow{\phi} \mathbb{C}[x,y]$, $u \mapsto x^2y^2$, $v \mapsto x^{2n} + y^{2n}$, $w \mapsto xy(x^{2n} - y^{2n})$

$\operatorname{Im}\phi = \mathcal{O}(X) = \mathbb{C}[x,y]^G$; i.e. these 3 generate all G-invar. polys.

Let $I = \ker\phi$. Then $Z(I) \subseteq \mathbb{A}^3_{u,v,w}$ is iso. to $\mathbb{C}^2/G$.

(b/c $\mathbb{C}[u,v,w]/\ker\phi \cong \operatorname{Im}\phi = \mathcal{O}(X)$)

$Z(I)$ a 2-dim. variety in 3-sp, so only looking for one eqn: I: $\boxed{w^2 - vu^2 + 4v^{n+1} = 0}$

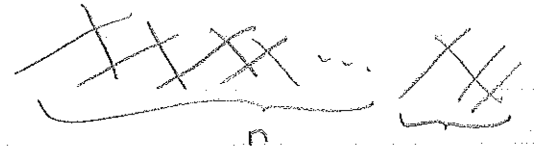
- This space only singular at origin, b/c only get singularities at fixed pts of action & for D_{4n}, that's only the origin. (G acts freely on $\mathbb{C}^2 \setminus \{0\}$, (of codim < 2)) so smooth.

- Start resolving the singularity at 0.

exceptional locus = $\pi^{-1}(0)$.

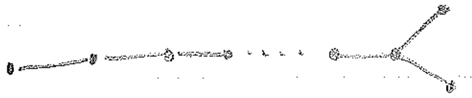
$$\begin{tikzcd} \tilde{X} \text{ final blowup} \arrow[d, "\pi"] \\ X \end{tikzcd}$$

The exceptional locus, $\pi^{-1}(0)$ under



} a bunch of curves that are $\mathbb{P}^1$ → each meets prev. in 1 pt

dual graph: (replace each $\mathbb{P}^1$ w/ Vertex & $\mathbb{P}^1$ meets $\mathbb{P}^1$ ⇒ edge)



Dinkin Diagram for D_{2n}

2 curves at each blowup w/ a sing pt on one of them until final blowup

10/17 "Minimal Model Program" = distinguished, "nice" representative in each birat'l class.

Goal: Prove that:
(a) Any q. proj. curve is birat'l to a smooth proj. one.
(b) Two smooth proj. curves ⇔ are isomorphic.
[(b) breaks down for surfaces b/c $\mathbb{P}^1$ & $Bl(\mathbb{P}^1)$ are birat'l but not isomorphic]

recall, need tr. deg 1 b/c want to get a curve (so deg 1)

Idea: Given a f.g. field ext'n K/k of transcendance deg. 1, produce a smooth proj. curve w/ fcn. field K.

Obs: If X is a curve s.t. $K(X) = K$, P a smooth pt on X, then $\mathcal{O}_{X,P}$ is a regular local domain of dim 1. (f.g. k-alg).
- domain b/c curve irred
- reg b/c P smooth pt
- local b/c $\mathcal{O}_{X,P}$ always local
- dim 1 b/c X has dim 1.

Thm: Let A be a noetherian local domain of dim 1, max'l ideal $\underline{m}$. TFAE:
(a) A is regular
(b) A is integrally closed
(c) $\underline{m}$ is principal
(d) A is a DVR.

Ex: $\mathbb{Z}_{\mathfrak{p}}$, $\mathfrak{p}$ a prime ideal → all ideals principal
→ $\mathbb{Z}$ int cl ⇒ all localizations are int cl.

- Similarity btwn $\mathbb{Z}$ & rings of curves.

Def: Let K be a field, G an ordered abel. gp. (like $\mathbb{Z}$)
A valuation on K is a fcn $v: K \setminus \{0\} \to G$ s.t.
(a) $v(xy) = v(x) + v(y)$
(b) $v(x+y) \geq \min\{v(x), v(y)\}$ if $x+y \neq 0$.
The valuation ring of v is $R_v = \{x \in K \setminus \{0\} \mid v(x) \geq 0\} \cup \{0\}$.
[or let $v(0) = \infty$ & $\infty > g \ \forall g \in G$]
R_v is a subring of K, local w/ max'l ideal
$\underline{m}_v = \{x \in K \setminus \{0\} \mid v(x) > 0\} \cup \{0\}$.

Ex: v a valuation on $\mathbb{Q}$: $v: \mathbb{Q} \setminus \{0\} \to \mathbb{Z}$
Fix p a prime. Let $v(x) =$ largest power of p which divides x.
(can be negative if p div. denom.)
So $v_2(3/16) = -4$
Note: $R_{v_2} = \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in 2\mathbb{Z}+1\} = \mathbb{Z}_{(2)}$

Def: • If $k \subseteq K$, a valuation v of K is a valuation of K/k if $v(x) = 0 \ \forall x \in k$.
• A valuation is discrete if $G = \mathbb{Z}$
• A ring R is a DVR if there is a discrete valuation v on the field of fracs K of R s.t. $R = R_v$.

Def: A domain is a Dedekind domain if it is an int. cl. noeth. dom. of dim 1.
[all localizations at primes are DVR's - like $\mathbb{Z}$]

Thm: Let A be a Dedekind dom, K its field of fracs, $L \supseteq K$ a fin. field ext'n. Then the int. closure of A in L is also a Dedekind dom.

Obs: If X is a smooth, $P \in X$, then $\mathcal{O}_{X,P}$ is a DVR, $\hookrightarrow$ b/c int. cl.
$\mathcal{O}_{X,P} \subseteq K = K(X)$.

Def: Let K be a f.g. field ext'n of tr. deg 1.
Define $C_K = \{v \mid v$ a discrete valuation rings of $K/k\}$
[like if $K = \mathbb{Q}$, then C_K = all nonzero prime ideals]
(& ignore k)

- If X is a smooth, qproj curve, get a map $X \xrightarrow{\Phi} C_K$ ($K = K(X)$) given by $P \mapsto \mathcal{O}_{X,P}$ (rather, the valuation that produces $\mathcal{O}_{X,P}$, i.e. v_P)
[$v_P : K \to \mathbb{Z}$]
v_P is answering $\longrightarrow$ "to what order does the fcn vanish at the pt?
(if doesn't vanish, then has a pole, so take neg order of pole.)
$\hookrightarrow$ [so $v_P(f)$ = order of 0
$\uparrow$ input is a rat'l fcn]

Claim: Φ is injective.

Lemma: If P, Q are smooth pts on a curve X, if $\mathcal{O}_{X,P} \subseteq \mathcal{O}_{X,Q}$, then $P = Q$.
"If every fcn that's regular at P is also reg at Q, then the pts must coincide."

Pf: Embed X in $\mathbb{P}^n$. Find a hyperplane H which avoids both P & Q. Then $X \setminus H$ is qaffine & $P, Q \in X \setminus H$, so wlog, can assume X is q-affine (in fact affine by taking closure in $\mathbb{A}^n$).
Now have $A = \mathcal{O}(X)$, & $\underline{m}_P, \underline{m}_Q$ max'l, &
$A_{\underline{m}_P} \subseteq A_{\underline{m}_Q} \Rightarrow \underline{m}_P \supseteq \underline{m}_Q$, but both max'l
$\Rightarrow \underline{m}_P = \underline{m}_Q \Rightarrow P = Q$ $\square$

(think of x as a rat'l fcn on curve)

Thm: Let K be a fcn field of dim 1 $/k$. Let $x \in K$. Then $\{R \in C_K \mid x \notin R\}$ is a finite.

"The set of pts where a rat'l fcn is not defined is finite"

$\to$ the valuation rings that don't contain x — but DVR's are the local rings from pts at which curve defined (& smooth).

Pf: $x \notin R \Leftrightarrow 1/x \in \underline{m}_R$ (from props of val ring. $x \notin R \Rightarrow v(x) < 0 \Rightarrow v(1/x) > 0 \Rightarrow 1/x \in \underline{m}_R$).

WTS: For $y = 1/x \in K$, $\{R \mid y \in \underline{m}_R\}$ finite.

wlog, $y \notin k$ (b/c for $y \in k$, $v(y) = 0 \Rightarrow$ set $\emptyset$)

k alg. cl $\Rightarrow y$ not alg $/k \Rightarrow k[y] \subseteq K$ is a polynomial ring. This is a Ded. dom. of dim 1.

K is a finite field ext'n of $k(y)$ (b/c K has tr. deg 1, & so does $k[y]$, so $K/k[y]$ must be alg)

Let B be the int. cl of $k[y]$ in K. Then by thm, B is a Ded. dom. B also a f.g. k-alg, & its field of fracs is K (ie $B_{(0)} = K$), & it has no zero divisors. If X is the affine var. w/ $\mathcal{O}_X = B$, then $k(X) = K$.

* Given field, have produced a smooth affine var w/ fcn field K.

X is smooth b/c B is a Ded. dom, & so all localizations are int cl $\Rightarrow \mathcal{O}_{X,P}$ int cl $\forall P$

If $y \in R$, for some $R \in C_K \Rightarrow k[y] \subseteq R \overset{R \text{ int cl.}}{\Longrightarrow} B \subseteq R$.

Let $\underline{n} = \underline{m}_R \cap B$. Then $\underline{n}$ is a max'l ideal of B since $\underline{m}_R$ prime $\Rightarrow \underline{n}$ prime & B has dim $1 \Rightarrow$ all ideals are max or 0 & $\underline{n} \neq 0$,

$B \subseteq R$ $\qquad \underline{n} = \underline{m}_R \cap B$ max'l $\qquad B_{\underline{n}}$ b/c Ded.

$\Rightarrow (B_{\underline{n}}, \underline{n})$ is dominated by $(R, \underline{m}_R)$ (both DVR's)

(i.e. $B_{\underline{n}} \subseteq R$ & $\underline{n} = \underline{m}_R \cap B$).

Thm: Valuation rings are max'l wrt domination relation.

$\Rightarrow B_{\underline{n}} = R$. $\Rightarrow y \in \underline{m}_R$ for some $R \Leftrightarrow \exists\, \underline{n} \in B$

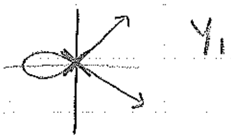
$\Leftrightarrow$ $\underline{\Omega}$ corresp. to a zero of y as a reg. fcn on X, but we've seen there are only fin. many of these $\square$

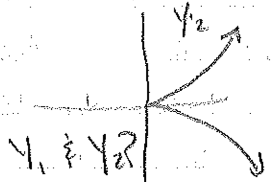
10/22 Last time: · For every field ext'n K/k f.g. of tr. deg 1, $\exists!$ smooth projective curve C w/ $K(C) = K$.
· We defined $C_K = \{R_p \subseteq K \mid R_p$ is a valuation ring of $K/k\}$

Ex: $K = k(x)$
$C = \mathbb{P}^1$: $\mathbb{P}^1 \supseteq \mathbb{A}^1$, $\mathcal{O}(\mathbb{A}^1) = k[x] \Rightarrow K(\mathbb{P}^1) = k[x]_{(0)} = k(x)$
↳ b/c can do it on affine subset

Thus any curve that's birational to $\mathbb{P}^1$ is $\cong$ to $\mathbb{P}^1$.
* If blow-up smooth pt on curve, nothing happens.

Digression: (Normal Cones)
Let $Y = Z(f)$ be a hypersurface in $\mathbb{A}^n$.
ex: $Y = Z(y^2 - x^3 + x^2) \subseteq \mathbb{A}^2$.
$T_0 Y = k^2$ (2 dim'l vect sp)
$Y_2 = Z(y^2 - x^3)$





($T_0 Y_2$ also 2-dim'l)

How do we distinguish Y_1 & Y_2?

Normal cone: look at Z(initial part of f) $\subseteq T_0 C$.
in(f) = throw away all monomials not of lowest deg.
$\text{in}(f_1) = y^2 - x^2$ (from $y^2 - x^3 - x^2$)
$\text{in}(f_2) = y^2$ (from $y^2 - x^3$)
in(f) is homogeneous $\Rightarrow$ its zero locus will be a cone $\subseteq T_0(C)$

Claim: If we blow up the origin in C, $\tilde{C} \xrightarrow{\pi} C$ ↖ blowup
then $\pi^{-1}(0) = \mathbb{P}(Z(\text{in} f)) \subseteq \mathbb{P}(T_0 C)$
↑ projectivization (exceptional locus is always the projetivization of the normal cone of the fcn)
(also true for arbitrary dimension C)

Ex: If we blow-up the origin in $\mathbb{A}^n$, the pts of the exceptional locus are identified w/ the projectivization, $\mathbb{P}(T_0 A)$.

$\mathbb{P}(Z(y^2-x^2)) = 2$ pts, b/c in affine sp, get 2 Lines, & each line corresp to a pt in $\mathbb{P}^2$ $(1,1), (-1,1)$

$\mathbb{P}(Z(y^2)) = 1$ pt, b/c only one sol'n $y=0$ $(x=1)$

- What is the exceptional locus if we blow-up a smooth pt on a variety of dim n? $\underline{\underline{\mathbb{P}^{n-1}}}$ (guess)
  - works for $\mathbb{A}^n$; for a hypersurface:

$X = Z(f) \subseteq \mathbb{A}^n$, X smooth at the origin

$\hookrightarrow \Rightarrow f$ must have linear term(s)

$\Rightarrow in(f)$ is a linear poly, $\neq 0$.

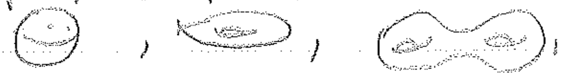
$\Rightarrow Z(in(f)) \cong \mathbb{A}^{n-1}$ $\swarrow$

$\Rightarrow E \cong \mathbb{P}(Z(in(f))) = \mathbb{P}(\mathbb{A}^{n-1}) = \mathbb{P}^{n-2}$ (ok, b/c dim $X = n-2$)

On a curve, preimage of pt is just the pt itself, b/c dim $C = 1$, so dim $\mathbb{P}(in(f)) = \mathbb{P}^0$.

Genus of a smooth proj. curve; the only discrete invariant, $\in \mathbb{Z}_{\geq 0}$

$\rightarrow$ over $\mathbb{C}$, end up w/ compact, smooth surface

so can get  , , , ...

genus 0 — genus 1 — genus 2 ...

genus 0 $\uparrow$ $\exists!$ $g=0$ curve, $\mathbb{P}^1$

genus 1 $\uparrow$ $\exists$ a whole $\mathbb{A}^1$ worth of $\neq$ curves w/ $g=1$, elliptic curves

$zy^2 = x(x-z)(x-az)$

$a \in \mathbb{C} \setminus \{0,1\}$

genus 2 $\nwarrow$ $\exists$ a whole 3-dim var. $(\underset{\top}{M_2})$ of curves (moduli space of gen. 2)

$$\begin{tikzcd}
C \arrow[d] & (x:y:z) \arrow[d, "2:1 \text{ cover}"] \\
\mathbb{P}^1 & (x:z)
\end{tikzcd}$$

(deg 2) ~~(cubic)~~ ext'n of $k(x)$

$$K(C) = \left(k[x,y] \big/ (y^2 - x(x-1)(x-a)) \right)_{(0)}$$

$\llcorner$ (in $g \geq 24$, cannot nicely parametrize the curves!)

- **Prop**: $\forall x \in K$, $x \neq 0$, $\{R_P \in C_K \mid x \notin R_P\}$ is finite.

Crucial part of proof: $y = 1/x$ $(y \notin k)$

$k[y] \subseteq B =$ int. cl. of $k[y]$ in $K \subseteq K$

$\uparrow$ Dedekind B f.g. ring whose localization of max. ideal is R_P.

We constructed a max ideal $\mathfrak{m}_P \subseteq B$ s.t. $B_{\mathfrak{m}_P} = R_P$ if $y \in R_P$.

Thm: Let C be any curve s.t. $k(C) = K$ & $P \in C$ smooth. Then $\mathcal{O}_{C,P} \in C_K$. Conversely, $\forall R_P \in C_K$, $\exists$ curve C & $P \in C$ smooth s.t. $\mathcal{O}_{C,P} \cong R_P$.

[Prev:

$$\begin{aligned} C &\longrightarrow C_K \\ P &\longmapsto \mathcal{O}_{C,P} \leftarrow \text{a val. ring of } K \Rightarrow \text{elt of } C_K. \end{aligned}$$

want: if C a sm. proj. curve, want map to be an iso.

Now have: $\forall R_P \in C_K$, it's hit by some curve w/ fcn field K.]

Pf: Pick $y \in R_P$, $y \notin k$. Take the affine var. corresp. to B (as above, the int closure of $k[y]$ in K, a f.g. k-alg w/ no zero divs) $\Rightarrow C$ a smooth affine curve. B can from construction w/ max'l ideal $\underline{m}$. $\updownarrow P \in C$

s.t. $B_{\underline{m}} = R_P$

$\parallel$

$\mathcal{O}_{C,P}$

Def: Put a topology on C_K by making finite sets closed. For $U \subseteq C_K$ open, define $\mathcal{O}(U) = \bigcap_{R_P \in U} R_P$

(parallel to $\mathcal{O}(U) = \bigcap_{\substack{\text{inside } K \\ P \in U}} \mathcal{O}_{C,P}$ for a curve C)

$\hookrightarrow$ reg fcns on U are reg at each pt, so lies in local ring at each $P \Rightarrow$ int.

If $f \in \mathcal{O}(U)$, can we produce a fcn $\bar{f} : U \to k$?
Let $P \in U$, i.e. $P = R_P$ is a valuation ring in K.
$\bar{f}(P) = f \bmod \underline{m}_P$, where $\underline{m}_P$ is the max'l ideal in $R_P = P$. $R_P/\underline{m}_P \simeq k$ (b/c a DVR of K/k)

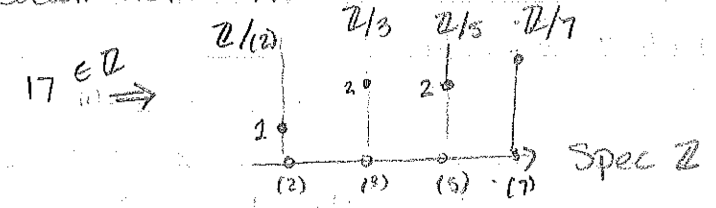
Ex: $C = \mathbb{P}^1$, $K = k(x)$, $f = x^2+1$, $U = \mathbb{P}^1 \setminus \{\infty\}$ (val. rings are in 1-1 corresp w/ $\mathbb{P}^1$)

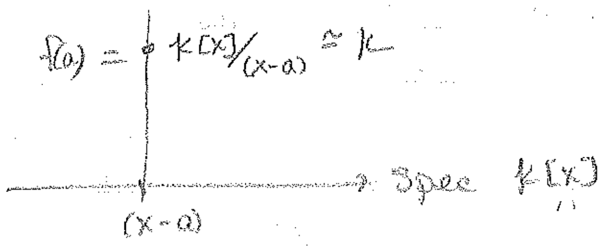
[Val. rings of $k(x)$ at p
- highest power of $(x-p)$ div. poly
and order of vanishing at ∞ (ie, degree)]

v = valuation at 0 "order of vanishing at 0"
$\updownarrow \hookrightarrow (0,1) \in \mathbb{P}^1$
$v(f) = 0$. $R_v = k[x]_{(x)}$ ← all polys in $k(x)$ that don't have pole at 0.
↖ quotients of polys but denom doesn't vanish at 0.

$\underline{m} = (x)$,
& $f \bmod \underline{m}$
$= x^2+1 \bmod (x) = 1$ $[= f(a)$ ← divide by $x-a$, the remainder is $f(a)]$
$\in k[x]_{(x)} / x k[x]_{(x)} \simeq k$

* Recall from 742:





$f \in k[x]$

Prop: The mapping $f \longmapsto \bar{f}$ is injective.
Pf: Assume $\bar{f} = \bar{g}$. Then $f - g \in K$, & $\bar{f} = \bar{g}$ on U
$\Rightarrow f - g \in \underline{m}_P \; \forall P \in U$. But then $f-g$ belongs to max ideal of ∞-ly many max ideals $\Rightarrow f - g = 0 \Rightarrow f = g$. □

Def: A fcn $g: U \to k$ is regular iff $g = \bar{f}$ for some $f \in \mathcal{O}(U)$.

10/24 K = f.g. field ext'n of k of transcendence deg. 1

Goal: Produce a curve w/ fcn. field K.

- If C were a smooth curve w/ $K(C) = K$, then how do we understand the relationship btwn pts on C & K?
  - If $P \in C$, we can look at its local ring, (ring of reg. fcns near P) $\mathcal{O}_{C,P} \subseteq K$. Call $\mathcal{O}_{C,P} = R_P$.
    - Then R_P is local
    - R_P is regular, since C smooth at P
    - dim $R_P = 1$, b/c on curve
    - R_P a domain b/c curve irred
    - $K = (R_P)_{(0)}$.

    But such rings are always valuation rings, by a thm of comm. alg.
  - Thus $\exists$ discrete valuation, $v: K\setminus\{0\} \to \mathbb{Z}$, with $\mathcal{O}_{C,P}$ = valuation ring of v
    - Why do 2 diff pts give 2 diff val. rings? B/c if $\mathcal{O}_{C,P} = \mathcal{O}_{C,Q}$, then $P = Q$, from prev. lemma [which just said if $\mathcal{O}_{C,P} \subseteq \mathcal{O}_{C,Q}$, then $P = Q$]
  - So we have a map:

$$\{P \in C\} \xrightarrow{\varphi} \{R_P \text{ a val. ring of } k\} = C_K$$

$$P \longmapsto R_P = \mathcal{O}_{C,P}.$$

  This map is injective by pt above.
  - Suppose we had a curve C s.t. the map above is surj. (i.e. an iso. of sets) Then what top. would be nec. on $\{R_P\}$ for the map to be homeo? The top. on $\{P \in C\}$ has finite sets as closed sets. So the nec. top. on $\{R_P\} = C_K$ is the top. where closed sets are finite (or everything).

- On C, we have a notion of regular fcns. How can we translate this to C_K?
  - Reg maps on C are encoded in $U \subseteq C \longleftrightarrow \mathcal{O}_U$ ring s.t. $\mathcal{O}(U) \subseteq K$, $\mathcal{O}(U)$ an int dom w/ field of fracs, K.

    for $\underset{\sim}{f} \in \mathcal{O}(U)$, $\bar{f}$ is a fcn on U, $\bar{f}: U \to k$.

    ↳ abstract elt of subring of K ⟨ map $U \to k$.

    [same elt, but thought of differently].
- On C, we have $\mathcal{O}(U) = \bigcap_{P \in U} \mathcal{O}_{C,P}$.
  - Thus, we can define $\mathcal{O}(U)$ for $U \subseteq C_K$: $\mathcal{O}(U) \subseteq K$ is $\mathcal{O}(U) = \bigcap_{R_P \in U} R_P$.
  - Note $U \subseteq C_K$ is just all but fin. many val rings of K (b/c U open)
- To construct fcns on $U \subseteq C_K$, we need to start w/ $f \in \mathcal{O}(U)$, & then associate an elt $f(R_P) \in k$ $\forall R_P \in U$.

  Take $\bar{f}(R_P) := f \pmod{\underline{m}_P} \in R_P/\underline{m}_P \cong k$.

  [essentially division w/ remainder alg - if have poly, to find value of f at a, we take $f \pmod{(x-a)}$, & $(x-a)$ is the max ideal assoc. to pt a]
  - If $R = k[x]$, $\underline{m}_P = (x-a)$, then there is a! k-alg iso $R/\underline{m}_P \cong k$, by $f \bmod \underline{m}_P \mapsto f(a)$.

Call $U \subseteq C_K$ the abstract nonsingular curve assoc. (like q-proj curve) to K, w/ the top. we defined & the notion of regular fcns above = it's a top. sp w/ a notion of a distingused set of fcns on its open sets. All notions of morphism from varieties extends to $U \subseteq C_K$.

Thus we have a category

Funny Varieties = Varieties
abstract nonsing. curves.

& we've defined morphisms.

We want to show Funny Varieties = Varieties, i.e. we haven't added any new curves.

Ex: Let $K = k(x)$. $C_K = \{v_a \mid va \in k\} \cup \{v_\infty\}$

Valuations on K:

- $v_0(f/g) = v_0(f) - v_0(g)$, where $v_0(f)$ = highest power of x dividing f.
  ex: $v_0\left(\frac{x^2+2x+1}{x(x-1)}\right) = -1$.
  - think of this as the order of the zero or pole at the origin.
- $v_a(f/g) = v_a(f) - v_a(g)$, where $v_a(f)$ = highest power of $(x-a)$ dividing f.
  → this gives one valuation for each pt in $\mathbb{A}^1$. But we know $\mathbb{P}^1$ has fcn field K, so we should have one more val.
- $v_\infty(f) = -\deg f$ (deg of pole or zero at ∞).

[Thm: These are all the valuations of $k(x)$.]

Take U = "finite $\mathbb{A}^1$" $= \mathbb{P}^1 \setminus \{\infty\}$. Want to write this inside C_K: So $U = \{v_a \mid a \in k\}$ is an open set $\subseteq C_K$. (open b/c threw away fin. many pts).

Then $\mathcal{O}(U) = \bigcap_{a \in k} R_{v_a} = \{f/g \mid v_a(f) - v_a(g) \geq 0 \;\forall a \in k\}$

(since $R_{v_a} = \{f/g \mid v_a(f) - v_a(g) \geq 0$ for this $a\}$)

Claim: $\mathcal{O}(U) = k[x]$ (i.e. $g = 1$)

If $g = 1$, $v_a \geq 0 \;\forall a$. Conversely, reduce f/g to lowest terms. If $g \notin k$, then $\exists$ a zero z of g, not a factor of f, $\Rightarrow v_z(f/g) = 0 - (\text{positive } \#) < 0$ ↯.

→ This is exactly the statement that reg fcns on $\mathbb{A}^1$ are just polynomials.

- If $U = C_k$, then $\mathcal{O}(U) = k$, b/c now have $(\bigcap_{a \in k} R_{v_a}) \cap R_{v_\infty} = k[x] \cap R_{v_\infty}$, so deg is non-neg, but also deg is non-positive. $\Rightarrow$ deg $= 0$.

  $\rightarrow$ This is: only reg fcns on proj. variety are constant.

- If $U = C_k \setminus \{v_0\}$, then $\mathcal{O}(U) = k[1/x]$

  $\mathcal{O}(U) = \{ f/g \mid v_a(f) - v_a(g) \geq 0 \ \ \forall a \in k \setminus \{0\}$, and $\underline{\deg g - \deg f} \geq 0 \}$

  $\hookrightarrow$ from $v_\infty(f/g) \geq 0$

  $\Rightarrow \deg g \geq \deg f$. But $\forall a \neq 0$, if $(x-a)^i \mid f$, then $(x-a)^j \mid g$ w/ $j < i$, so f can have more $(x-a)$'s than g, except at $a = 0$, so can add as many x's to g as we want. so if reduce f/g, $g = x^n$. $\Rightarrow f/g = \frac{f(x)}{x^n}$ s.t. $n \geq \deg f$.

  Claim: this is $h(1/x)$ for some $h \in k[x]$.

  Thus $\mathcal{O}(U) = k[1/x]$, so get another copy of $\mathbb{A}^1$, b/c reg fcns are polys in $y = 1/x$.

- Now, we want to do the same thing for an arbitrary K.

Thm: Every smooth curve is isomorphic to an abstract non-sing. curve.

Pf: Let X be a smooth curve. Let $K = C(X)$ (rat'l fcns on X) & let $\phi : X \to C_K$, $P \mapsto \mathcal{O}_{X,P}$. ϕ clearly a reg fcn, so ϕ will be an iso if we prove $C_K \setminus \phi(X)$ is finite.

i.e. the image is an open subset, which is what we called an abs. non-sing. curve.

Take an affine open in X. If this maps to an open set, so will X, so wlog, assume X is affine. Call $\mathcal{O}(X) = A$. Then $\phi(X) = \{ A_{\underline{m}} \mid \underline{m}$ max'l in $A \}$. (b/c a pt in X corresp to max'l ideal, & the local ring $\mathcal{O}_{X,P} = A_{\underline{m}}$.)

But $\forall \underline{m}$, $A \subseteq A_{\underline{m}}$ & $A_{\underline{m}}$ is a DVR b/c X smooth

Thus $\Phi(X) = \{R \subseteq K \mid R \text{ DVR}, A \subseteq R\}$.

$\subseteq$ obvious

$\supseteq$: given $A \subseteq R \subseteq K$, R a DVR, $R = A_{\underline{m}}$ for some max'l ideal $\underline{m}$: Look at $\underline{b} \cap A = \underline{m} \subseteq A$, for $\underline{b}$ the ! max. ideal of R. [uses that DVR's are max'l subrings].

Recall, A a f.g. k-alg, so let $x_1, \dots, x_n$ be gens.

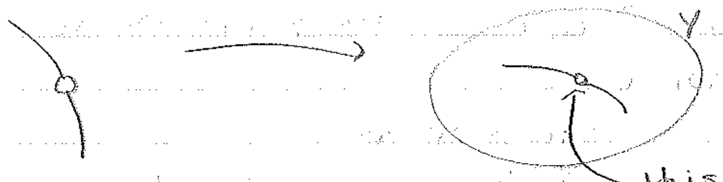
Thus $\phi(X) = \{R \subseteq K \mid x_1 \in R, x_2 \in R, \dots, x_n \in R\}$.

$$= \bigcap_{i=1}^{n} \underbrace{\{R \in C_K \mid x_i \in R\}}_{\text{open in } C_K \text{ so/c int. of fin many opens}}$$

$\Rightarrow \Phi(X)$ is open in C_K. □

<u>Thm</u>: Let X be an abstract non-sing curve, Y a proj. variety, $P \in X$. Then, if $\phi : X \setminus \{P\} \to Y$ is a morphism, $\exists!\ \Phi : X \to Y$ s.t. $\Phi|_{X \setminus \{P\}} = \phi$, Φ a morphism.

→ Given a proj. var., & know how to map all but one pt to Y, there's a ! way to extend it.



this pt exists in Y, i.e. Y is complete.

→ This lets us take one-directional limits

- Really sequential cptness (every Cauchy seq. has limit in sp) → This is the alg. geom. equivalent of compactness.

$\Rightarrow$ called separability.

- Hausdorffness: (In a Hausdorff sp, every seq. w/ a limit has a ! limit)
  Same thm, but remove $\exists$ $\rightarrow$ so the fcn may not be able to be extended, but if you can, then it can be extended uniquely.

Ex: Look at $f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$, $x \longmapsto 1/x$. Cannot be extended at $\{0\}$.

But, if $f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$, $x \longmapsto 1/x$ can be extended by $0 \longmapsto \infty$.

[$\mathbb{P}^1$ covered by $\mathbb{A}^1_x$ & $\mathbb{A}^1_{1/x}$, they int at $\mathbb{P}^1 \setminus \{0,\infty\}$]

- Ok for f to be defined on affine sp, b/c choose affine cover around P. $\hookrightarrow X \setminus \{P\}$

In general, to extend in $\mathbb{P}^1$, on the patch of $\mathbb{P}^1$ whose coord. is $1/x$. Then the pull-back $f^*(1/x) = x$, regular not just on $X \setminus \{P\}$, but also on X.

Idea of Pf: $\Phi : X \setminus \{P\} \Rightarrow \mathbb{P}^n$ given, X curve, find patch $U_i \subseteq \mathbb{P}^n$, $U_i \cong \mathbb{A}^n$ s.t. $\Phi^*(x_j)$ is regular on all of X, where x_i are coords on U_i.

10/29 (not quasi)

Thm: Let C be an abstract nonsingular curve, $P \in C$, Y a proj. variety, $\Phi : C \setminus \{P\} \to Y$ a morphism. Then $\exists! \, \overline{\Phi} : C \to Y$ extending Φ.

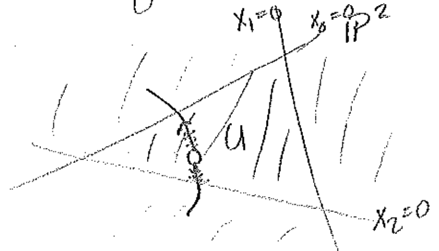
Pf: ① Replace Y by $\mathbb{P}^n$ b/c:

$$\begin{tikzcd} C \setminus \{P\} \arrow[r, "\Phi"] \arrow[d, phantom, "\subseteq" rotate=-90] & Y \arrow[r, hook] & \mathbb{P}^n \\ C \arrow[rru, "\overline{\Phi}\ ?"'] & & \end{tikzcd}$$

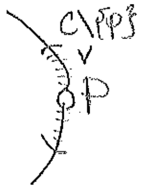
$\leftarrow$ if we construct this, then b/c Y closed, $\overline{\Phi}(C) \subseteq Y$.
(b/c im(closure) $\subseteq$ cl(image)
$\rightarrow \overline{\Phi}(\overline{C \setminus \{P\}}) \subseteq \overline{\overline{\Phi}(C \setminus \{P\})} \subseteq \overline{Y} = Y$.
$\overline{C \setminus \{P\}} = C$, so $\overline{\Phi}(\overline{C \setminus \{P\}}) = \overline{\Phi}(C)$

Wlog, $Y = \mathbb{P}^n$.

② let $U \subseteq \mathbb{P}^n$ be the open set $U = \{x_1 \neq 0, \; \& \; x_2 \neq 0 \; \& \cdots \& \; x_n \neq 0\}$.
Wlog, can assume $\exists$ nbhd V of P s.t. $\Phi(V \setminus \{P\}) \subseteq U$.



(these are the boundary lines)



How could this fail? i.e. when does such a V not exist? → When $C\setminus\{P\}$ lands inside one of the lines. (one of the coordinate hyperplanes) $_\wedge$ say inside $x_n=0$. But if this is the case, $\{x_n=0\}\cong\mathbb{P}^{n-1}$. By induction, repeat w/ $\mathbb{P}^n$ replaced by $\mathbb{P}^{n-1}$. At some pt, this must stop.

③ On $\mathcal{U}$, we have regular fcns $\frac{x_i}{x_j}$ $\forall i,j$. (b/c $x_j\neq 0$ on $\mathcal{U}$)

$\phi: \underset{C\setminus P}{\overset{\cap\!\shortmid}{V}} \longrightarrow \mathcal{U}$ regular, $f_{ij}=\underbrace{\frac{x_i}{x_j}\circ\phi}_{\text{rat'l fcn.}\Rightarrow\in K}: V\to k$ regular.

$P\in C \Leftrightarrow v_P$, a valuation on K.

Define $a_{ij}=v_P(f_{ij})\in\mathbb{Z}$. Define $r_i=a_{i0}$.

$f_{ij}\cdot f_{jk}=f_{ik} \Rightarrow a_{ij}+a_{jk}=a_{ik}$ (from prop's of valuations)

In particular, $a_{ij}=a_{ik}-a_{jk}=r_i-r_j$ (set $k=0$)

Pick k s.t. $r_k=\min\{r_i\}$ (only fin. many & they're int's, so $\exists$ a smallest)

$\Rightarrow a_{ik}\geq 0$ $\forall i$.

(b/c $a_{ik}=r_i-r_k\geq 0$, since r_k smallest)

$\Rightarrow f_{ik}$ actually defined at P, b/c the valuation is non-neg, so fcn doesn't have a pole.

$\Rightarrow f_{ik}$ extends to a regular fcn $\bar{f}_{ik}: V\cup\{P\}\to k$.

[regular fcns on an open set are elts of K whose valuations at all pts of the open set are nonneg]

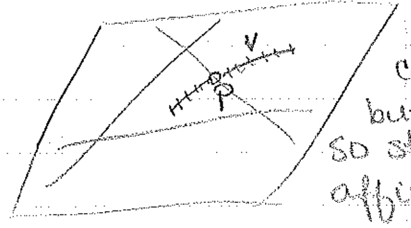
$\Rightarrow$ The map $\Phi: V\cup\{P\}\to \mathbb{A}^n_{\{x_k\neq 0\}}\subseteq\mathbb{P}^n$

$x\longmapsto \left(\bar{f}_{0k}(x), \bar{f}_{1k}(x), \ldots, \widehat{\bar{f}_{kk}(x)}, \ldots, \bar{f}_{nk}(x)\right)$

is regular. In V, the coords were exactly x_i/x_k, and at P, we have the ext'n.

Conclusion: We have extended $\phi|_V: V\to\mathcal{U}$ to

$$\begin{tikzcd}
\Phi: V\cup\{P\} \arrow[r] & \mathbb{A}^n_{\{x_k\neq 0\}}\subseteq\mathbb{P}^n \\
V \arrow[u, phantom, "\cup\!\shortmid"] \arrow[r, "\phi|_V"'] & \mathcal{U} \arrow[u, phantom, "\cup\!\shortmid"]
\end{tikzcd}$$



P on a coord hyp-plane but only on one, so still in an affine patch.

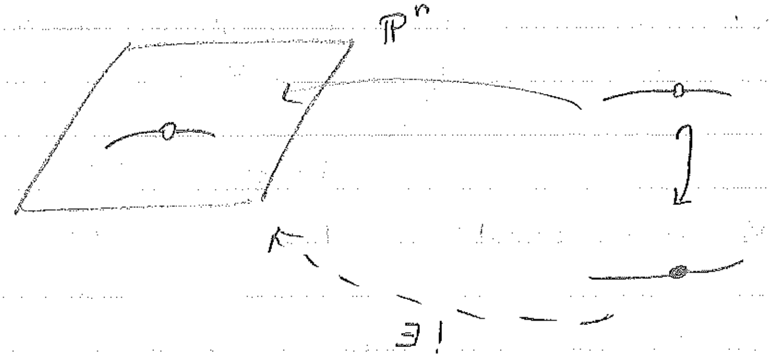
Done □

→ the valuation allowed us to choose an affine patch

R a DVR. Consider Spec R:

It's the germ of a smooth pt on a curve. $\underline{m}$

↳ b/c every DVR is the local ring of a pt on curve.



-every map from the germ of a curve w/o the pt maps to a full curve.

What we've done:

birational class of curves
↓
Fcn field, K
↓
Abs. non-sing curve, C_K
↓
smooth proj curve

<u>Thm</u>: Every abstract nonsingular curve C_K is isomorphic to a smooth projective curve.

<u>Pf</u>: ① C_K can be covered by finitely many open sets, $\{U_i\}_{i=1}^{k}$ each of which is isomorphic to an affine curve.
[From earlier construction w/ Dedekind domain (B)
$P \in C_K \to B$ a f.g. k-alg, so corresp to an affine var]
Only need fin. many b/c take one affine patch - then complement is finite, so take the affine patches of those pts.

② Each $U_i \subseteq \mathbb{A}^{n_i} \subseteq \mathbb{P}^{n_i}$. Define $Y_i = \overline{U_i} \subseteq \mathbb{P}^{n_i}$. (projective closure) ↙not nec. smooth

③ We have a map

$$\begin{tikzcd} U_i \arrow[r, hook, "\phi_i"] \arrow[d, equal] & Y_i \text{ proj.} \\ C\setminus\{P_1,\dots,P_k\} \arrow[r, phantom, "\subseteq"] & C \arrow[u] \end{tikzcd}$$

(by using prev. thm repeatedly)

So we get $\overline{\Phi}_i : C \to Y_i$, which are isomorphism on U_i.

④ Get a map

$$\Phi: C \longrightarrow \prod_{i=1}^{k} Y_i \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k} \subseteq \mathbb{P}^N$$
$$P \longmapsto (\overline{\Phi}_1(P), \overline{\Phi}_2(P), \dots, \overline{\Phi}_k(P))$$

Claim: This map is an isomorphism onto the closure of its image $Y \subseteq \prod Y_i \subseteq \mathbb{P}^N$.
[Y smooth, b/c reg. prop & $\mathcal{O}_{Y,P} \cong \mathcal{O}_{C,P} \leftarrow$ a DVR]

Pf: We have a comm. diagram of morphisms:

$$\begin{tikzcd}
P \in C \arrow[r, "\phi"] \arrow[dr, dashed, "\bar{\Phi}_i"] & Y \ni \phi(P),\ Y \subseteq \prod Y_i \arrow[d, "\pi_i"] \\
P \in U_i \arrow[u] \arrow[u, bend left, "\text{inclusion}"] \arrow[r, hook, "\phi_i"'] & Y_i \ni \phi_i(P)
\end{tikzcd}$$

(Y a curve b/c image of curve)

π_i = proj. onto i^{th} component

all maps are dominant. (all nonconstant maps of curves are dominant)
So we get inclusions

$$\mathcal{O}_{\phi_i(P), Y_i} \hookrightarrow \mathcal{O}_{\phi(P),Y} \hookrightarrow \mathcal{O}_{P,C}$$ b/c dom. maps induce inclusions of local rings

But $\mathcal{O}_{\phi_i(P),Y_i} \cong \mathcal{O}_{P,C} \Rightarrow$ all inclusions are iso's:
$\mathcal{O}_{P,C} \cong \mathcal{O}_{\phi(P),Y} \cong \mathcal{O}_{\phi_i(P),Y_i}$.

We have:

- $\forall P \in C$, $\mathcal{O}_{\phi(P),Y} \cong \mathcal{O}_{C,P}$ by map $\phi \Rightarrow \mathcal{O}_{\phi(P),Y}$ a regular local ring $\Rightarrow$ C maps by ϕ into smooth part of Y.
- ϕ is surj: Let $Q \in Y$. Then $\exists P \in C \overset{=C_K}{} $ s.t. $\mathcal{O}_P$ dominates $\mathcal{O}_{Y,Q}$. Look at $\phi(P) = Q' \in Y$. $\mathcal{O}_{Y,Q'} \cong \mathcal{O}_{P,C}$ dominates $\mathcal{O}_{Y,Q} \Rightarrow Q' = Q$. $\Rightarrow Q \in \mathrm{Im}(\phi)$.

$\Rightarrow$ (1) ϕ is surj
(2) ϕ^* is an isomorphism on local rings
(3) Y is a smooth proj curve
(4) ϕ is injective, [b/c distinct pts of U_i map injectively through diagram into Y. if they're in 2 U_i's, map to 2 diff components of Y
$\Rightarrow \phi$ is a homeomorphism & induces iso on local rings $\Rightarrow \phi$ is an iso (by an exercise)

Clean argument for injectivity

$P_1 \neq P_2 \in C \Rightarrow$
$\mathcal{O}_{P_1,C} \neq \mathcal{O}_{P_2,C}$
$\Rightarrow \mathcal{O}_{\phi(P_1),Y} \neq \mathcal{O}_{\phi(P_2),Y}$
$\Rightarrow \phi(P_1) \neq \phi(P_2)$.

2 distinct pts $\Leftrightarrow$ 2 dist. DVR's $\Leftrightarrow \phi$ maps to 2 diff pts w/ distinct DVR's b/c ϕ^* iso. $\square$

10/31 **Fact**: Every morphism btwn smooth proj. curves is either onto or constant.

"**Reason**": Projective varieties are "compact."

X cpt $\Rightarrow$ f(X) cpt if f cts & Y Hausdorff $\Rightarrow$ f(X) closed inside Y. (Top argument, but it fails in our case) But:

Thm: If X projective, $f: X \to Y$ a morphism, then f(X) closed in Y.

Pf of Fact: X proj curve $\Rightarrow$ X proj & X irred $\Rightarrow$ f(X) closed & irreducible in Y. But Y a curve $\Rightarrow$ f(X) is a pt or Y. (these are the only cl. irred subsets of a curve)

Cor 1: Every abstract nonsingular curve is isomorphic to a quasiproj. curve.

↑ open set in C_k

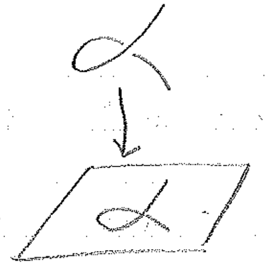
↑ open set of proj. curve.

Cor 2: Every curve is birat'l to a smooth proj curve.

Pf: Let X be a curve, $K = K(X) \Rightarrow$ f.g. field of tr. deg 1/k. Let $C = C_K$. C is iso. to a smooth proj. curve, Y. $K(C_K) = K \Rightarrow K(Y) = K(C_K) = K \Rightarrow K(X) = K(Y) \Rightarrow$ X is birat'l to Y. □

[More intuitive idea – embed X in proj sp & take closure. Then blow up singular points, repeatedly, which doesn't change the birational type, & hope the process terminates]

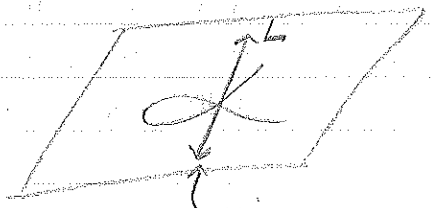
- For affine varieties, $X \subseteq \mathbb{A}^n$. If $R = \mathcal{O}(X)$, let $\overline{R}$ be the int. closure of R in $k = k(X)$. If R = domain, f.g. /k $\Rightarrow$ $\overline{R}$ = domain, f.g. /k $\Rightarrow$ $\overline{R} = \mathcal{O}(Y)$ for some affine Y. $R \hookrightarrow \overline{R} \Rightarrow$ get a map of varieties $Y \to X$ dominant. Y is the <u>normalization</u> of X. (A variety Y is <u>normal</u> is $\mathcal{O}_{Y,P}$ is int. closed $\forall P \in Y$) In dim ≥ 2, normalization need not be smooth, but in dim 1, Y is smooth, b/c int cl. local domain of dim 1 $\Rightarrow$ regular.



$$\begin{tikzcd} k[t] \\ k[x,y]/(y^2+x^2-x^3) \arrow[u, hook] \end{tikzcd}$$

(the int closure of ?)

Ex:



via an automorphism of $\mathbb{P}^2$, can make this line a line at ∞. Then in $\mathbb{P}^2 \setminus L$, an affine patch, the curve is smooth, but it's projective closure is not.

Divisors & Maps to $\mathbb{P}^n$ (Ch. 2 of Hartshorne)

- Maps from $X \to \mathbb{A}^n$ are just n regular fcns on X.

  B/c: $\underset{Y \text{ affine}}{\mathrm{Maps}(X;Y)} = \mathrm{RingMaps}_k(\mathcal{O}(Y), \mathcal{O}(X))$

  Then $\mathrm{Maps}(X, \mathbb{A}^n) = \mathrm{Ring\,maps}_k(k[x_1, \dots, x_n], \mathcal{O}(X))$

  so this map defined by where $x_1, \dots, x_n$ go, & this is same as picking n elts of $\mathcal{O}(X)$ to send them.
- Maps from proj sp to $\mathbb{A}^n$ must be constant, since $\mathcal{O}(X)$ const.

Let V be a f.d. vector sp, $\mathbb{P} = \mathbb{P}(V)$ be the corresp. proj. sp.

Claim: The set $\{L \subseteq \mathbb{P} \mid L \text{ hyp-plane}\} \cong \mathbb{P}(V^*)$

Pf: A hyperplane $L = Z(f)$, f homogeneous & linear

$\{L\} \xleftrightarrow{1-1} \{f \text{ homog. of deg } 1, f \neq 0,\}/\sim \qquad f \sim \lambda f \ \forall \lambda \neq 0.$

But such f are elts of $V^* \Rightarrow \{L\} \longleftrightarrow (V^* \setminus \{0\})/\sim = \mathbb{P}(V^*)$,
But the dual of the dual is the sp (if V fin dim).

Call $\mathbb{P}(V^*) = (\mathbb{P}V)^*$. Then

$$\mathbb{P}V = (\mathbb{P}V^*)^*$$

$$\overset{\Psi}{P} \longmapsto \{H \in \mathbb{P}V^* \mid P \in H\}. \quad \underline{\text{WTS:}} \text{ in } (\mathbb{P}V^*)^*.$$

Let $P = (t_0 : \cdots : t_n)$, let $H \in \mathbb{P}V^*$, $H = Z(f)$

$$f = a_0x_0 + \cdots + a_nx_n : \underset{x_0,\dots,x_n}{V} \longrightarrow k \text{ a lin. fcnal}$$

$P \in H \Rightarrow a_0t_0 + \cdots + a_nt_n = 0$. If $\{t_i\}$ fixed, then this is an eqn in the $\{a_i\}$, & it's a lin eqn on coords of hyperplane. Thus $\{H \in \mathbb{P}V^* \mid P \in H\}$ = hyperplane in $\mathbb{P}V^*$.

$\Rightarrow$ pt in $(\mathbb{P}V^*)^* =$ $[\subseteq \mathbb{P}V^*$, & actually linear$^{\nwarrow}]$
$\{$Linear subsets in $\mathbb{P}V^*\}$

[pts in original sp are hyperplanes in dual]

<u>Ex</u>: Let C be a smooth curve in $\mathbb{P}^2$, ($C \xhookrightarrow{f} \mathbb{P}^2$) If $L \subseteq \mathbb{P}^2$ is a line, let $D_L = C \cap L \subseteq C$. Let $\deg C = d$

Generally $D_L = d$ pts on C

$$L \longmapsto D_L \subseteq C$$

We have a map $(\mathbb{P}^2)^* \xrightarrow{\Phi} \{d\text{-pt subsets of } C\}$

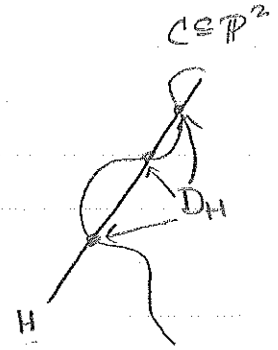
(i.e, these d-pt subsets are parametrized by a dual hyperplane)

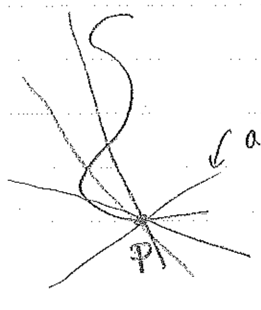
Φ allows us to recover the original map $C \xhookrightarrow{f} \mathbb{P}^2$.

<u>How</u>? Given $P \in C \longmapsto \{H \in (\mathbb{P}^2)^* \mid \Phi(H) \ni P\}$

[all lines through P in $\mathbb{P}^2$

this is a hyperplane in $(\mathbb{P}^2)^*$ & so a pt in $\mathbb{P}^2$.





all D_H's that hit P.

If H_1, H_2 are 2 hyperplanes in $\mathbb{P}^n$, then there exists a rat'l fcn, f, on $\mathbb{P}^n$ s.t. zeros$(f) = H_1$ & poles$(f) = H_2$; outside of $H_1 \cap H_2$:

Write $H_1 = Z(g)$, where g is linear homog.
$H_2 = Z(h)$, ——— " ———

Take $f = g/h$. (ok b/c g and h have same degree)
$\in K(\mathbb{P}^n)$

If $L, L' \in (\mathbb{P}^2)^*$, then $(D_L, D_{L'})$ has the property that
(*) ∃ rat'l fcn f on C s.t. zeros$(f) = D_L$ & poles$(f) = D_{L'}$.

Given a fixed D_L, look at all other $D_{L'}$ s.t. (*) holds. The space of all such $D_{L'}$ is a projective space!! & all such $D_{L'}$ must have come from some embedding of the curve in a projective space.

"A map to $\mathbb{P}^n$ will be given by a choice of a divisor."
(statement must be qualified, of course).
↑ Big Idea

hyp-plane will cut out a bunch of codim 1 sub-varieties by Krull's Hauptidealsatz

Def: Let X be smooth (or later w/ ok sing's). A prime divisor Y on X is an irreducible codimension 1 sub-variety of X. A Weil divisor on X is a formal finite sum w/ integer coeff. of prime divisors, i.e. an elt of the free abel. gp gen'd by prime divisors. D is effective if all its coeffs are ≥ 0.

Ex: If C is a curve, a divisor D on C is of the form $n_1p_1 + n_2p_2 + \cdots + n_kp_k$, where $p_i \in C$.
[coefficients will deal w/ the multiplicities of $L \cap C$]

If f is a rat'l fcn on X, $(\in K(X))$ $Y \subseteq X$ is irred & codim 1, define $V_Y(f)$ = order of vanishing of f along Y.
$Y \leftrightarrow$ valuation ring of K.

$Y \longleftrightarrow$ valuation ring.

Let $U \subseteq X$ affine open which intersects Y.

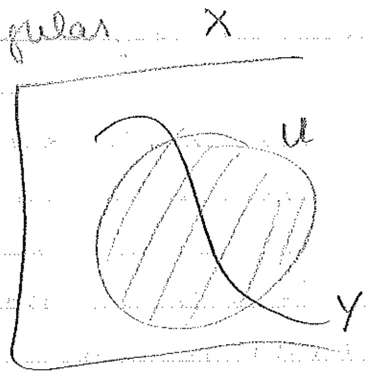
$\mathcal{O}(U)$. int cl. b/c all local rings regular

$\exists$ prime ideal of ht 1 $\mathfrak{p} \subsetneq \mathcal{O}(U)$

b/c irred subsets corresp. to prime ideal & dim of subset = ht of $\mathfrak{p}$

$\Rightarrow \mathcal{O}(U)_{\mathfrak{p}}$ is a dim 1 $\xleftarrow{\text{b/c } \mathfrak{p} \text{ ht } 1.}$ int. cl. local ring $\Rightarrow \mathcal{O}(U)_{\mathfrak{p}}$ a val. ring of k.

Call the valuation v_Y.



<u>Def</u>: If $f \in k(X)$, define the <u>divisor of f</u>

$$(f) = \sum_{Y \text{ prime div.}} v_Y(f) \cdot Y$$

<u>Ex</u>: • On $\mathbb{A}^1$, $(x^2) = 2 \cdot O$, where O = origin.

$$(x(x-1)) = \underset{\uparrow \text{pt at } 0}{O} + \underset{\nwarrow \text{pt at } 1}{1}$$

$$\left(\frac{1}{x}\right) = -O$$

• On $\mathbb{P}^1_{x_0:x_1}$, $\left(\frac{x_0}{x_1}\right) = \underset{\uparrow = [0:1]}{0} - \underset{\nwarrow = [1:0]}{\infty}$

11/5 <u>Divisor</u>: • $Y \subseteq X$ (X smooth) is a <u>prime divisor</u> if Y irred. of codim 1.

• A <u>divisor</u> $D = \sum_{i=1}^{n} a_i Y_i$, $a_i \in \mathbb{Z}$, Y_i prime div's.

Came from observation: If $X \subseteq \mathbb{P}^n$, then $X \cap H \overset{= D_H}{}$ is a divisor on X for every $H \subseteq \mathbb{P}^n$ hyperplane.
(Think of a curve intersecting a line in $\mathbb{P}^2$ - then $X \cap H$ is a bunch of pts)

• $\{D_H\}_{H \in (\mathbb{P}^n)^*}$ is a family of effective divisors parametrized by pts of the dual $\mathbb{P}^n$.
  - D is <u>effective</u> if $a_i \geq 0$.

• If $f \in K(X)$, we should construct $(f) = (\text{zeros of } f) - (\text{poles of } f)$, a <u>principal divisor</u>.

• $D_{H_1} - D_{H_2} = (f)$

• If $X = \operatorname{Spec} \mathbb{Z}$, pts = max ideals in $\mathbb{Z}$, i.e. prime #'s, (p).
  dim $X = 1$ ($0 \subsetneq (p) \to$ only chain)
    $(2), (3), (5), (7), \ldots$ are codim 1 subsets (pts)
  A divisor is: $(2)a_2 + (3)a_3 + \cdots$ (finite)
  A rat'l fcn on X is: $K(X) = \mathbb{Q}$, & for $f \in \mathbb{Q}$,
  $(f) =$ [ex: $\left(\frac{16}{75}\right) = 4(2) - 1(3) \cdot 2(5)$]

• If $X = \mathbb{P}^1$, prime divisors = pts of $\mathbb{P}^1$, either $x \in k$ or ∞.
  A divisor is a lin. comb: ex: $3 \cdot \text{"}5\text{"} + 1 \cdot \text{"}6\text{"} - 7 \cdot \text{"}\infty\text{"}$ ← the pt 5 on $\mathbb{P}^1$
    - this is not a principal divisor b/c
  A rat'l fcn on $\mathbb{P}^1$: $\frac{(x-1)(x-3)}{(x+2)(x-7)} = f(x)$ $\left[\text{or } \frac{(x-y)(x-3y)}{(x+2y)(x-7y)} \right]$

  $(f) = 1 \cdot \text{"}1\text{"} + 1 \cdot \text{"}3\text{"} - 1 \cdot \text{"}-2\text{"} - 1 \cdot \text{"}7\text{"}$ (no pole or zero at ∞ b/c $\lim_{x \to \infty} = 1$)

  - $f(x) = \frac{x-1}{(x+2)^2}$: $(f) = 1 \cdot \text{"}1\text{"} - 2 \cdot \text{"}-2\text{"} + 1 \cdot \text{"}\infty\text{"}$
    ↑ goes to ∞ like $1/x$.

- $f(x) = x$, $(f) = 1\cdot$"0" $- 1\cdot$"∞".

In all examples, the sum of coeffs is 0 in all principal divisors on $\mathbb{P}^1$ ↑ $\underline{\text{degree}}$ of divisor

- If $f \in K(X)^*$?

If $Y \subseteq X$ is a prime divisor, pick some affine $U \subseteq X$ s.t. $U \cap Y \neq \phi$. Want to define a_Y = "order of vanishing of f along Y."

$a_Y = v_Y(f)$, where $v_Y : K(X) \setminus \{0\} \to \mathbb{Z}$ a valuation

(b/c we want $a_{YZ} = a_Y + a_Z$)

If $\mathcal{O}(U) = R$, R int dom, $R_{(0)} = K(X)$. $Y \cap U$ = codim 1 irred in $U \Rightarrow Y \cap U = Z(\mathfrak{p})$, $\mathfrak{p} \subseteq U$ prime of ht 1

Then $R_{\mathfrak{p}}$ is a DVR of $K(X)$. Take v_Y = valuation assoc. to $R_{\mathfrak{p}}$.

Ex: If $X = \mathbb{P}^1$, $K(X) = k(x)$

P = "0". Take usual finite $\mathbb{A}^1$ nbhd of "0" for U

$R = k[x]$, $P \leftrightarrow \underline{m} \subseteq k[x]$, $\underline{m} = (x)$

$R_{\mathfrak{p}} = k[x]_{(x)}$ is a DVR of $k(x)$ for valuation "order of vanishing at 0"

- The correct condition on X is "smooth in codim 1," i.e. $\text{Sing}(X) \subsetneq X$ has codim ≥ 2. Then our construction will ensure $R_{\mathfrak{p}}$ will be a DVR.

- For $f \in K(X)$, define $(f) = \sum_{Y \text{ prime div.}} v_Y(f) \cdot Y$.
↑ essentially the multiplicities

Ex: $f = xy$ on $\mathbb{A}^2$

$(f) = 1\cdot$"x-axis" $+ 1 \cdot$"y-axis"
↑ need prime divisor to have codim 1.

$k[x,y]_{(x)}$ local ring w/ $\underline{m}$. Q: Find smallest k s.t. $f \in \underline{m}^k$. $\underline{m} = (x) \cdot k[x,y] \Rightarrow k = 1$

$\Rightarrow v_{(x)}(xy) = 1$.

Why does $(f) = \sum_{Y} v_Y(f) \cdot Y$ have finite support?

<u>Prop</u>: $v_Y(f) \neq 0$ for only finitely many Y's.

<u>Pf</u>: $f \in K(X)$. Pick U affine, along which f is regular. (b/c f reg. on some open set, & in every open set there's an affine).

(1) $X \setminus U$ closed, so it has only fin. many prime divisors contained in it.

(2) Wlog, restrict attn to divisors Y s.t. $Y \cap U \neq \emptyset$.

Note: f reg on $U \Rightarrow v_Y(f) \geq 0$ (b/c no poles on U), & $v_Y(f) > 0 \Leftrightarrow \underbrace{Y \subseteq Z(f)}_{}$.

proper closed subset of U, so has fin. many components. □

<u>Def</u>: The <u>class group</u> $Cl(X) = {}^{\text{Div}(X)}/_{\text{Princ. Div.}(X)}$

- $\text{Princ. Div}(X) \leq \text{Div}(X)$ b/c $(f/g) = (f) - (g)$ so cl. under subtraction.

<u>Prop</u>: If X is affine, $R = \mathcal{O}(X)$ is a UFD $\Leftrightarrow$ X is normal & $Cl(X) = 0$.

<u>Pf</u>: $(\Rightarrow)$: prime divisors in X corresp. to primes $\mathfrak{p}$ of ht 1 in R. But R UFD $\Rightarrow$ such $\mathfrak{p}$ are principal $\overset{\mathfrak{p}=(f)}{\Rightarrow}$ prime div's on X are principal $((f) = 1 \cdot \mathfrak{p})$ so all divisors are principal. □

<u>Cor</u>: $Cl(\mathbb{A}^n) = 0$: b/c $\mathbb{A}^n$ affine & $\mathcal{O}(\mathbb{A}^n) = k[x_1, \dots, x_n]$ is a UFD.

<u>Thm</u>: For $D \in \text{Div}(\mathbb{P}^n)$, $D = \sum a_i Y_i$, define $\deg D = \sum a_i \cdot \deg(Y_i) \in \mathbb{Z}$. Then

(1) $\deg D = 0$ if D is principal $\Rightarrow \deg : Cl(\mathbb{P}^n) \to \mathbb{Z}$

(2) $\deg : Cl(\mathbb{P}^n) \to \mathbb{Z}$ is an iso.

__Pf__: ~~Given a homog. poly $g \in k[x_0, \dots, x_n]$, define $(g) \in \mathrm{Div}(X)$ by: write $g = g_1^{\alpha_1} \cdots g_k^{\alpha_k}$ (b/c UFD) where g_i's irred & homog. Then $Z(g_i) \subseteq \mathbb{P}^n$ is a hypersurface in $\mathbb{P}^n$ of deg equal to deg g_i.~~

(1):

① If $Y \subseteq \mathbb{P}^n$ is a prime divisor, define $\deg(Y) = \deg f$ s.t. $Y = Z(f)$, f irred.

② If g is homogeneous, define
$(g) = \sum a_i Z(g_i)$ if $g = \prod g_i^{a_i}$, g_i irred (UFD)
Then $\deg(g) = \deg g$ (obvious)

③ If $f = g/h \in k(\mathbb{P}^n)$, then
$(f) = (g) - (h)$. Now since $\deg g = \deg h$, then $\deg(f) = 0$.

(2): $\deg: \mathrm{Cl}(\mathbb{P}^n) \to \mathbb{Z}$ is surjective (b/c $1 \cdot H$ has deg $\underset{\uparrow \text{gen of } \mathbb{Z}}{\underline{1}}$)

If D has deg 0, then it is principal.
If $D = \sum a_i \cdot Y_i$, write each Y_i for which $a_i \geq 0$ as $Z(g_i)$, & $a_i < 0$ as $Z(h_j)$. Take $f = \prod \frac{g_i^{a_i}}{h_j^{a_j}}$, a quotient of homog. polys of same deg since $\deg D = 0$, so $f \in K(\mathbb{P}^n)$ w/ $(f) = D$. □

__Def__: $D \sim D'$ (__linearly equivalent__) if $D = D'$ in $\mathrm{Cl}(X)$
$\Leftrightarrow D - D' \in \mathrm{Princ.Div}(X)$.

__Thm__: (Equivalently) $\forall D \in \mathrm{Div}(\mathbb{P}^n)$, $D \sim (\deg D) \cdot H$,
H = hyperplane.

- If D given, $D = \sum a_i Y_i$, form same f as before, but $\deg g \neq \deg h$: $\deg g - \deg h = \deg D$.
$\Rightarrow f / x_0^{\deg D} \in K(\mathbb{P}^n)$, & $(f / x_0^{\deg D}) = D - (\deg D) \cdot H$
$\Rightarrow D \sim (\deg D) \cdot H$.

Ex: $f \in K(\mathbb{P}^2)$, $f = \frac{x_0^2 - x_1x_2}{x_0x_1}$

$(f) = 1 \cdot Q - 1 \cdot H_0 - 1 \cdot H_1$

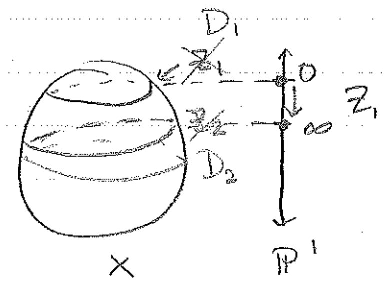
$Q = Z(x_0^2 - x_1x_2)$ $(\deg Q = 2)$

$H_0 = Z(x_0)$ $(\deg H_0 = \deg H_1 = 1)$

$H_1 = Z(x_1)$

$\Rightarrow Q \sim 2H$, b/c if $f = \frac{x_0^2 - x_1x_2}{x_0^2}$, then $(f) = Q - 2H$.

11/7



$Z_1 \sim Z_2$ (homologous) b/c $\exists$ cts deformation $\overset{f_t}{\wedge}$ of Z_1 onto Z_2.

$D_1 \sim D_2$ (linearly equiv), both effective divisors

$\Rightarrow \exists f \in K(X)$ s.t. $Z(f) = D_1$ & poles$(f) = D_2$

(b/c $D_1 - D_2 \in \mathrm{Div}(f)$)

$f \in K(X) \longleftrightarrow f: X \to \mathbb{P}^1$ (use finite part if f defined, else f has pole, so send it to ∞)

*so really, 2 effective divisors are lin. equiv if there's a def. retr. btwn them parametrized by $\mathbb{P}^1$. (by a rat'l fcn)

On $\mathbb{P}^n$, any 2 hyperplanes H_0, H_1 are linearly equivalent.

ex: $H_1 = (\Sigma a_i x_i = 0)$, $H_0 = (\Sigma a_i' x_i = 0)$

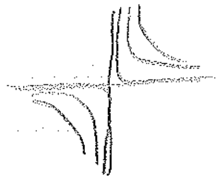
Let $H_t = (\Sigma (t a_i + (1-t) a_i') x_i = 0)$

t a family of hyp. planes: $\{H_t\}_{t \in \mathbb{A}^1}$

change coords so that $0 \to 0$ & $1 \to \infty$,

& have interpolations

Ex: look at $Z(xy + t(x^2+y^2)) = Q_t$, $t \in \mathbb{A}^1$



$Q_1 = Z(x^2+xy+y^2)$ = smooth quadric

$Q_0 = Z(xy)$ = union of 2 lines

$\Rightarrow Q \sim H_1 + H_2 \subseteq Cl(\mathbb{P}^2)$

i.e, a smooth quadric can be transformed into the union of 2 lines.

(rat'l fcn would be $\frac{x^2+xy+y^2}{xy}$ in $\mathbb{A}^2$)

→ On $\mathbb{P}^2$, as long as 2 equations have same degree, the 2 varieties are lin. equiv.

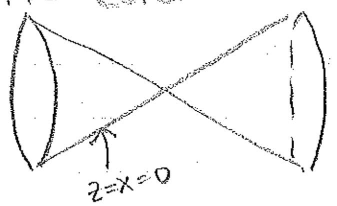
Ex: $Z(x^2 + t(y^2+z^2))$

$\uparrow$ double line "fat line" $2 \cdot H$ as a divisor

$\uparrow$ smooth quadric

→ first degenerate into 2 lines & then transform so one line lies on top of other.

Ex: $X = Z(z^2 - xy) \subseteq \mathbb{A}^3$ the quadric cone



smooth in codim 1, b/c sing. is a pt. of codim 2.

$Cl(X) = \mathbb{Z}/2\mathbb{Z}$ gen. by $(z=x=0) = L$

$(x) = 2L$ → the plane $x=0$ is tangent to cone

$x=0 \Rightarrow z^2=0$, so get zero of order $\underline{\underline{2}}$.

$2L \sim 0$ in $Cl(X)$, but $L \nsim 0$ in $Cl(X)$

i.e. $2L$ is a principal divisor ← need to show this.

Thus $Cl(X)$ has torsion.

$(z) \Rightarrow z=0$ gives $xy=0 \Rightarrow$ 2 lines

Claim: $I(L)$ is not gen by a single elt (i.e. not a principal ideal in $R = \mathcal{O}(X) = k[x,y,z]/(z^2-xy)$ → not a UFD

→ R has ht 1 prime ideal that's not principal, b/c R not a UFD.

In particular, $I=(x,z)$ is one example.

R not a UFD: $z^2 = z\cdot z$

$z^2 = x\cdot y$

$\left.\begin{array}{l} R/I = k[y] \text{ an int dom} \\ \text{or } I \text{ cuts out line} \Rightarrow I \text{ irred} \end{array}\right\} \Rightarrow I$ prime.

I is ht 1 b/c X dim 2, line dim 1 $\Rightarrow$ ht $I = 1$.

I is not principal: $\dim T_{X,0} = 3$

≤ 3 b/c quadric in 3-space

$\neq 2$ b/c then it would be = variety's dim & thus X smooth ↯ $\Rightarrow$ > 2 & $\leq 3 \Rightarrow = 3$

$\dim T_{L,0} = 1$ b/c L a line. (& so smooth $\Rightarrow \dim L = \dim T_{L,0}$)

- if L were cut out by f, $T_{L,0} = \{ \underset{\uparrow \text{ linear part of } f}{f'} = 0 \text{ inside } T_{X,0} \}$
  (i.e., if I were principal)

[- if smth cut out by 1 eqn, dim of $T_{X,0}$ can go down by $\underbrace{1 \text{ or by } 0}_{\text{from } \dim X}$.]

$\Rightarrow \dim T_{L,0} \geq 2$ ↯.

We've shown $\mathbb{Z}/2\mathbb{Z} \subseteq Cl(X)$.

$(P \neq \infty)$

Look at $X = \mathbb{P}^1$, $P \in \mathbb{P}^1$, $D = 2P$. Consider the set $\{D' \mid D' \text{ effective } \& \ D' \sim D\}$. $\mathcal{L}(D)$, the linear system of D.

- D' is $2P'$ or $P'+Q$. b/c only the degree matters on $\mathbb{P}^1$.

$$\mathcal{L}(D) = \{ f \in k(X) \mid (f) = D' - D,\ D' \text{ effective} \}$$

$f(x) = \dfrac{(x-A)(x-B)}{(x-P)^2}$ ← must be deg 2, else would be pole or zero at ∞.

P fixed, A, B free.

$\Rightarrow \mathcal{L}(D) = \{\text{quadratic polys in } x \text{ \& } y\}$, a f. dim linear v. sp.

$\dim \mathcal{L}(D) = 3$, $\underbrace{\{x^2, xy, \& \ y^2\}}$ basis — remove 0

Then $|\mathcal{L}(D)| = \{D' \mid D' \text{ effective}\} = \mathbb{P}\mathcal{L}(D)$, b/c if you mult. f by a constant, zeros & poles don't change.

① Start w/ D effective

$\mathcal{L}(D) = \{ f \in K(X) \mid (f) = D' - D,\ D' \text{ effective} \}$ is a v.s.

$|\mathcal{L}(D)| = \mathbb{P}\mathcal{L}(D) = \{ D' \mid D' \text{ effective},\ D' \sim D \}$.

Also, $|\mathcal{L}(2P)| = \mathbb{P}^2$

or $\mathcal{L}(D) = \{ f \in K(X) \mid (f) + D \text{ is effective} \}$

Ex: Find all mero. fcns on $\mathbb{P}^1$ w/ pole of order at most 2 at origin, & no other poles

$\frac{f(x)}{x^2}$ no other poles $\Rightarrow$ deg $f \leq 2$ (else would have pole at ∞)

$\dim\{f\} = 3$, b/c a poly of deg ≤ 2 given by 3 coeffs.

Ex: $\dim(\mathcal{L}(nH))$ on $\mathbb{P}^m$?

$H = (x_0 = 0)$

$\mathcal{L}(nH) = \{ f \in K(\mathbb{P}^m) \mid (f) + nH \text{ is effective (i.e. } \geq 0) \}$

$f = g/h$, $g, h \in k[x_0, \dots, x_m]$ of same degree.

$f = \frac{g}{x_0^n}$ w/ deg $g = n$ (ok for g to have x_0's, since $(f) + nH \geq 0$ not just $= 0$)

↑ b/c $(f) = \text{effective} - \underline{nH}$ $\Rightarrow x_0^n$

$\Rightarrow$ only g important

So $\mathcal{L}(nH) = \{ g \in k[x_0, \dots, x_m] \mid \deg g = n \}$

$\Rightarrow \dim \mathcal{L}(nH) \overset{?}{=} \binom{m+n}{n}$

$\mathbb{P}^1$, $2H$, $m = 1$, $n = 2$: $\binom{3}{2} = 3$ ✓

On $\mathbb{P}^1$, $\dim \mathcal{L}(nH) = n+1$ (b/c have $\underbrace{x_0^n, x_0^{n-1}x_1, \dots, x_1^n}_{n+1}$)

Idea: Start w/ a divisor D. Look at $\mathcal{L}(D)$, & construct a map $X \rightarrow |\mathcal{L}(D)|$.

Ex: (from above) $\frac{f(x)}{x^2}$

$X = \mathbb{P}^1$, $D = 2P$, $|\mathcal{L}(D)| = \mathbb{P}^2$ (b/c $\dim \mathcal{L}(D) = 3$).

So there's a map

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ → 2-uple embedding.

$(x_0, x_1) \longmapsto (x_0^2; x_0x_1; x_1^2)$

↑ exactly the basis of the lin. vect sp. we found

want divisors in lin. sys. to be sections of image w/ hyperplanes? back to original ex: lines through pt to get divisors.

If $D = nH$, this gives the n-uple embedding of $\mathbb{P}^m$ into $\mathbb{P}^N$, $N = \binom{m+n}{n} - 1$.

4 Steps:

(1) Choose a divisor, D

(2) Construct $\mathcal{L}(D)$

(3) Pick $L \subseteq \mathcal{L}(D)$ a vector subspace

(4) Do 'black' magic & get pretend map $X \to \mathbb{P}L$

L is called a linear system - a linear subsp of sp. of all effective divisors.

L is a complete linear system if $L = \mathcal{L}(D)$.

Troubles:

(1) $\mathcal{L}(D) = 0$ sometimes. i.e. no fcn s.t. $(f) + D$ effective.

ex: $\mathcal{L}(-H)$ on $\mathbb{P}^n$ is 0.

((f) on $\mathbb{P}^n$ is never effective, has some + & some - ⇒ add more -, can't be +)

The divisor does not have enough sections

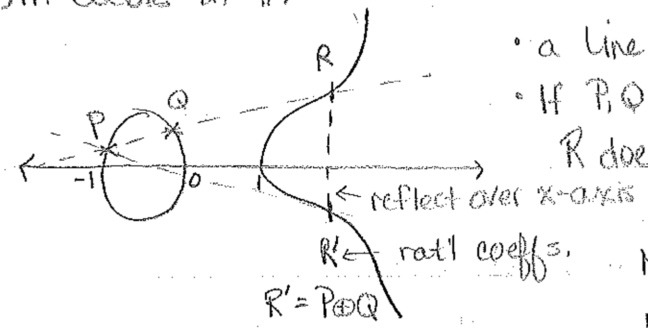
(2) ex: $\mathcal{L}(H)$ on $\mathbb{P}^n$

$\mathcal{L}(H) = \langle x_0, x_1, \dots, x_n \rangle$
$\cup\!|$
$L = \langle x_0, \dots, x_{n-1} \rangle$

$\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ projection from $(0:0:\cdots:0:1)$
$(x_0:\dots:x_n) \longmapsto (x_0:\cdots:x_{n-1})$

- not defined at $(0:\cdots:0:1)$. → base locus of a linear system.

11/12 Ex: Cubic $y^2 = x(x-1)(x+1) \subseteq \mathbb{A}^2$
or better, projectivize: $y^2z = x(x-z)(x+z) \subseteq \mathbb{P}^2$
Smooth cubic in $\mathbb{P}^2$. $\times 0 = \infty$ (identity)



- a line will meet curve in 3 pts
- If P, Q have rat'l coeffs, then R does as well.

Now repeat, & get a wealth of new pts.

- $(P, Q) \mapsto P \oplus Q$ (defined geometrically) is a group operation. identity = pt at ∞

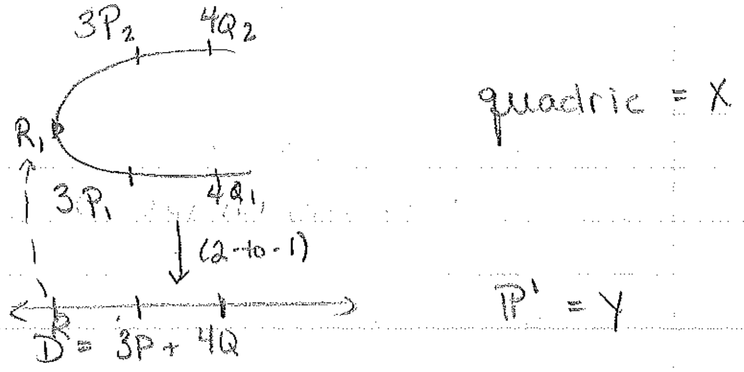
So C, or any elliptic curve, is a gp, abelian.
What is $\ominus R$? the refl. of R across x-axis, b/c $R \oplus R = 0$, i.e. 3rd pt of int. is ∞.
$0 \oplus R = R$.

Pic = Picard gp

Let's prove associativity using divisors:
Let $\mathrm{Pic}^\circ(X) = \{ D \in Cl(X) \mid \deg D = 0 \}$, where $\deg D = \sum a_i$ if $D = \sum a_i P_i$.
(on any curve)
$\mathrm{Pic}^\circ(X)$ is the gp of line bundles of degree 0.
① $\mathrm{Pic}^\circ(X)$ is an abel. gp.: well-def: $\deg(f) = 0$ on any curve b/c f defines a map $f: X \to \mathbb{P}^1$. On $\mathbb{P}^1$ have divisor $0 - \infty$, w/ deg 0.
Thm: $\exists f^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$ for any dominant

map of smooth proj. curves s.t. $\deg(f^*D) = \deg D \cdot \deg f$, where $\deg f = [K(X):K(Y)]$

Intuitively:



$$f^*D = \underbrace{3P_1 + 3P_2 + 4Q_1 + 4Q_2}_{\deg 14}$$
$\uparrow$ deg 7

- Intuitive only, b/c need $f^*R = 2R_1$ to get an iso

Then, $(f) = f^*(0-\infty)$, so $\deg f = \deg(0-\infty)\deg f = 0 \cdot \deg f = 0$

That $\text{Pic}^0(X)$ is abel gp is obvious

$\deg: \text{Cl}(X) \to \mathbb{Z}$ a hom, &

$\text{Pic}^0(X) = \ker(\deg) \Rightarrow$ subgp.

② The map $X \to \text{Pic}^0(X)$, pt at ∞, the origin

$P \mapsto P - O$

is surjective on a cubic.

i.e., any divisor $D = \sum a_i P_i$ s.t. $\sum a_i = 0$ is $(\sim)$ linearly equivalent to $P - O$ for some P.

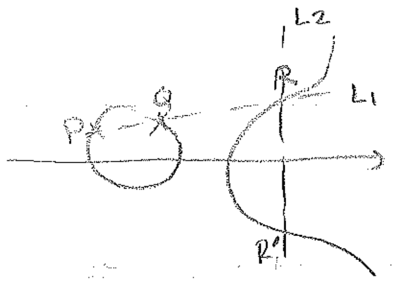
Pf: Claim: Any divisor $D = \sum_{i=1}^{n} P_i$ (may have repetitions) can be made deg 0 by $D = \sum_{i=1}^{n} P_i - nO$. This is $\sim$ to $(P_1 \oplus \cdots \oplus P_n) - O$. Then done, since

Given $D = \sum a_i P_i$, look at $\sum a_i (P_i - O) = D$ (b/c $\sum a_i = 0$)

But $D \sim P_1 \oplus \cdots \oplus P_n - O = Q - O$. (may need some $\ominus$ instead of $\oplus$)

Pf of Claim: The induction step is

$(P - O) + (Q - O) \sim (P \oplus Q) - O$ (next pg)



In $\mathbb{P}^2$, $L_1 \sim L_2$, so $\exists$ rat'l fcn $h \in K(\mathbb{P}^2)$ s.t. $L_1 - L_2 = (h)$.
Restrict h to C. (h defined everywhere but L_2, so get $h|_C \in K(C)$.

$$(h) = \underbrace{(P+Q+R)}_{\substack{\text{zeros of } h \\ \text{are } L_1}} - \underbrace{(O+R+R')}_{\substack{\text{poles of } h \\ \text{are } L_2}} = (P-O)+(Q-O)+(R'-O)$$

$\Rightarrow (P-O)+(Q-O) \sim (R'-O)$

& $R' = P \oplus Q$.

So we have constructed a map

$$\begin{aligned} C &\xrightarrow{\Phi} \mathrm{Pic}^0(C) \\ P &\longmapsto P-O \end{aligned}$$

- surjective
- a gp map, b/c $\Phi(P \oplus Q) = \Phi(P) + \Phi(Q)$ → if C is a gp

Claim: Φ is injective.

Prop: Let C be a smooth proj. curve, $P \neq Q \in C$. Then $P \sim Q \Leftrightarrow C \cong \mathbb{P}^1$ (i.e. C is rat'l)
[So if C not rat'l, any 2 distinct pts are distinct in the class gp.]

Pf: $(\Leftarrow)$ obvious

$(\Rightarrow)$ Assume $(f) = P-Q$, $f \in K(C)$. Then $f: C \to \mathbb{P}^1$ s.t.
$f^*(0-\infty) = P-Q$. $\Rightarrow f^*(0) = P \Rightarrow \underset{\substack{\| \\ \deg P \\ \| \\ 1}}{\deg f^*(0)} = \underset{\substack{\| \\ 1}}{\deg(0)} \cdot [K(C):K(\mathbb{P}^1)]$

$1 \cdot [K(C):K(\mathbb{P}^1)]$

$\Rightarrow [K(C):K(\mathbb{P}^1)] = 1 \Rightarrow K(C) = K(\mathbb{P}^1) \Rightarrow C \cong \mathbb{P}^1$, both smooth proj. □

Pf of Claim: Assume $\Phi(P) = \Phi(P')$. Then $P-O \sim P'-O$
$\Rightarrow P \sim P' \Rightarrow P = P'$ b/c C not $\mathbb{P}^1$. □

Line Bundles in Differential Geometry

If M is a manifold, a line bundle L over M is a manifold w/ a proj. map $\pi : L \to M$ s.t.

(a) $\exists$ cover $\{U_i\}$ of M s.t. $\pi^{-1}(U_i) \overset{\phi_i}{\cong} U_i \times \mathbb{R}^1$

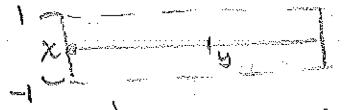
(b) The isomorphism $\phi_j \circ \phi_i^{-1} : U_i \cap U_j \times \mathbb{R}^1 \to U_i \cap U_j \times \mathbb{R}^1$ is linear in the fibers.

(a) & (b) is called a trivialization

Ex 0: $M \times \mathbb{R}^1$ (cover is M) Called the trivial line bundle

Ex 1: The möbius strip: $M = S^1$

$L =$ Möbius strip (open)



$(-1,1) \cong \mathbb{R}^1$

One open set : $\underset{U_1}{S^1 \setminus \{x\}}$, 2nd is $\underset{U_2}{S^1 \setminus \{y\}}$ both $\cong$, ok

$U_1 \cap U_2$: $\underbrace{(\quad) \cup (\quad)}_{\text{the gluing map}}$ is Id on one interval & -1 on other interval

Ex 2: Instead of manifold, just look at a space M, & replace $\mathbb{R}^1$ w/ k.

$\mathbb{P}^n$ has a natural line bunde on it, $\mathcal{O}(-1)$.

$\{\text{A pt } x \in \mathbb{P}^n\} \longleftrightarrow \{\text{line } L_x \subseteq \underset{V}{\underset{\parallel}{k^{n+1}}}\}$

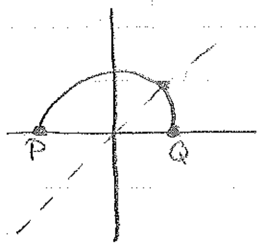
Look at $\mathbb{P}^n \times V$.

Then $\mathcal{O}(-1) \subseteq \mathbb{P}^n \times V$, by defining

$$\begin{tikzcd} \mathcal{O}(-1) = \{(x,v) \mid v \in L_x\} \arrow[d, "\pi"] & (x,v) \arrow[d, mapsto] \\ \mathbb{P}^n & x \end{tikzcd}$$

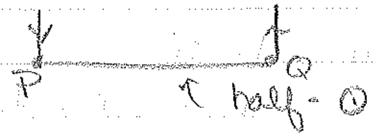
Claim: restricting π to $\pi^{-1}(U_i) \to U_i \subseteq \mathbb{P}^n$, U_i a basic affine open, trivializes this bundle.

Ex: For $\mathbb{RP}^1 \cong S^1$, this is the Mobius strip. Called tautological line bundle.



P=Q, b/c come from same line

But if start w/ line to Q, & rotate, get line in other dir when get to P.



⇒ mobius strip

- If X is smooth,

$\mathrm{Pic}(X) = \{\text{line bundles on } X\}$

$\cong$

$\mathrm{Cl}(X) = \{\text{Weil divisors}\}/\text{lin. equiv.}$

$\cong$

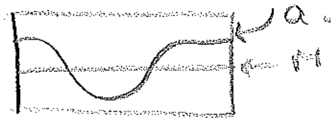
$\mathrm{CaCl}(X) = \{\text{Cartier divisors}\}/\text{lin. equiv.}$

11/14 Let $\begin{matrix} L \\ \downarrow \pi \\ M \end{matrix}$ be a line bundle. A <u>section</u> of L is a map $s: M \to L$ s.t. $\pi \circ s = \mathrm{id}$.

(assoc. to each pt in M a pt in the line that is over it, in a cts way)

Ex: If L is the trivial line bundle, $M \times \mathbb{R}^1 \to M$, then sections of L are the same as fcns on M

L:

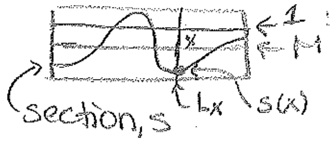


a section $f: M \to \mathbb{R}^1$

← M

Given a section $s: M \to M \times \mathbb{R}^1$, take $f = \pi_2 \circ s$, a fcn

In trivial line bundle, have a distinguished fcn 1,



& given a section s, can compare value of s to 1 at each pt.

A section, s, always gives a pt $s(x) \in L_x = \pi^{-1}(x)$ $\forall x \in M$. But if line bundle not trivial L_x is a 1 dim'l vector sp w/o a basis; ie there is no distinguished pt. So one pt $\not\Rightarrow$ number. But 2 pts in a 1 dim'l vect. sp can yield a #, b/c the ratio of 2 pts is well-def. even w/o basis (units of measure) [i.e. y is 2x as far from origin as x].

* If s_1, s_2 are sections of a line bundle, the ratio s_1/s_2 is a well-def quantity in $\mathbb{R}^1 \cup \{0\}$ (or $\mathbb{k}$), even though s_1, s_2 do not give values in $\mathbb{R}^1$.

→ similar to $f(x), g(x)$ not fcns on $\mathbb{P}^n$, but $\frac{f}{g}$ is.

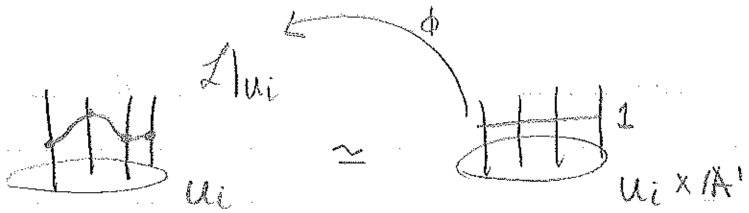
- To construct a map $X \to \mathbb{A}^n$, all we need are n (regular fcns) sections of the trivial line bundle on X. OR an n-dim'l vector subsp of the space of sections of the $\underbrace{\text{trivial line bundle}}_{\hookrightarrow \mathcal{O}}$ on X.

- To construct a map $X \to \mathbb{P}^n$, all we need are a line bundle $\mathcal{L}$ on X & n+1 sections of $\mathcal{L}$ which do not vanish simultaneously.
  - Given this data, cover X by open sets U_i s.t. $\mathcal{L}$ is trivial along U_i, i.e. $\mathcal{L}|_{U_i} \overset{\Phi}{=} U_i \times \mathbb{A}^1$. You get n+1 sections $s_0, \ldots, s_n$ of $\mathcal{L}|_{U_i}$, i.e. n+1 regular fcns on U_i. Map $U_i \hookrightarrow \mathbb{P}^n$

$$P \longmapsto (s_0(P) : \ldots : s_n(P))$$

this iso. exists but is not unique.

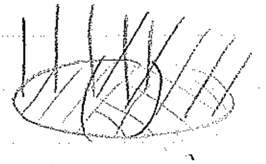
* If change the trivialization ϕ, all fcns $s_i(P)$ get scaled by the same amt, so get same pt in $\mathbb{P}^n$, so the map $U_i \to \mathbb{P}^n$ does not change. The maps $U_i \to \mathbb{P}^n$ glue to $X \to \mathbb{P}^n$, b/c each agree on $U_i \cap U_j$.



- this isomorphism is determined by where 1 is sent.

Transition Functions

Given a line bundle $\mathcal{L}$ on X, pick a trivialization of $\mathcal{L}$, i.e. a cover of X by open sets U_i with isomorphisms $\phi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{A}^1$. Now on $U_i \cap U_j$, look at $\phi_i \circ \phi_j^{-1} : U_j \cap U_i \times \mathbb{A}^1 \xrightarrow{\sim} U_j \cap U_i \times \mathbb{A}^1$. This map is linear in the fibers (for each pt in $U_j \cap U_i$, the fcn is linear in $\mathbb{A}^1$)



- linear autos of $\mathbb{A}^1$ are k^* (GL_1),

& don't require one to pick a basis. So this map is a fcn $\phi_{ij} : U_{ij} = U_i \cap U_j \to k^*$.

ϕ_{ij} is called the transition fcn from U_i to U_j.

Properties:

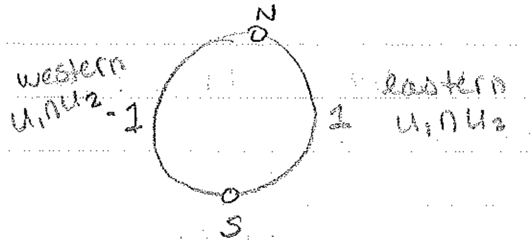
ⓐ $\phi_{ij} \cdot \phi_{jk} \cdot \phi_{ki} = 1$ on $U_{ijk} = U_i \cap U_j \cap U_k$. ($\Rightarrow \phi_{ii} = 1$, $\phi_{ij} = \phi_{ji}^{-1}$)

ⓑ If we change the trivialization $\{\phi_i\}$ to $\{\phi_i'\}$ on same U_i's, the ϕ_{ij}'s change by $\phi_{ij}' = \phi_{ij} \cdot \frac{\phi_i'}{\phi_i} \cdot \frac{\phi_j}{\phi_j'}$.

Claim: The data of the $\{\phi_{ij}\}$'s satisfying ⓐ gives a nat'l construction of a line bundle $\mathcal{L}$, & changing the ϕ_{ij}'s as in ⓑ gives the same line bundle.

Ex#0: Trivial Line Bundle: Cover X by $X = U_1$.
$\phi_{11} = 1$.

Ex#1: Mobius band: (Bundle over S^1)
Cover S^1 by $U_1 = S^1 \setminus \{\text{north pole}\}$, $U_2 = S^1 \setminus \{\text{south pole}\}$
Need $\phi_{11} = \phi_{22} = 1$, $\phi_{12} : U_1 \cap U_2 \to \mathbb{R}^*$. Take
$$\phi_{12} = \begin{cases} 1 & \text{on eastern } U_1 \cap U_2 \\ -1 & \text{on western } U_1 \cap U_2 \end{cases}$$



(If $\phi_{12} = 1$ or $\phi_{12} = -1$, we get trivial line bundle)

Ex#2: On $\mathbb{P}^1$, take the covering $U_0 =$ finite $\mathbb{P}^1 = \mathbb{P}^1 \setminus \{\infty\} \cong \mathbb{A}^1$, $U_1 = \mathbb{P}^1 \setminus \{0\} \cong \mathbb{A}^1$. Pick $\phi_{00} = \phi_{11} = 1$, & $\phi_{01} : \underset{\overset{\cong}{k^*}}{U_0 \cap U_1} \to k^*$

- $\phi_{01} = 1 \Rightarrow$ Trivial line bundle. $\mathcal{L} = \mathcal{O}$
- $\phi_{01} = x$ (id $k^* \to k^*$) $\Rightarrow \mathcal{L} = \mathcal{O}(1)$
- $\phi_{01} = 1/x \Rightarrow \mathcal{L} = \mathcal{O}(-1)$

Want to understand sections of $\mathcal{O}(1)$:
- space of sections is a vector sp.
- Let $\Gamma(X, \mathcal{L})$ = space of all sections s of $\mathcal{L}$, a v.s./k.

Here $\Gamma(\mathbb{P}^1, \mathcal{O}(1)) =$ 2-dim'l $\mathbb{C}$.
ex: $\Gamma(\mathbb{P}^1, \mathcal{O}) = k$, $\dim \Gamma(\mathbb{P}^1, \mathcal{O}) = 1$ (b/c only const fcns on $\mathbb{P}^1$)

A section s of $\mathcal{O}(1)$ will give sections of $\underset{\overset{\cong}{\mathcal{O}|_{U_i}}}{\mathcal{O}(1)|_{U_i}}$ $\forall i$.

trivial $\Rightarrow$ sections are fcns $\rightarrow$

So s gives rise to 2 fcns, s_0 on U_0 & s_1 on U_1, & $s_0 \in k[x]$, $s_1 \in k[1/x]$ (b/c $U_1 = \mathbb{P}^1 \setminus 0$).
If $\phi_{01} = 1$, then $s_0 = s_1$ in $k(x) = k[x, 1/x] \Rightarrow s_i$ constant $\in k$
(confirms $\Gamma(\mathbb{P}^1, \mathcal{O}) = k$)

If $\phi_{01} = x$, then $\frac{s_0}{s_1} = x$ (i.e., ratio of sections at 7 must be 7)

$\Rightarrow \frac{s_0}{x} \in k[1/x] \Rightarrow s_0$ is linear, $s_0 = a + bx$

The space of such s_0's is 2-dim'l. & once s_0 is fixed, s_1 is completely determined

$\Rightarrow \Gamma(\mathbb{P}^1, \mathcal{O}(1)) \simeq \langle 1, x \rangle$

If $\phi_{01} = 1/x$, then $\frac{s_0}{s_1} = 1/x \Rightarrow x \cdot s_0 \in k[1/x]$, & no such polys sat. this $\Rightarrow \Gamma(\mathbb{P}^1, \mathcal{O}(-1)) = 0$.

- $\Gamma(\mathbb{P}^1, \mathcal{O}(1)) = \{ f \in k(x) \mid \text{"}\infty\text{"} + (f^1) \geq 0 \}$ (i.e., is effective)

$= \{ D \sim \text{"}0\text{"} \mid D \text{ effective} \}$

$= |\mathcal{L}(\text{"}0\text{"})|$

Have correspondence

$$\underset{\text{(Weil divisor)}}{\text{"}0\text{"}} \rightsquigarrow \underset{\text{"}x\text{"}}{\text{Cartier divisor}} \rightsquigarrow \underset{\mathcal{O}(\text{"}0\text{"})}{\text{Line bundle}}$$

s.t. $\Gamma(\mathcal{O}(\text{"}0\text{"})) = |\mathcal{L}(\text{"}0\text{"}$

11/19 There is a natural line bundle, $\mathcal{O}(-1)$, on $\mathbb{P}^n$, called the tautological line bundle:

$\mathcal{L} \subseteq \mathbb{P}^n \times \mathbb{A}^{n+1}$

$\mathcal{L} = \{ (P, x) \in \mathbb{P}^n \times \mathbb{A}^{n+1} \mid x \text{ lies on the line } P)$

$= \{ ([y_0, \ldots, y_n], (x_0, \ldots, x_n)) \mid$

$\text{rank}\begin{pmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{pmatrix} < 2 \}$ (really $= 1$ (i.e. the vects are dependent)

[a pt in $\mathbb{P}^n$ is a line in $\mathbb{A}^{n+1}$

b/c at least one x-coor is $\neq 0$, although ok for all $y_i = 0$)

- This is the blowups of $\mathbb{A}^{n+1}$ at the origin.

$\pi : \mathcal{L} \to \mathbb{P}^n$ is projection on the 1st factor

What is $\pi^{-1}(P)$ = line spanned by P in $\mathbb{A}^{n+1}$ (or P_x (line)

Obs: If $\mathcal{L}$ is a line bundle on X, $\exists$ another line bunde on X called $\mathcal{L}^\vee$ s.t. the fiber $(\mathcal{L}^\vee)_x = (\mathcal{L}_x)^\vee$

- dual of a line is another line, but not same line, so new line bundle in gen is not same as old,

∴ dual of fiber of orig. line bundle.

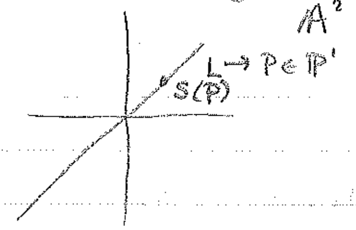
Denote by $\mathcal{O}(1) = \mathcal{O}(-1)^\vee$

Claim: The variables $x_0, \ldots, x_n$ on $\mathbb{P}^n$ can be thought of as sections of $\mathcal{O}(1)$. In fact, any linear fcn f on $\mathbb{A}^{n+1}$ is a section of $\mathcal{O}(1)$.

- Think of sections of a line bunde as generalized fcns on $\mathbb{P}^n$

coord fcns on $\mathbb{A}^n$ are the coords)

- Fix i. why wasn't x_i a fcn on $\mathbb{P}^n$? B/c only defined up to scaling.



- What is the x-coord on this line? Not defined
- What is the x-coord of given pt on line? Defined

- If change pt on line, x changes in linear fashion.

Indeed, $\underline{x}_i$ gives a linear fcn on the fibers of $\mathcal{O}(-1)$

$\underline{x}_i(P, x) = x_i$ (i^{th} coord of pt $x \to$ b/c x is on line P) on the line P, i.e. $\pi^{-1}(P)$, this is a linear fcn.

$\to$ a linear functional on $\pi^{-1}(P)$ for each $P \Rightarrow$

$(\underline{x}_i)_P \in (\mathcal{O}(-1)_P)^\vee$

$\in \mathcal{O}(1)_P \leftarrow$ a section of $\mathcal{O}(1)$

[So $\mathcal{O}(1)$ comes w/ $n+1$ sections. & the space $\mathbb{P}^n$ is naturally endowed w/ a line bundle & $n+1$ sections]

If $\mathcal{L}_1$ & $\mathcal{L}_2$ are line bundles on X, we can form $\mathcal{L}_1 \otimes \mathcal{L}_2$, whose fiber at P is $(\mathcal{L}_1)_P \otimes (\mathcal{L}_2)_P$.
[Given 2 lines, tensoring gives a new line]

Think about: $\mathrm{Pic}(X) := \{\mathcal{L} \mid \mathcal{L}$ a line bundle on $X)$ (Picard gp) is a gp. The operation is $\otimes$, id is the trivial line bundle, $\mathcal{O}$, & the inverse is $^\vee$.

- $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, so $\mathcal{O}(1)$ notation not random!

We'll now do all of this in terms of transition fcns:

- If $\mathcal{L}$ is a line bundle on X, then a trivialization of $\mathcal{L}$ is:
  (a) A cover of X by open sets U_i
  (b) A set of iso's $\phi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{A}^1$
- Given a trivialized line bundle $\mathcal{L}$ as above, the transition fcns are

$$\phi_{ij} : U_i \cap U_j \longrightarrow k^* \text{ defined by}$$
$$\phi_{ij} = \phi_i \circ \phi_j^{-1} : (U_j \cap U_i) \times \mathbb{A}^1 \longrightarrow (U_i \cap U_j) \times \mathbb{A}^1$$

- over each pt of $U_i \cap U_j$, get a linear automorphism of $\mathbb{A}^1$, b/c ϕ_i & ϕ_j preserve projection down to base pt, & a lin. auto just mult by scalar.
- these will satisfy $\phi_{ij} \cdot \phi_{jk} \cdot \phi_{ki} = 1$.

Claim: If $\mathcal{L}$ is trivialized, then a section of $\mathcal{L}$ is nothing but a set of fcns $f : U_i \to k$ s.t.
$f_i = \phi_{ij} \cdot f_j$ on $U_i \cap U_j$, both give #'s, so can multiply them.

Pf: Let $s : X \to \mathcal{L}$ be a section.

$$\begin{tikzcd} U_i \arrow[r, "s|_{U_i}"] \arrow[rrr, bend right, "f_i"'] & \mathcal{L}|_{U_i} \arrow[r, "\phi_i", "\sim"'] & U_i \times \mathbb{A}^1 \arrow[r, "\pi_2"] & \mathbb{A}^1 = k \end{tikzcd}$$

↑from triv.

this map is a section of the trivial line bundle.

On $U_i \cap U_j$

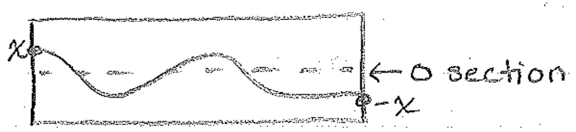
$$\begin{tikzcd}
U_i \cap U_j \arrow[r, "s|_{U_i}"] \arrow[rrr, bend left, "f_i|_{U_i\cap U_j}"] \arrow[dr, "s|_{U_j}"'] \arrow[drrr, bend right, "f_j|_{U_i\cap U_j}"'] & (\mathcal{L}|_{U_i})|_{U_i\cap U_j} \arrow[r, "\phi_i"] \arrow[d, phantom, "//"] & (U_i \times \mathbb{A}^1)|_{U_i\cap U_j} \arrow[r] & \mathbb{A}^1 \\
 & (\mathcal{L}|_{U_j})|_{U_i\cap U_j} \arrow[r, "\phi_j"] & (U_j \times \mathbb{A}^1)|_{U_i\cap U_j} \arrow[r] & \mathbb{A}^1 \arrow[u, bend left, "\phi_{ij}"]
\end{tikzcd}$$

So $f_i = \phi_{ij} \circ f_j$

Ex: Sections of the Möbius band, M

- How do we know sections of Möbius band not trivial?

B/c on M, all sections have at least one zero:



So by IVT, this function must have a zero.
But on triv. line bundle, have fcn ≡ 1 (b/c sec's on $\mathcal{O}$ are fcns on S^1 & have a fcn ≡ 1 on S^1)

Zeros of Sections

If $s \in \Gamma(\mathcal{L})$, we get a well-defined locus $Z(s) \subseteq X$ (closed subset)
(b/c on each open set, have triv, so s is a fcn - if move to diff U_i, use nowhere 0 fcn to transition, so $Z(s)$ still defined)

- s gives a pt in 1-dim'l vect sp. can't tell its magnitude, but can say if 0 or not w/o choosing a basis.

$Z(s) = \{P \in X \mid s(P) = 0\}$

Same goes for a few sections $s_1, \dots, s_n$:

$Z(s_1, \dots, s_n) = Z(s_1) \cap \dots \cap Z(s_n)$

Define $\mathcal{O}(1)$ on $\mathbb{P}^n$ to be the line bundle trivialized over the cover $\{U_i\}_{i=0\ldots n}$ of $\mathbb{P}^n$ (nat'l cover) w/ transition fcns

$$\phi_{ij} : U_i \cap U_j \to k^*$$

$$\phi_{ij}([x_0 : \cdots : x_n]) = x_j/x_i$$ (b/c $x_i \neq 0$ on U_i, $x_j \neq 0$ on U_j & need ratio to get fcn on $\mathbb{P}^n$)

If $i_0 \in \{0, \ldots, n\}$, we get a section $\underline{x}_{i_0}$ of $\mathcal{O}(1)$ by

$f_i = \frac{x_{i_0}}{x_i} : U_i \to k$ (ratio gives honest fcn on U_i; i_0 fixed, i runs through all indices)

<u>Note</u>: $\phi_{ij} \cdot f_j = \frac{x_j}{x_i} \cdot \frac{x_{i_0}}{x_j} = \frac{x_{i_0}}{x_i} = f_i$. ✓

$\Rightarrow$ The collection $\left\{ \frac{x_{i_0}}{x_i} \right\}$ gives a section $\underline{x}_{i_0}$ of $\mathcal{O}(1)$.

<u>Exercise</u>: Prove that all sections of $\mathcal{O}(1)$ are lin. combs of $\underline{x}_i$'s.

i.e. $\dim \Gamma(\mathbb{P}^n, \mathcal{O}(1)) = n+1$

↑ space of sections on a line bundle.

[look at example from 2 classes ago on $\mathbb{P}^1$ w/ trans. fcns x (or $1/x$?)]

<u>Q</u>: What is $Z(\underline{x}_{i_0}) \subseteq \mathbb{P}^n$

A: This is $Z(x_{i_0})$ in the usual sense – the i_0th hyperplane (b/c $f_i = 0 \Leftrightarrow x_{i_0} = 0$]

<u>Claim</u>: If $\mathcal{L}$ is a line bundle on X & $s \in \Gamma(X, \mathcal{L})$, & X is non-singular in codim 1.

(ex: if X is smooth) Then $Z(s)$ is naturally an effective Weil divisor on X.

[gives $\mathcal{L}$ w/ sects $\Rightarrow$ divisors]

– the divisor of the difference of 2 sec's is principal:

(not a principal divisor, b/c $Z(x_{i_0})$ = hyperplane, deg 1, not deg 0)

Different sections $s_1, s_2 \in \Gamma(X, \mathcal{L}) \setminus \{0\}$ have linearly equivalent divisors of zeros.

On each U_i in a trivialization, s gives a fcn. f_i;
& $(f_i) \in \mathrm{Div}(U_i)$. Because $f_i = \varphi_{ij} \cdot f_j$ & φ_{ij} nowhere zero w/ no poles, then $(f_i)|_{U_i \cap U_j} = (f_j)|_{U_i \cap U_j}$
(b/c divisors only care about zeros & poles)
So these (f_i)'s patch together to give a divisor.

Note: s_1/s_2 is a well-def mero. fcn on X ($\in K(X)$)
$\uparrow$ not on patches
and locally $(s_1) = (s_2) + (s_1/s_2)$
$\Rightarrow Z(s_1) = Z(s_2) + (s_1/s_2)$
$\Rightarrow Z(s_1) \sim Z(s_2)$ [recall $\sim$ means they differ by a nonzero fcn in $K(X)$]

Dictionary (On a nice variety)

more common notation

• divisor D	• line bundle $\mathcal{L}(D) = \mathcal{O}(D)$
• $D_1 \sim D_2$	• $\mathcal{O}(D_1) \cong \mathcal{O}(D_2)$
• divisor D', effective, s.t. $D' \sim D$	• section of $\mathcal{O}(D)$
• To get a map to $\mathbb{P}^n$, want divisor D & $n+1$ elts of $\lvert\mathcal{L}(D)\rvert = \{D' \geq 0 \mid D' \sim D, D'\}$, a vect sp.	• To get a map to $\mathbb{P}^n$, want Line bundle $\mathcal{L}(D)$ & $n+1$ sections in $\Gamma(X, \mathcal{L}(D))$.

Geometric Structure vs. Divisors
to get map to $\mathbb{P}^n$

11/21

Cartier Divisors

Def: A cartier divisor is given by:

(a) A cover $\{U_i\}$ of X by open sets

(b) On each U_i, a rat'l fcn $f_i : U \dashrightarrow k$, $f_i \neq 0$

s.t. on $U_{ij} = U_i \cap U_j$, the rat'l fcn f_i/f_j is regular & has no zeros. We write $f_i/f_j \in \mathcal{O}^*(U_{ij})$. ($* \Rightarrow$ units)

(up to an equivalence)

Ex: If $\underset{f \neq 0}{f \in K(X)}$, then $(f) = \begin{cases} \text{cover is } \{X\} \\ \text{fcn } f : X \dashrightarrow k \text{ is } f. \end{cases}$

Ex: If $X = \mathbb{P}^1$, $f = X_0/X_1 = x$ (coord on finite part, ie $X_1 = 1$)

$(\mathbb{P}^1, x) =$ principal divisor (x) [will corresp to $0 - \infty$]

Ex: If $X = \mathbb{P}^1$, $U_0 = \mathbb{P}^1 \setminus \{\infty\}$, $U_1 = \mathbb{P}^1 \setminus \{0\}$.

$(U_0, x = X_0/X_1)$, $(U_1, 1)$

On $U_{01} = \mathbb{P}^1 \setminus \{0, \infty\}$, $x/1$ has no poles or zeros, so this is a Cartier divisor. [Weil: $1 \cdot$"0"]

Ex: $X = \mathbb{P}^1$, U_0, U_1 as above.

$(U_0, 1/x)$, $(U_1, 1)$. On U_{01}, $1/x$ has no poles or zeros, so a Cartier divisor [Weil: $-1 \cdot$"0"]

Ex: $X = \mathbb{P}^1$, U_0, U_1 as above

$(U_0, \underset{\substack{\uparrow \\ \text{one zero at } 0 \\ \text{no poles}}}{x})$, $(U_1, x) \nwarrow$ no zeros, one pole at ∞ [x not regular on U_1, b/c $\infty \in U_1$]

[1"0" − 1"∞"]

Equivalence Relation:

(1) Refining the cover gives the same divisor (as in 2^{nd} & last example).

$D = (\{U_i\}, \{f_i\})$. Cover each U_i by $\{V_{ij}\}$, & take $g_{ij} = f_i$, then $D' = (\{V_{ij}\}_{i,j}, \{g_{ij}\}_{i,j}) = D$.

[any cover that's fine enough should do]

→ Take the equivalence relation gen. by this.

(2) If $g_i \in \mathcal{O}^*(U_i)$ (i.e. g_i are reg. fcns that are nowhere 0) then $(\{U_i\}, \{f_i\}) = (\{U_i\}, \{g_i f_i\})$

[i.e. all we care about are zeros & poles of f_i].

ex: $((U_0, x), (U_1, 1)) = ((U_0, 2x), (U_1, 3))$

Claim: Cartier Divisors form an abel. gp.:

Given D, D', 1^{st} find a common refinement of their covers. Then $D = (\{U_i\}, \{f_i\})$ & $D' = (\{U_i\}, \{g_i\})$.

$D + D' = (\{U_i\}, \{f_i \cdot g_i\})$.

ex: $((U_0, x), (U_1, 1)) + ((U_0, f), (U_1, f)) = ((U_0, xf), (U_1, f))$

- Principal divisors form a subgp.
- Define the Cartier class gp as $Ca(X)/\mathrm{Princ}\,Ca(X)$ (Cartier divisors mod principal divisors).

Observations:

① $\exists$ map $Ca(X) \to \overset{\text{weil div's}}{Div(X)}$ which takes principal divisors to princ. div's.

Let $Y \subseteq X$ be a prime div (codim 1 subvar.) in $Div(X)$. Pick some i s.t. $Y \cap U_i \neq \emptyset$, & make U_i smaller to be affine.

Take $a_Y = v_Y(f_i)$ (valuation of f_i, the fcn on U_i)

If we had picked another U_j, then $a_Y = v_Y(f_j)$.

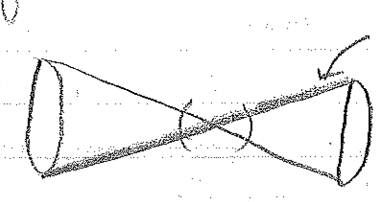
But $f_j/f_i \in \mathcal{O}^{*} \overset{\text{(unit)}}{\Rightarrow} v_Y(f_j/f_i) = 0 \Rightarrow v_Y(f_j) = v_Y(f_i)$

$\Rightarrow a_Y$ the same.

Take the divisor $\sum a_Y \cdot Y$ as the image of the Cartier divisor.

Thm: Assume X is covered by U_i's which are affine & $\mathcal{O}(U_i)$ a UFD. Then the map is an isomorphism, ie. $Ca(X) \simeq Div(X)$.

ex: The map is not always surj. Recall the ex of the quadric cone.



Weil div. that not locally principal b/c near X cannot be cut out by a single eqn

But $Ca(X)$ always gives loc. princ. b/c by def of Car-div, you have one fcn on each open set, so $Ca(X)$ says here's an open set, & take a princ. div. on it, then glue then together.
So $\mathcal{O}(U_i)$ a UFD means the Weil div. is loc. princ.

Pf: The map is always injective, so we need to show surjectivity. Let $D = \sum a_i Y_i$ be a Weil div on X. On each U_i, look at the restr. of D to U_i. Since U_i affine & $\mathcal{O}(U_i)$ a UFD, then $D|_{U_i}$ is principal [on an affine var. whose ring of fcns is a UFD, all div's are princ.] $\Rightarrow D_{U_i} = (f_i)$ for some $f_i \in K(U_i) = K(X)$. Take the Car. div $(\{U_i\}, \{f_i\})$. But on $U_i \cap U_j$, $(f_i) - (f_j) = 0$ b/c are same restr. of D, $\Rightarrow f_i/f_j \in \mathcal{O}^*$ □

Note: The conditions of the thm are sat. for X smooth.

[A Car. div is princ $\Leftrightarrow$ it is equiv. to one whose cover is whole sp]

From Cartier divisors to line bundles:

Given $(\{U_i\}, \{f_i\})$, take $\varphi_{ij} = f_i/f_j \in \mathcal{O}^*(U_{ij})$ to be the transition fcns:

(a) $\varphi_{ij} \cdot \varphi_{jk} \cdot \varphi_{ki} = f_i/f_j \cdot f_j/f_k \cdot f_k/f_i = 1$ ✓

(b) What about the equiv. rel? If we triv. a l.b. on a cover, & then refine the cover, we get a l.b. that's already triv, so refining doesn't change line bundle.

- What if we look at $\{g_i f_i\}$? This is the ambiguity in the triv - recall; we had to choose a triv., & if we chose another, we got these extra products.

<u>Ex</u>: $X = \mathbb{P}^1$, $D = 1 \cdot$"0".

- Car. Div: ① Cover $\mathbb{P}^1$ by affines whose ring is a UFD, U_0, U_1.
  ② $D|_{U_0} = 1 \cdot$"0" on $\mathbb{A}^1$, so $D|_{U_0} = (x)$
  $D|_{U_1} = 0$ on $\mathbb{A}^1$, so $D|_{U_1} = (1)$
  ③ Take the Car. div. $((U_0, x), (U_1, 1))$
- Line Bundle: $\varphi_{01} = x/1 = x$ (transition fcn) $\Rightarrow \mathcal{O}(1) = \mathcal{O}(D)$ $(= \mathcal{L}(D))$

  Global sections:

  $\dim(\Gamma(\mathbb{P}^1, \mathcal{O}(1)) = 2$

  $\uparrow$ = Lin. homog. polys in x_0 & x_1, as we saw last time.

<u>Ex</u>: $X = \mathbb{P}^1$, $D =$ "0" + "1"

- ① Cov by U_0, U_1
  ② Car. Div $\{(U_0, x(x-1)), (U_1, \frac{x-1}{x})\}$

  $\uparrow$ need 0 at "1", & need rat'l fcn w/ zero at 1 but no poles ($0 \notin U_1$).
- Line Bundle:

  $\varphi_{01} = \frac{x(x-1)}{(x-1)/x} = x^2$

$\Rightarrow \mathcal{O}(D) = \mathcal{O}(2)$

* Tensoring on an open set multiplies the trans. fcn.
So $\mathcal{O}(1)\otimes\mathcal{O}(1) = \mathcal{O}(2)$ has trans. fcn ϕ_{01} $= x \cdot x = x^2$

Thus $\mathcal{O}(\text{"}0\text{"}+\text{"}1\text{"}) = \mathcal{O}(2\text{"}0\text{"})$ (b/c trans. fcn same)

In gen. $\mathcal{O}(D) = \mathcal{O}(D')$ if $D \sim D'$

"0"+"1" $\sim$ 2"0" b/c dif is "0"−"1" & so the global fcn is $x/x-1$.

A section of $\mathcal{O}(\text{"}0\text{"}+\text{"}1\text{"})$ is a selection of reg fcns f_0, f_1 on U_0, U_1 s.t. $f_j = \phi_{ji} f_i$. ($\phi_{10} = 1/x^2$)

$f_1 = 1/x^2 f_0$, , $f_0 \in k[x]$, $f_1 \in k[1/x]$

$\Rightarrow$ deg $f_0 \le 2$. $\Rightarrow \dim \Gamma(\mathbb{P}^1, \mathcal{O}(2)) = 3$.

__exercise__: Check $h^0(\mathbb{P}^1, \mathcal{O}(k)) = \begin{cases} k+1 & \text{if } k \ge 0 \\ 0 & \text{if } k < 0 \end{cases}$

where $h^0 := \dim \Gamma(\mathbb{P}^1, \mathcal{O}(k))$.

__Prop__: D is principal $\Leftrightarrow \mathcal{O}(D) = \mathcal{O}$, the trivial line bundle.

__Pf__: $D = (f) = \{(X, f)\} \Rightarrow$ only trans. fcn is $\phi_{00} = 1$ $\square$

PrincCa $\in$ kernel of

__Cor__: $\mathrm{Div}(X) \cong \mathrm{Ca}(X)$ as abel. gps., and

$$\begin{tikzcd} \mathrm{Ca}(X) \arrow[d, "\text{gp hom.}"] \\ (\mathrm{Pic}(X), \otimes) \end{tikzcd}$$

b/c $\frac{f_i g_i}{f_j g_j} = \frac{f_i}{f_j} \cdot \frac{g_i}{g_j} = \mathcal{O}(D_1)\otimes\mathcal{O}(D_2)$

$$\frac{f_i g_i}{f_j g_j} \overset{\shortparallel}{} \mathcal{O}(D_1+D_2)$$

* $\mathrm{Ker}(\mathrm{Ca}(X) \to \mathrm{Pic}(X)) = \mathrm{Princ\,Ca}(X)$

$\Rightarrow \mathrm{Cl}(X) \cong \mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$ for X any locally factorial variety (i.e. sat the prop that if can be covered by affines whose rings are UFD's)

__Def__: $D = (\{U_i\}, \{f_i\})$ is effective if it can be represented w/ all $f_i \in \mathcal{O}(U_i)$ [instead of $K(U_i)$] − i.e. if all fcns are regular.

Prop: If D is any (Weil or Cartier) divisor,
$\Gamma(X, \mathcal{O}(D)) \simeq \{ f \in k^*(X) \mid D + (f)$ is effective$\} \cup \{0\}$

Ex: $X = \mathbb{P}^1$, $D = 2 \cdot "0"$.
$\{ f \in k(x) \mid (f) + 2 \cdot "0" \geq 0 \} = \{ \frac{g(x)}{h(x)}$ reduced $\mid (g,h) = 1$,
$\underbrace{h \mid x^2}$, $\deg g \leq \deg h \}$
i.e. only zeros or poles is ≤ 2 at 0.
no pole at ∞

$h(x) = 1, x, x^2$
OR $= \{ \frac{g(x)}{x^2} \mid \deg g \leq 2 \} \to \dim = 3$ ✓

11/26 **Thm**: Let D be a Weil divisor and $\mathcal{O}(D)$ the assoc. line bundle. Then $\exists$ a 1-1 correspondence btwn
$\{ f \in K(X)^* \mid (f) = D' - D, D' \geq 0$ (i.e. D' effective) $\} \cup \{0\}$
$\longleftrightarrow \Gamma(X, \mathcal{O}(D))$ (i.e. global sections of the line bundle).

Pf: $(\leftarrow)$: Let $\{(U_i, f_i)\}$ be the Cartier divisor corresp. to D. Define $\mathcal{O}(D)$ to have transition fcns, $\phi_{ij} = f_i / f_j$ on U_{ij}. A section of $\mathcal{O}(D)$ is a collection of fcns $\{s_i\}$, $s_i : U_i \to k$ regular s.t. $s_j = \phi_{ji} s_i$.
Note that $\{(U_i, s_i)\}$ is a Cartier divisor [as $s_j / s_i = \phi_{ji}$, which has no zeros & no poles]
Let D' be the Weil divisor corresp. to it. $D' \geq 0$ b/c s_i are regular [ie no poles on any U_i].
Define $g_i : U_i \dashrightarrow k$ rat'l fcn, $g_i = s_i / f_i$.
On U_{ij}, $s_j = \phi_{ji} s_i = f_j / f_i \, s_i \Rightarrow g_j = s_j / f_j = s_i / f_i = g_i$.
$\Rightarrow \{g_i\}$ are actually the restrictions of a single rat'l fcn g on all of X. $\Rightarrow (g) = D' - D$
(b/c get (zeros of s_i) $-$ (zeros of f_i) $= D' - D$)
[Thus get $\Gamma(X, \mathcal{O}(D)) \longrightarrow g \in K(X)^*$ s.t. $(g) = D' - D$, $D' \geq 0$.
$(\rightarrow)$: Exercise. $\square$

On $\mathbb{P}^2$, $|\mathcal{O}(2H)|$ — all quadrics in $\mathbb{P}^2$
$\uparrow$ 2x a hyperplane
→ i.e. $H: x_0 = 0$, need rat'l fcn w/ x_0^2 in denom, so choose deg 2 poly in num, which is a quadric

Thus $\dim \Gamma(\mathbb{P}^2, \mathcal{O}(2H)) = 6$ b/c $\exists$ 6 quadrics

$|\mathcal{O}(3H)|$ = cubics in $\mathbb{P}^2$, w/ dim 10 $\binom{5}{2} \leftarrow \binom{n+d}{d}$

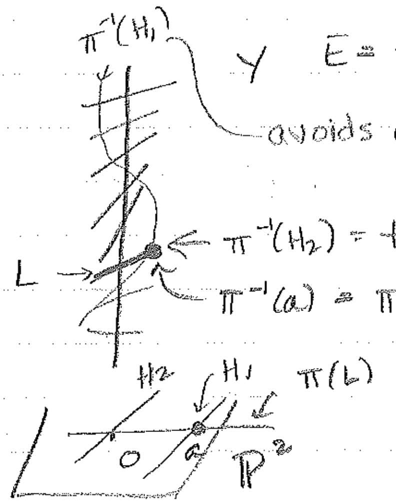
→ any quadric can be deformed into any other (use linear deformation)

Ex: Let $Y = Bl_0(\mathbb{P}^2)$ (smooth ⇒ Pic, Cl same)

In $\mathbb{R}^2$, you'd get Y = Mobius strip.

- Claim: $\mathrm{Pic}(Y) = \mathrm{Cl}(Y) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}H \oplus \mathbb{Z}E$

$Y \xrightarrow{\pi} \mathbb{P}^2$, so $H = \pi^{-1}(H \subseteq \mathbb{P}^2)$ doesn't matter which b/c lin. equiv.
$\uparrow$ any line in $\mathbb{P}^2$



$E = \pi^{-1}(0)$, exceptional divisor

avoids 0

$\leftarrow \pi^{-1}(H_2)$ = this line and E.

$\pi^{-1}(a) = \pi^{-1}(H_1) \cap L \neq \emptyset$

$H - E = L$ is linear equivalent to an effective div.

On a projective smooth surface X, we have a good intersection theory: There is a well-defined int. pairing, $D_1 . D_2 \in \mathbb{Z}$ for any 2 divisors $D_1, D_2 \in Pic(X)$, s.t.

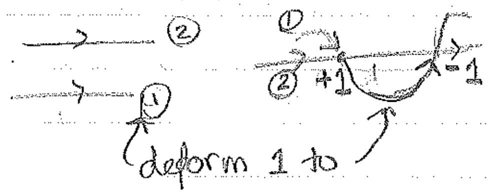
ⓐ $(D_1 + D_2) . D_3 = D_1 . D_3 + D_2 . D_3$

ⓑ If D_i, D_j are effective & have no common components, then $D_i . D_j \geqslant 0$

[$H.H = 1$, deform on H

Poincaré Duality

<u>Topology</u>: deform one until they're transversal, then count the # of ints

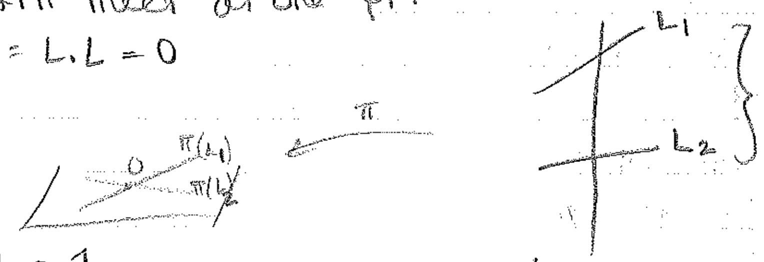


positively oriented vect. sys ⇒ ⊕, neg. oriented ⇒ ⊖

Find $E^2 = E . E = ?$ $\boxed{-1}$

warmup: ① Find $H^2 = H . H = 1$ b/c deform one to hit at one pt & lift — make them not intersect 0 b/c $\mathbb{P}^2 \setminus \{0\} \underset{homeo}{\cong} Bl_0(\mathbb{P}^2) \setminus E$ so still meet at one pt.

② $L^2 = L . L = 0$



no pts in common: diff slopes ⇒ diff hts

③ $H . L = 1$

$$\Rightarrow E = H - L \Rightarrow E^2 = (H - L)^2 \underset{\uparrow \text{from prop ⓐ}}{=} H^2 - 2HL + L^2 = 1 - 2 + 0 = -1 \checkmark$$

What does neg. intersection # mean? E is rigid; it cannot move, so cannot make the 2 E's transversal.

- E is an effective divisor, but any other effective div. lin. equiv. to E is E itself.
  $\Rightarrow \dim \Gamma(Y, \mathcal{O}(E)) = 0$, i.e. no rat'l fcn that has poles at E & zeros elsewhere.

  $Y \xrightarrow{\pi} \mathbb{P}^2$ birat'l iso $\Rightarrow K(Y) = K(\mathbb{P}^2)$, so rat'l fcn on Y is pull back of one on $\mathbb{P}^2$, so can look on $\mathbb{P}^2 \rightarrow \nexists$ fcn that has zeros or poles at just 0, b/c zeros/poles form a codim 1 subset, but 0 a codim 2 subset.

If $E \sim D$, $D \geq 0$, then E must appear as a component of D (b/c of prop ⓑ) so $D = E + D'$, $D' \geq 0 \Rightarrow \underset{\substack{\uparrow \\ \text{zeros \& poles} \\ \text{of rat'l fcn}}}{D' \sim 0}$ & $\underset{\substack{\text{zeros \& poles} \\ \text{of reg fcn}}}{D' \geq 0} \Rightarrow D' = 0$ (b/c the only reg rat'l fcns are const & need zeros, so 0)

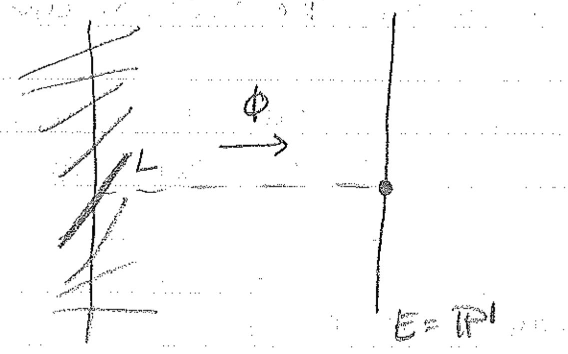
Def: Let D be a divisor on some X. The <u>base locus</u> of D, $BP(D) = \bigcap_{\{D' \sim D,\, D' \geq 0\}} D' \subseteq X$

<u>Ex</u>:
- $X = \mathbb{P}^2$, $D = H$
  $BP(D) = \emptyset$ We say D is <u>base pt free</u> (bpf)
- $Y = Bl_0(\mathbb{P}^2)$, $D = H$
  $BP(H) = \emptyset$
- $Y = Bl_0(\mathbb{P}^2)$, $D = L$
  $BP(L) = \emptyset$ L is bpf
- $Y = Bl_0(\mathbb{P}^2)$, $D = E$
  $BP(E) = E$

<u>Obs</u>: If $X \xrightarrow{f} \mathbb{P}^n$ and $H \in \mathrm{Pic}(X)$ is $f^{-1}(H \subseteq \mathbb{P}^n)$, then H is bpf.
Choose $x \in X$ & $H \subseteq \mathbb{P}^n$ that misses $f(x)$. $f^{-1}(H)$ will miss x.

<u>Fact</u>: Given any divisor D on X, $n = h^0(X, D) - 1$ $[\dim \Gamma(X,D) - 1]$. Then this gives a map $X \setminus BP(D) \xrightarrow{f} \mathbb{P}^n$ regular s.t. $f^{-1}(H) = D$ for some hyperplane $H \subseteq \mathbb{P}^n$.

<u>Thus</u>: Here, if $D = H$, we get the map $\pi : Y \to \mathbb{P}^2$. If $D = L$, get map to $\mathbb{P}^1$, ϕ



(project to $\mathbb{P}^1$)

$h^0(Y, \mathcal{O}(H)) = 3$ (b/c $\xrightarrow{\pi} \mathbb{P}^2$ so $2+1$)
$h^0(Y, \mathcal{O}(L)) = 2$ (b/c $\xrightarrow{\phi} \mathbb{P}^1$, so $1+1$)

- Neither map, π or ϕ, is an embedding.

<u>Note</u>: A curve C on Y get contracted under the map given by $D \Leftrightarrow C.D = 0$
- If there are no curves C s.t. $C.D = 0$, then get (almost) an embedding
  ex: $D = H + L$ will give an embedding.

12/3
- BPF - at any given pt in space, $\exists$ a divisor that avoids it. $\Rightarrow$ gives map to proj. sp
- separates pts $\Rightarrow$ map will be injective
- separates tangent vectors $\Rightarrow$ map will be an embedding

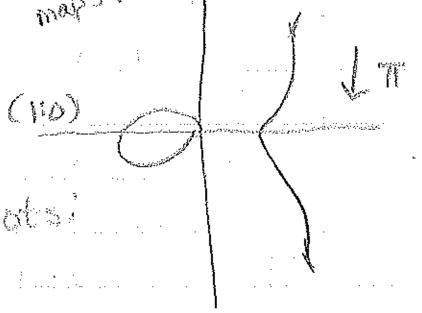
Def: Let L be a linear system (i.e. a subvector sp of the effective divisors equiv. to some D). We say L separates pts if $\forall P \neq Q$ on X, $\exists D \in L$ s.t. $P \in \mathrm{Supp}(D)$ and $Q \notin \mathrm{Supp}(D)$.

Ex: $L = \{H \subseteq \mathbb{P}^n \mid H \text{ a hyperplane}\}$ (i.e. set of sections of $\mathcal{O}(1)$)
L separates pts, b/c given P & Q, can find H_1 s.t. $P \in H_1$ & $Q \notin H_1$.

Ex: Let $C = Z(y^2z - x(x-z)(x+z)) \subseteq \mathbb{P}^2$
Consider $\pi : C \to \mathbb{P}^1$, projection on the x-axis: $(x:y:z) \mapsto (x:z)$ if $(x:y:z) \neq (0:1:0)$; $(0:1:0) \mapsto (1:0)$
$\pi^{-1}(P)$ is generically 2 pts (b/c solving $y^2 \cdots$) except at 4 spots: $x = 0, 1, -1$ & $(1:0)$

[think of $z=1$, the finite part

π not defined here a priori, b/c maps to 0 $\to$ so this extends it



This map, π, exhibits C as a double cover branched over 4 pts.
We have 2 maps $C \xhookrightarrow{\Phi_1} \mathbb{P}^2$, $C \xrightarrow{\pi} \mathbb{P}^1$, & each comes from a lin. syst
For C embedded in $\mathbb{P}^2$, what is the linear system?
deg 3, b/c a hyperplane in $\mathbb{P}^2$ (line) pulls back to a triple of pts under Φ_1 (the int pts), so
$L_1 = \{D \in \mathrm{Div}(C) \mid D = H \cap C \text{ for } H \text{ a hyperplane in } \mathbb{P}^2\}$
$\to$ BPF, separating pts
↑ all lin. systems that come from an embedding, must be BPF & sep. pts!
actually just need inj.
$L_2 = \{D \in \mathrm{Div}(C) \mid D = \pi^{-1}(x), x \text{ a pt in } \mathbb{P}^1\}$
$\to$ deg $L_2 = 2$, BPF, not sep pts (2 on same vert line)
↑ all lin sys's that come from a map to proj. sp are BPF

Now, map $C \to \mathbb{P}^2$, via proj from $(0:0:1:0)$

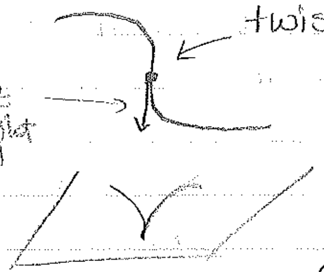
(this pt not on curve b/c $s^3 = t^3 = 0$, so $st^2 \neq 1$)

- regular b/c defined $\forall P \neq (0:0:1:0)$, but $P \notin C$, so ok.

$\mathbb{P}^1 \to \mathbb{P}^2$

$(s:t) \longmapsto (s^3 : s^2t : t^3)$

Image is



twisted cubic in $\mathbb{P}^3$ ($= \mathbb{P}^1$)

tangent sp at origin pts straight down.

$x^2 = y^3$ (b/c $(s^3)^2 = (s^2)^3$ when $t=1$)

(for any value of y, there are 2 values of x)

L is basept free & does separate pts b/c injective but does not separate tangent vectors, b/c that vert. tang. vect gets smushed to one pt.

<u>Ex</u>: What are the automorphisms of $\mathbb{P}^n$?

$PGL_{n+1} = GL_{n+1}/k^* \leftarrow$ diagonal $\overset{\neq 0}{\text{scalar}}$ matrices.

$\uparrow$ center of GL_{n+1}

$\dim_k GL_{n+1} = (n+1)^2$

$\dim_k PGL_{n+1} = (n+1)^2 - 1$ b/c mod out by 1-dim'l sp

GL_{n+1} is an affine variety
have $\mathbb{A}^{(n+1)^2} \leftarrow$ all ...
then get rid of det=0
which is a single eqn, so it's compl. of hyperplane

Given $M \in PGL_{n+1}$, $x \in \mathbb{P}^n$, just multiply $M\vec{x}$.

well-def: if mult by k, get $kM\vec{x} = M\vec{x}$

Given $\Phi \in \text{Aut}(\mathbb{P}^n)$, $\Phi: \mathbb{P}^n \to \mathbb{P}^n$

Φ must come from a lin. system,

$\Phi^{*}(H) = H'$ is another hyperplane ($\leftarrow$ inverse)

so pick basis of target $\mathbb{P}^n$, & look at where they pull back to: equiv. to a change-of-basis matrix $\in PGL_{n+1}$.

[each section gives coord. in target $\mathbb{P}^n$]

- On an elliptic curve, ⓐ if $\deg D = 3$, get an embedding into $\mathbb{P}^2$, i.e. $h^0(D) = 3$

  ⓑ if $\deg D = 2$, get a 2-to-1 map to $\mathbb{P}^1$, i.e. $h^0(D) = 2$

  ⓒ if $\deg D = 1$, [recall, if $C \ncong \mathbb{P}^1$, if $P \sim Q$, then $P = Q$], then $D = 1$"P", so if $D' \sim D$, $D' \geq 0$, $D' = 1$"Q" & $P \sim Q$ $\Rightarrow Q = P \Rightarrow D' = D$, so $L = \{D\}$.

  L is <u>not</u> BPF $\Rightarrow$ no map to $\mathbb{P}^n$ $\forall n$.

  ↑ cannot miss P.

  * As you increase the degree of a divisor, you are more likely to get an embedding.

<u>Ex</u>: Let $X = Bl_0(\mathbb{P}^2)$, $\pi : X \to \mathbb{P}^2$

$L = \{\pi^{-1}(H) \mid H \in \mathbb{P}^2\}$

L is BPF b/c we have the map π or we can see it in the picture from last wk: proj point down, choose $H \subseteq \mathbb{P}^2$ w/ $P \notin H$, pull back.

L does not sep. pts $\rightarrow$ choose P, Q on exceptional divisor.

* If L does not sep. pt, then each set of pts that cannot be sep. will map to same pt.

<u>Ex</u>: Consider the curve $C \subseteq \mathbb{P}^3$ which is the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$

$$(s:t) \longmapsto (s^3 : s^2t : st^2 : t^3)$$

- This map comes from the linear system (deg 3 b/c curve is twisted cubic, & any div. of deg 3 is equiv — b/c $\text{Pic}(\mathbb{P}^1) = \mathbb{Z}$, so only deg matters) $D = 3H$ on $\mathbb{P}^1$. Is this lin sys. complete? map to $\mathbb{P}^3$, so must be 4 sections, & $h^0(\mathbb{P}^1, \mathcal{O}(3H)) = 4$ ← from before, so is complete. If map had been to $\mathbb{P}^2$, then would be 3 sections, so not complete.

Ex: $\mathrm{Aut}(\mathbb{P}^1) = PGL(2)$ has dim 3 $\Rightarrow$ Mobius Transformations

$(s:t) \longmapsto (as+bt : cs+dt)$ s.t. $ad-bc \neq 0$

in finite coord, $s \mapsto \frac{as+b}{cs+d}$

* for any 3 pts in $\mathbb{P}^1$, $\exists!$ auto of $\mathbb{P}^1$ which maps them to $0, 1, \infty$. [i.e. any 3 pts are same as any other 3]

* 4 pts give some info, the cross-ratio, i.e, where does the 4th pt go under the above! auto?

Thm: (Riemann-Roch) Let C be a smooth proj. curve of genus g. Then $h^0(D) - h^0(K-D) = \deg D + 1 - g$, $\forall D \in \mathrm{Div}(C)$, where K = canonical divisor.

Ex: $D=0$: $h^0(D) = 1$ (global sections on trivial line bundle, are global fcns on C & so constant, so there are a k's worth)

$\overset{RR}{\Rightarrow}$ $1 - h^0(K) = 0 + 1 - g \Rightarrow \boxed{h^0(K) = g} \leftarrow$ definition of genus g.

Ex: $D=K$: $\underbrace{h^0(K)}_{g} - \underbrace{h^0(0)}_{1} = \deg K + 1 - g$

$\Rightarrow g - 1 = \deg K + 1 - g \Rightarrow \boxed{\deg K = 2g-2}$

Ex: $K_{\mathbb{P}^1}$ = canon. div. on $\mathbb{P}^1$. Let's try to understand the line bundle ω of differential 1-forms on $\mathbb{P}^1$.

[coeff of d/dy times x coord — $x\,dx$, or $x\,dy$

$a \cdot d/dx + b \cdot d/dy$ = tan. vector]

[i.e. smth that, at each pt of sp, eats a tan. vect & produces number — so an elt of dual of vect sp. "global" $\Rightarrow$ can do it $\forall$ pts in a smooth/differentiable manner]

$\rightarrow$ Line bundle b/c get a line at each pt on curve.

dz is dual of d/dz

To understand which divisor it comes from, we'll take a meromorphic section & look at its zeros & poles.

Cover $\mathbb{P}^1$ by U_0, U_1. A regular diff. 1-form on U_0 is: dz. On U_0, dz has no zeros & no poles. Try to write dz as a 1-form (maybe w/ poles) on U_1.

coords $\Rightarrow$ (z), $(1/z)$

$z\,dz$ has 0 at origin, $\frac{1}{z}dz$ has pole at origin.

Claim: $d(1/z) = -\frac{1}{z^2} dz$

Conclusion, dz has pole of order 2 at ∞.
$\Rightarrow$ $|\omega = \mathcal{O}(-2H)$. [i.e. $\exists$ pole of order 2 somewhere]
$\Rightarrow$ genus of $\mathbb{P}^1 = 0$ by R.R.
& true, that $h^0(\mathcal{O}(-2H)) = 0$ b/c $-2 < 0$, i.e $\nexists$
fcn w/ no poles but zero of order 2

12/5 $K_{\mathbb{P}^1} = \mathcal{O}(-2)$:
Cover $\mathbb{P}^1$ by U_0 w/ coord z & U_1 w/ coord. $1/z$.
<u>Note</u>: On U_1, there is the nowhere vanishing 1-form $\omega = d(1/z)$. On $U_1 \cap U_0$, we have 2 possible coords, z & $1/z$, therefore dz & $d(1/z)$ are coords for the space of 1-forms.
Apply Leibnitz rule: $d(1/z) = -1/z^2 \, dz$, so the 1-form ω on U_1 has a pole of order 2 when we try to extend it over 0.
$\Rightarrow$ A global meromorphic section of $K_{\mathbb{P}^1}$ has no zeros & a pole of order 2 at the origin
$\Rightarrow$ $K_{\mathbb{P}^1} = \mathcal{O}(-2 \text{"}0\text{"}) = \mathcal{O}(-2)$
$\Rightarrow$ $h^0(K_{\mathbb{P}^1}) = 0$ (A divisor of negative degree can never have sections)
$\Rightarrow$ genus of $\mathbb{P}^1 = 0$ b/c $g(C) := h^0(K_{\mathbb{P}^1})$

<u>Ex</u>: $K_{\text{elliptic curve}} = \mathcal{O}$ (triv.) (genus should be 1, so deg should be 0 $\rightarrow$ Know triv. line bundle has deg 0)
ell. curves <u>gp</u> $\Rightarrow$ tangent bunde trivial $\Rightarrow$ cotan. bundle (its dual) also triv.
$\hookrightarrow = K$
triv $\Rightarrow$ $\exists$ global way to ID fiber at one pt w/ fiber at any other pt w/o choices — in a gp, can use gp law to get from origin to any other pt.

$\Rightarrow h^0(K) = 1 \Rightarrow g(\text{ell. curve}) = 1.$
(only fcn's are const)

* If a curve has a gp structure, then its genus must be 1 * (smooth, proj)

The converse is also true:

(· every smooth cubic in $\mathbb{P}^2$ has gp)

<u>Idea of pf</u>: WTS: every smooth curve of genus 1 can be embedded in $\mathbb{P}^2$ as a smooth cubic. Then we'd be done.

We need a linear system to get a map to $\mathbb{P}^2$, & that needs a divisor - what degree should the divisor have? deg of divisor = intersection # of image w/ hyperplane in $\mathbb{P}^2$, so we want deg 3.

Pick a divisor D on C of deg 3. Take the linear system which has all sections of D (this gives the best shot of map being an embedding).

How many are there?

RR: $\underset{\substack{\nearrow \\ \#\text{ of sections of } D}}{h^0(D)} - h^0(K-D) = 1 - \underset{1}{\underset{\shortparallel}{g}} + \underset{3}{\underset{\shortparallel}{\deg D}} = 3$

$\deg K = 2g-2 = 0$

$\deg(K-D) = -3 \Rightarrow K-D$ cannot have sections, so $h^0(K-D) = 0$

$\Rightarrow h^0(D) = 3 \Rightarrow 3$ sections, so if D is BPF, get map to $\mathbb{P}^2$, & the 3 sections will be the coords of $\mathbb{P}^2$.

D is BPF, sep pts, & sep tan $\Rightarrow$ get embedding.
Black Box

topological genus 3

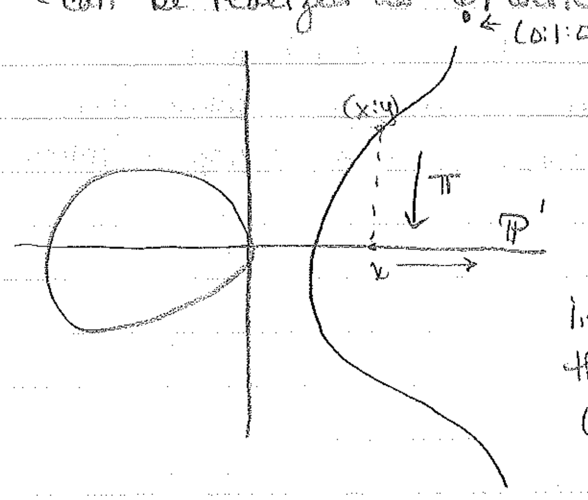
What's the connection btwn our notion of genus & this top. notion?

Over $\mathbb{C}$, a curve is a surface

If C is over $\mathbb{C}$ (i.e. endow $\mathbb{CP}^n$ w/ $\mathbb{C}$-topology instead of Zariski), we can regard it as a real surface & can find its genus.

$C = \mathbb{P}^1 \rightsquigarrow$ sphere $= S^2$, genus $= 0$

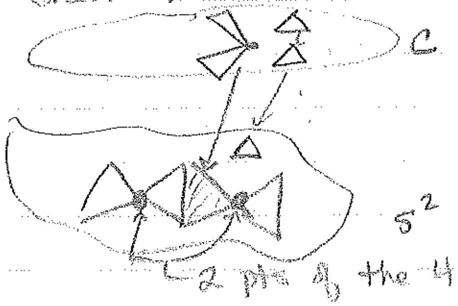
$C =$ ell. curve $y^2 z = x(x-z)(x+z) \subseteq \mathbb{P}^2$
↑ can be realized as $\mathbb{C}/$lattice $\Rightarrow$ torus.



We can extend π by sending P to ∞, as before:

as $x \to \infty$, $y \to$ which ∞?
i.e what is the limit of the slope of the line through $(x:y)$ & 0: $y \approx \sqrt{x^3} = x^{3/2}$
$\Rightarrow \underbrace{y/x}_{\text{slope}} = x^{1/2} \to \infty$ as $x \to \infty$
$\Rightarrow$ goes up faster than to right
$\Rightarrow$ goes to ∞ at top.

<u>Recall</u>: π exhibits C as a double cover of $\mathbb{P}^1$ branched over 4 pts
In top, C cover $\mathbb{P}^1 = S^2$ 2-1 except at 4 pts. Calculate the Euler char $\chi(C)$



↖ 2 pts of the 4 branching pts

Call $F_{S^2} =$ # of triangles
$E_{S^2} =$ # of edges
$V_{S^2} =$ # of vertices

easiest triangulation
↓
$S^2 =$ tetrahedron
$F=4$ $E=6$
$V=4$
$4-6+4=2$

$$F_{S^2} - E_{S^2} + V_{S^2} = 2 \quad \Rightarrow \quad \left.\begin{array}{l} F_C = 2F_{S^2} \\ E_C = 2E_{S^2} \\ V = 2V_{S^2} - 4 \end{array}\right\} \Rightarrow \chi(C) = 2(\chi(S^2)) - 4 = 0$$

Thm in top: $\chi(C) = 2-2g$, g = top genus
$\Rightarrow g(C) = 1$. $\Rightarrow C$ a torus.
[Form of Hurvitz Thm in alg. geom]

Ex: Consider the map $\mathbb{P}^1 \to \mathbb{P}^1$
$[x:y] \longmapsto [x^2:y^2]$
in finite part, it's $x \mapsto x^2$ $(y=1)$
This map is 2:1 except it is branched over $[0:1]$ & $[1:0]$ (preimage of ∞ only ∞)
From previous calculation, we get
$\chi(\mathbb{P}^1) = 2\chi(S^2) - 2 = 0 \Rightarrow \chi(\mathbb{P}^1) = 2 \Rightarrow g(\mathbb{P}^1) = 0$,
as we know.

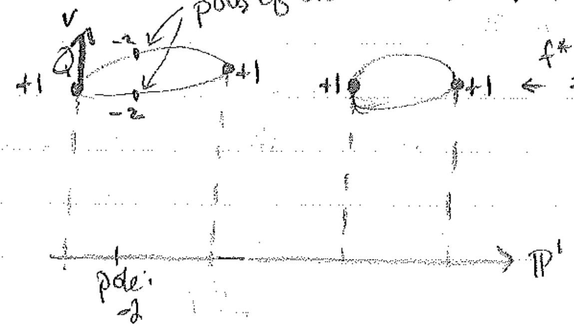
- If $X \to Y$ is a 2:1 covering of curves w/ k branching pts,

$\chi(Y) = F_Y - E_Y + V_Y = 2 - 2g_Y$
$2-2g_X = F_X - E_X + V_X = 2\cdot\chi(Y) - k \Rightarrow k$ must be even
$= 2(2-2g_Y) - k$
$\Rightarrow 2g_X = k - 2 + 4g_Y \Rightarrow g_X = 2g_Y + \frac{k-2}{2}$

Ex: Say $C \xrightarrow{f} \mathbb{P}^1$ 2:1 branched over k pts. Let's calculate $\deg K_C$. From top, should be $2g_C - 2 = \overset{\text{since } g_{\mathbb{P}^1}=0}{\overset{\downarrow}{2(\tfrac{k-2}{2})}} - 2 = k-4$
Calculate directly:
Pick a meromorphic diff one form ω on $\mathbb{P}^1$ w/ no zeros ($d(1/z)$) so that its poles are not at the branching pts.
Look at $f^*\omega$ a mero. 1-form on C.

given map of manif & diff form on target, can pull back to diff form on 1st mani.

poles of order 2 b/c f iso off branch pts,



$f^*\omega$ has zeros of order 1 at these pts:
Evaluate $f^*\omega$ on tangent vector at Q. $f^*\omega(v) = \omega(f_* v) = \underset{\substack{\uparrow \\ \text{tan vect becomes pt}}}{\omega(0)} = 0$

$f^*\omega$ has 2× as many poles & has k zeros

$\Rightarrow \deg f^*\omega = \underset{\uparrow\, 2(-2)}{k-4}$ (zeros minus poles)

<u>Ex</u>: Curves of genus 2

- First we want to embed it in a space so we need to pick a divisor.

- all curves of genus 0 are $\cong \mathbb{P}^1$
- curves of genus 1 are all cubics in $\mathbb{P}^2$ & always form a gp.

Take a divisor $\overset{D}{}$ of deg 2:

$RR \Rightarrow h^0(D) - h^0(K-D) = 1-g+\deg D$

$= 1-2+2$

$= 1$ unlikely this will work

$\deg K-D = 0$:

$\deg K = 2g-2 = 4-2 = 2$

$\deg D = 2$

$\Rightarrow h^0(K-D) = \begin{cases} 1 \\ 0 \end{cases}$

If $\mathcal{O}(K-D)$ has a section (holomorphic) then # of zeros − $\underset{0\,\leftarrow \text{ no poles b/c holo.}}{\overset{}{\#\text{ of poles}}}$ = 0 (deg of K−D)

$\Rightarrow$ no zeros $\Rightarrow \mathcal{O}(K-D)$ is trivial $= \mathcal{O}$, which has only one section. ← i.e. $D \sim K$.

Thus $h^0(D) = \begin{cases} 2 \Rightarrow \text{map to } \mathbb{P}^1 \\ 1 \Rightarrow \text{map to } \mathbb{P}^0 = \text{pt.} \end{cases}$

$\Rightarrow$ every curve of genus 2 admits a map to $\mathbb{P}^1$, since D is basept free, ← black box which is 2:1 w/ 6 branching pts. (from $2g_C - 2 = k - 4$)

We've proved that every curve of genus 2 is "hyper-elliptic" i.e. has a 2:1 map to $\mathbb{P}^1$.

- How many curves of genus 2 are there?

$\dim M_0 = 0$

$\dim M_1 = 1$

$\dim M_2 = ?$

If we believe that all curves of genus g are parametrized by the pts of a space, M_g, what is $\dim M_g$?

dim $M_2 = 3$ b/c only need to pick branch pts → 6.
6 pts have 3 pts of info b/c ∃ automorphisms & the 1st 3 can be sent to $0, 1, \infty$, & look where last 3 go.
[can use same reasoning for 4 pts to prove dim $M_1 = 1$]

Thus dim $M_g^{\text{hyperell.}} = 2g+2-3 = 2g-1$.
genus g ⇒ branch over $2g+2$ pts
In higher genuses, not all curves are hyperelliptic.

12/10 There exists a space M_g $(g \geq 2)$ whose points parametrize all curves of genus g.

<u>Fun Fact</u>: Not all curves can be embedded in $\mathbb{P}^2$
- genus 0 $\hookrightarrow \mathbb{P}^2$ as lines or quadrics
- genus 1 $\hookrightarrow \mathbb{P}^2$ as cubics

Idea: A curve in $\mathbb{P}^2$ has a degree, d. The genus of such a curve is: $g = \frac{(d-1)(d-2)}{2}$. (notice: gives 0 for line or quadric, get 1 for cubic) But this expression skips integers ($d=4 \Rightarrow g=3$), so genus 2 cannot be embedded in $\mathbb{P}^2$. But can always embed in $\mathbb{P}^3$ (all curves can) & project down, but this won't be an iso, b/c any line through random pt in $\mathbb{P}^3$ will hit curve twice, yielding a node.

Ex: Finding genus of curve C of deg d in $\mathbb{P}^2$. To do so, we need to find $\deg K_C$ [$=2g-2$, so done].

Adjunction Formula: If $X \subseteq Y$ is a hypersurface (all smooth) then $K_X = (K_Y + X).X$.

$\hat{}$ lin comb of subvar's of codim 1 on X b/c divisor

$K_Y.X$ will be codim 1 in X if things suff. nice

$X.X$ – move X around inside Y so it becomes transversal to X, then intersect. This will be a hypersurface in X.

Let C be a curve in $\mathbb{P}^2$ (so is a hypersurface) of deg d

$K_C = (K_{\mathbb{P}^2} + C).C$

- What is $K_{\mathbb{P}^2}$? Answer: $\mathcal{O}(-3H)$
  – in general $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$
- $\deg K_{\mathbb{P}^2}.C = -3d$ (-3 hyperplanes int. curve, each at d pts)

$\deg C.C = d^2$

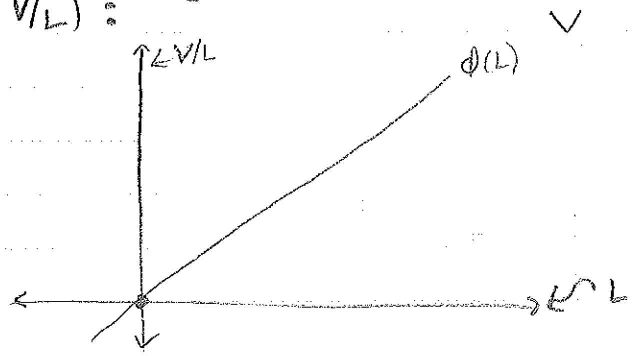
$\Rightarrow \deg K_C = -3d + d^2$

$\Rightarrow 2g - 2 = -3d + d^2 \Rightarrow g = \frac{d^2-3d+2}{2} = \frac{(d-1)(d-2)}{2}$ ✓

How do we understand the tangent space to $\mathbb{P}^n$?

Idea: Fix a vector sp V of dim $n+1$ s.t. $\mathbb{P}^n = \mathbb{P}(V)$, i.e. $= \{L \subseteq V \mid L$ has dim 1 is a linear subsp$\}$

Observation: The tangent space to $\mathbb{P}^n$ at a pt $[L]$ is canonically isomorphic to the vect sp. $\mathrm{Hom}(L, V/L)$:



Given $\phi: L \to V/L$, draw its graph. These give all lines in V except V/L ($\perp$ to L). But these lines are far away from L, & the tangent vector at L

only cares about lines close to $\mathbb{P}^n$. $\swarrow d(\nu)$

- There is a SES of vector bundles on $\mathbb{P}^n$

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

(global version of the observation)

Another version of this would be:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}^{n+1} \to T_{\mathbb{P}^n}(-1) \to 0$$

(tautological bundle) →

$\mathcal{O}^{n+1}$ ⟶ fiber over a pt $[L]$ is V (doesn't change from pt to pt, so triv line bundle)

$\mathcal{O}_{\mathbb{P}^n}(-1)$ ↓ fiber over a pt $[L]$ of $\mathcal{O}(-1)$ is L (recall from previous class)

$L \subseteq V$, so this first map is an inclusion: L moves inside V.

Since $T_{\mathbb{P}^n}(-1) = \mathcal{O}^{n+1}/\mathcal{O}_{\mathbb{P}^n}(-1)$, the fiber of $T_{\mathbb{P}^n}(-1)$ over $[L]$ is V/L. The -1 comes from fiber $= V/L$ & not $\mathrm{Hom}(L, V/L) = T_{\mathbb{P}^n}$.

Call $T_{\mathbb{P}^n}(-1) = E$

At each pt $[L]$, the tangent sp at $[L]$ is

$\mathrm{Hom}(L, V/L) = \mathrm{Hom}(\text{fiber of } \mathcal{O}(-1) \text{ at } [L], \text{fiber of } E \text{ at } [L])$

$\Rightarrow T_{\mathbb{P}^n} = \mathrm{Hom}(\mathcal{O}(-1), E) = E \otimes \mathcal{O}(1)$ ↩ dual of $\mathcal{O}(-1)$

[$\mathrm{Hom}(\mathcal{L}, E) = E \otimes \mathcal{L}^\vee$ ($\mathcal{L}$ a line bundle)
(globalized version of $\mathrm{Hom}(V, W) = V^\vee \otimes W$)]

$\Rightarrow$ 1st SES from 2nd SES can be obtained by $\otimes \mathcal{O}(1)$ & tensoring is exact

The 1st SES is called the <u>Euler Exact Sequence</u>.

Now dualize: $0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to 0$ ← cotangent sp.

(b/c dual an exact contravariant funtor of vector sp's)

Fact: If $0 \to V' \to V \to V'' \to 0$ is a SES of v.s.,

Then $\Lambda^{top} V \simeq \Lambda^{top} V' \otimes \Lambda^{top} V''$

↑ top exterior power

xpress top exterior power of one in terms of top 1-dim exterior power of other

$\mathcal{O}(K_{\mathbb{P}^n}) = \Lambda^{top}\,\Omega'_{\mathbb{P}^n} \otimes \underbrace{\Lambda^{top}\,\mathcal{O}}_{=\mathcal{O}} \cong \Lambda^{top}(\mathcal{O}(-1)^{n+1}) = \mathcal{O}(-n-1)$

$\Rightarrow \mathcal{O}(K_{\mathbb{P}^n}) = \mathcal{O}(-n-1)$ ✓

<u>Riemann-Roch</u>: $h^0(D) - h^0(K-D) = \deg D + 1 - g$

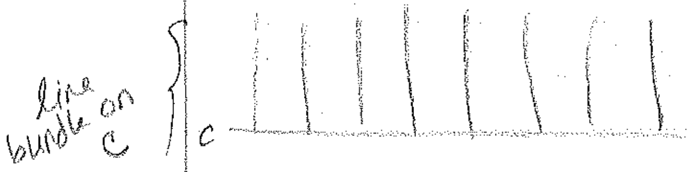
- 1^{st} pretend $h^0(K-D) = 0$. Then we can say:
  if $D = \mathcal{O}$, know formula: $\underset{1}{\underset{\shortparallel}{h^0(\mathcal{O})}} \underbrace{-g}_{\text{correction factor}} = 0 + 1 - g$ ✓
- if add 1 more pt to D RR says # of sections goes up by 1.
  "Prove" $h^0(D+P) = h^0(D) + 1$, if we hope $h^0(K-D)$ goes away for D large enough
  [so # of sections of a divisor goes linear w/ deg of D, corrected by smth connected to genus]
  $h^0(D+P)$ are mero. fcns on C, allowed to have poles at D & at P
  $h^0(D)$ are mero fcns on C, allowed poles at D only.

In fact, have an injective map of line bundles
$0 \to \mathcal{O}(D) \longrightarrow \mathcal{O}(D+P)$ & it gives rise to an inj map of global sections $0 \to \Gamma(\mathcal{O}(D)) \to \Gamma(\mathcal{O}(D+P))$
To know dim increases by 1, we need to make a SES:

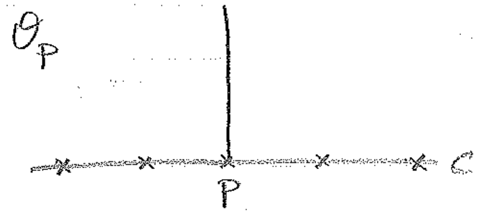
$0 \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}(D+P) \longrightarrow \mathcal{O}_P \longrightarrow 0$

(skyscraper sheaf)
↖ no longer a vector bundle

Think of this as:

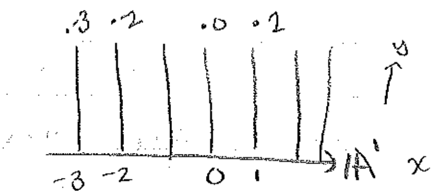


vs.



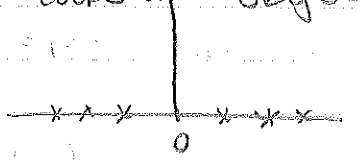
0-dim'l vect bundle at all pts except P, then a line at P

Ex: On $\mathbb{A}^1$, consider the map of line bundles $\mathcal{O} \to \mathcal{O}$ which at the pt x is multiplication in the fiber by x.



$\mathbb{A}^2 \to \mathbb{A}^2$ cts.
$(x,y) \mapsto (x, xy)$

This map is an $\cong$ when $x \neq 0$, so Ker is trivial in almost all fibers. But at $x=0$, Ker & coker is 1 dim. So Ker looks like skyscraper



Fcns on $\mathbb{A}^1$ is $k[x]$, so can think of this map

$0 \to k[x] \xrightarrow{\cdot x} k[x] \to k[x]/(x) \to 0$

$\exists$ corresp. btwn line bundles & proj modules

proj. module

not proj. module

12/12 Try to relate $\mathcal{O}(D)$ & $\mathcal{O}(D+P)$, P a pt on curve C.
Serre, [F.A.C] (algebraic coherent sheaves):

form an abel. category

Embed vector/line bundles into a larger category, "Coherent sheaves" which is abelian.

$\exists$ SES

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D+P) \to \mathcal{O}_p \to 0$$

skyscraper sheaf

Q: Is $h^0(D+P) = h^0(D)+1$? (cannot always be true b/c $h^0(D) \geq 0$, & if $h^0(D+p=0)$, problem.)

- but we expect this to be true for D big enough but may fail for small D.
- Know $h^0(\mathcal{O}_p) = 1$.

$\exists$ functor $\Gamma : Coh(X) \longrightarrow Vect$ (coherent sheaves to vector sp

Is Γ exact? No. If it were, from SES,

$$\Gamma(SES) \Rightarrow 0 \to \underset{\overset{\uparrow}{\dim = h^0(D)}}{\Gamma(X, \mathcal{O}(D))} \to \underset{\overset{\uparrow}{\dim = h^0(D+P)}}{\Gamma(X, \mathcal{O}(D+P))} \to \underset{\overset{\uparrow}{\dim = 1}}{\Gamma(X, \mathcal{O}_p)} \to 0$$

but this is not always true. The failure is the 0 on the right. We always have:

$$0 \to \Gamma(X, \mathcal{O}(D)) \to \Gamma(X, \mathcal{O}(D+P)) \to \Gamma(X, \mathcal{O}_p) \to \underset{\overset{\uparrow}{\text{cohomology}}}{H^1(X, \mathcal{O}(D))}$$

$$\to H^1(X, \mathcal{O}(D+P)) \to H_1(X, \mathcal{O}_p) = 0$$, a LES

Define $\chi(D) = h^0(D) - h^1(D) = \dim(\Gamma(\mathcal{O}(D))) - \dim H^1(\mathcal{O}(D))$

$\chi(D+P) = h^0(D+P) - h^1(D+P)$

$\chi(\mathcal{O}_p) = 1$

$\Rightarrow \chi(D+P) = \chi(D) + 1$ ← Always True

R.R. really says: ① $\chi(D) = \deg D + 1 - g$

② $\chi(D) = h^0(D) - h^1(D)$

RR: LHS: $h^0(D) - h^0(K-D)$

By Serre Duality, $\underset{\overset{\uparrow}{\text{very similar to Poincaré duality}}}{h^0(K-D) = h^1(D)}$

in gen. SD say $H^i(X, \mathcal{O}(D)) \cong H^{n-i}(X, \mathcal{O}(K-D))^\vee$

for an n-dim'l cpt variety X.

* Correction & Addition to HW

② It is well known that if ~~$g \geq 2$~~ $g \geq 3$, $2K_C$ gives an embedding

Addition (E.C.): Show that it could not be true for $g = 2$.

Grassmanians & Their Tangent Bundles

Fix a f.d. vect. sp. V of dim n, fix $0<k<n$.

$Gr(k,V) = \{L \subseteq V \mid \dim L = k\}$

$\left(\overset{\shortparallel}{Gr}(k,n)\right)$

- $Gr(1,V) = \mathbb{P}V$ (i.e., set of lines in V)

<u>Claim</u>: If $[L] \in Gr(k,V)$, then $T_{[L], Gr(k,V)} \cong \underset{\uparrow \dim k(n-k)}{Hom(L, V/L)}$

<u>Pf</u>:

① (Intuitive) Fix a fixed vect. sp. W of dim k.
(think of $L \subseteq V$ as the image of W under a linear map $W \to V$)

Any injective map $\phi: W \to V$ gives a pt of $Gr(k,V)$
(take $L = Im(\phi)$)

The space $Hom(W,V)$ is a vect. sp ($\cong W^\vee \otimes V$)

$\rotatebox{90}{$\subseteq$}$

$\mathcal{U} = \{\phi \mid \phi \text{ injective}\}$, Zariski open $\left(\begin{smallmatrix}\text{dense \&}\\ \text{open}\end{smallmatrix}\right)$

b/c $Hom(W,V)$ consists of a $k \times k$ matrix. Those that are inj have full rank $\Rightarrow$ no det of minors vanish. So $\mathcal{U}$ is cut out by the $\binom{n}{r}$-eqns that are the dets of minors $= 0$.

We have a map $\mathcal{U} \to Gr(k,V)$, surjective.

Not injective: We can think of pts of $\mathcal{U}$ as pairs $(L \subseteq V, \phi: \underbrace{W \xrightarrow{\cong} L}_{})$ (while $Gr(k,V)$ doesn't need ϕ ..)

there's a gp acting on this set of choices!

$\exists\ GL(W)$ action on $\mathcal{U}$: for $\psi \in GL(W)$ (i.e. automorph of W)

$\psi: (L, \phi) \to (L, \phi \circ \psi)$ (image doesn't change but map does change)

Thus $Gr(k,V) \cong \mathcal{U}/GL(W)$ action is free

$\dim Gr(k,V) = nk - k^2 = k(n-k)$.

Assume given a pt $[L] \in Gr(k,V)$. Think of $L = \mathrm{Im}(\phi)$ for some $\phi : W \to V$.

ϕ can be deformed to 1st order by picking a map $\psi : W \to V$ & taking $\phi + \varepsilon\psi$ for ε small ($\varepsilon \ll 1$)

$\Rightarrow T_{\phi,U} \simeq \mathrm{Hom}(W,V)$

Under this change of ϕ, L does not nec. change: If ψ happens to map W to L, then $\mathrm{Im}(\phi + \varepsilon\psi) = \mathrm{Im}(\phi) = L$. ($\Leftrightarrow$ statement, actually)

$\Rightarrow T_{[L],Gr(k,V)} = \mathrm{Hom}(W,V) / \mathrm{Hom}(W,L)$ } these are the ψ's that will actual change L ...

$0 \to L \to V \to V/L \to 0$ SES

$\Rightarrow 0 \to \mathrm{Hom}(W,L) \to \mathrm{Hom}(W,V) \to \mathrm{Hom}(W,V/L) \to 0$

[Recall $\mathrm{Hom}(W,\cdot)$ a covariant exact functor]

$\Rightarrow \mathrm{Hom}(W,V)/\mathrm{Hom}(W,L) \simeq \mathrm{Hom}(W,V/L) \underset{\text{via } \phi}{\simeq} \mathrm{Hom}(L, V/L)$ $\square$

* This is still a bit hand-wavy.

To be rigorous, look at

$Gr(k,V) = U/GL(W)$

$\Rightarrow T_{[L],Gr(k,V)} \simeq T_{\phi}U / \underset{\uparrow \text{Lie alg of } GL(W) \,=\, T_{GL(W),0}}{\mathfrak{gl}(W)} = \mathrm{Hom}(W,V)/\mathrm{End}(W) \simeq \mathrm{Hom}(W,V$

In general, on $Gr(k,V)$, there are 2 natural bundles: $\mathbb{T}$ = tautological & Q = quotient, defined by

$\mathbb{T}_{[L]} = L$ $\qquad Q_{[L]} = V/L$

$\uparrow$ fiber at L

The fact that at each pt of $Gr(k,V)$ we have a SES of v.s.'s:

$0 \to L \to V \to V/L \to 0$

becomes a SES of vect. bundles:

$0 \to \mathbb{T} \to V \otimes \mathcal{O} \to Q \to 0$

$\uparrow$ trivial vector bundle w/ fiber V

$\Rightarrow T_{Gr} = \mathrm{Hom}(\mathbb{T}, Q) \simeq \mathbb{T}^\vee \otimes Q$ $\Rightarrow$ have a SES ($\otimes \mathbb{T}^\vee$)

$0 \to \underbrace{\mathbb{T} \otimes \mathbb{T}^\vee}_{\mathrm{End}(\mathbb{T})} \to V \otimes \mathbb{T}^\vee \to T_{Gr} \to 0$

For proj. $\overset{\mathbb{P}^n}{\text{sp}}$, $\mathbb{T} = \mathcal{O}(-1)$. (now a line bundle b/c $k = 1$)

So this becomes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, equal] & \underbrace{\mathcal{O}(1) \otimes V} \arrow[r] \arrow[dr] & T_{\mathbb{P}^n} \arrow[r] & 0 \\
 & \mathcal{O}(-1) \otimes \mathcal{O}(1) & & \mathcal{O}(1)^{n+1} &
\end{tikzcd}$$

This is the Euler exact sequence.